# CIRCULAR FREE SPECTRAHEDRA

ERIC EVERT, J. WILLIAM HELTON[1], IGOR KLEP[2], AND SCOTT MCCULLOUGH[3]

Abstract. This paper considers matrix convex sets invariant under several types of rotations. It is known that matrix convex sets that are free semialgebraic are solution sets of Linear Matrix Inequalities (LMIs); they are called free spectrahedra. We classify all free spectrahedra that are circular, that is, closed under multiplication by $e^{it}$: up to unitary equivalence, the coefficients of a minimal LMI defining a circular free spectrahedron have a common block decomposition in which the only nonzero blocks are on the superdiagonal.

A matrix convex set is called free circular if it is closed under left multiplication by unitary matrices. As a consequence of a Hahn-Banach separation theorem for free circular matrix convex sets, we show the coefficients of a minimal LMI defining a free circular free spectrahedron have, up to unitary equivalence, a block decomposition as above with only two blocks.

This paper also gives a classification of those noncommutative polynomials invariant under conjugating each coordinate by a different unitary matrix. Up to unitary equivalence such a polynomial must be a direct sum of univariate polynomials.

## 1. Introduction

For square matrices $A, B$, write $A \preceq B$ (resp. $A \prec B$) to express that $B - A$ is positive semidefinite (resp. positive definite). Given a $g$-tuple $A = (A_1, \dots A_g) \in M_d(\mathbb{C})^g$, let $\Lambda_A(x)$ denote the linear matrix polynomial

$$\Lambda_A(x) = \sum_{j=1}^{g} A_j x_j \tag{1.1}$$

and let $L_A$ denote the **(symmetric monic) linear pencil**

$$L_A(x) = I_d - \sum_{j=1}^{g} A_j x_j - \sum_{j=1}^{g} A_j^* x_j^* = I_d - \Lambda_A(x) - \Lambda_A(x)^*. \tag{1.2}$$

2010 *Mathematics Subject Classification.* 47L07, 52A05 (Primary); 46N10, 46L07, 32F17 (Secondary).

*Key words and phrases.* linear matrix inequality, LMI, spectrahedron, matrix convex set, free convexity, circular domain, invariant polynomial, free analysis.

[1]Research supported by the NSF grant DMS 1201498, and the Ford Motor Co.

[2]Supported by the Marsden Fund Council of the Royal Society of New Zealand. Partially supported by the Slovenian Research Agency grants P1-0222 and L1-6722.

[3]Research supported by the NSF grant DMS-1361501.

The **spectrahedron** $\mathscr{S}_A$ is the set of all $x \in \mathbb{C}^g$ satisfying the **linear matrix inequality** (**LMI**) $L_A(x) \succeq 0$. Spectrahedra and LMIs are ubiquitous in control theory [SIG97, BGFB94] and optimization [BPR13]. Indeed LMIs are at the heart of the subject called semidefinite programming.

This article investigates spectrahedra from the perspective of the emerging areas of free convexity [DDSS+, Eff09, EW97, Far12, HKM+, WW99, Wit84, Zal+] and free analysis [AM14, BMV+, HKM12, KVV14, KŠ+, Pop08, Tay72, Voi10]. In free analysis we are interested in matrix variables and evaluate a linear pencil on $g$-tuples $X = (X_1, \dots, X_g) \in M_n(\mathbb{C})^g$ according to the formula

$$L(X) = I_d \otimes I_n - \sum_{j=1}^{g} A_j \otimes X_j - \sum_{j=1}^{g} A_j^* \otimes X_j^*. \tag{1.3}$$

For positive integers $n$, let

$$\mathcal{D}_A(n) = \{X \in M_n(\mathbb{C})^g : L_A(X) \succeq 0\}. \tag{1.4}$$

The sequence $\mathcal{D}_A = (\mathcal{D}_A(n))_n$ is called a **free spectrahedron**. It is the set of all solutions to the ampliated LMI corresponding to $L_A$. In particular, $\mathcal{D}_A(1) = \mathscr{S}_A$. Free spectrahedra are closely connected with operator systems for which [FP12, KPTT13, Arv08] are a few recent references. In a different direction they provide a model for convexity phenomena in linear system engineering problems described entirely by signal flow diagrams [dOHMP09].

The main results of this article characterize free spectrahedra and free polynomials that are invariant under various natural types of circular symmetry. A core motivation for this article comes from classical several complex variables where the study of maps on various types of domains is a major theme. There an important class is the *circular domains*. These behave very well under bianalytic mappings as described e.g. by Braun-Kaup-Upmeier [BKU78].

### 1.1. **Main Results.**

This subsection contains a summary of the main results of the paper. Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_{n\in\mathbb{N}}$ of $g$-tuples of $n \times n$ matrices with entries from $\mathbb{C}$. A subset $\Gamma \subseteq M(\mathbb{C})^g$ is a sequence $(\Gamma(n))_n$ where $\Gamma(n) \subseteq M_n(\mathbb{C})^g$.

#### 1.1.1. *Rotationally invariant free spectrahedra.*

A subset $\mathcal{D} \subseteq M(\mathbb{C})^g$ is **circular** if $Z \in \mathcal{D}$ implies $e^{it} Z \in \mathcal{D}$ for all $t \in \mathbb{R}$ and is **free circular** if $UZ \in \mathcal{D}$ for each $n$, each $Z \in \mathcal{D}(n)$, and each $n \times n$ unitary matrix $U \in M_n(\mathbb{C})$. Here $UZ = (UZ_1, \dots, UZ_g)$. Geometric and analytic properties of circular subsets of $\mathbb{C}^n$ and their generalizations, such as Reinhardt domains, are heavily investigated in several complex variables [Kra01], cf. [BKU78].

Given a tuple $A \in M_d(\mathbb{C})^g$, if there is an orthogonal decomposition of $\mathbb{C}^d$ such that with respect to this decomposition $A = A^1 \oplus A^2$, then $L_A(x) = (L_{A^1} \oplus L_{A^2})(x)$. In this case each $L_{A^i}$ is a **subpencil** of $L_A$. If $\mathcal{D}_A = \mathcal{D}_{A^i}$, then $L_{A^i}$ is a **defining subpencil** for $\mathcal{D}_A$. Say the pencil $L_A$ is a **minimal defining pencil** for $\mathcal{D}_A$ if no proper subpencil of $L_A$ is a defining subpencil for $D_A$.

Theorem 1.1 below says the tuple $A$ in a minimal defining pencil $L_A$ of a circular free spectrahedron is (up to unitary equivalence) block superdiagonal. It also says, if the domain is free circular, then there are just two blocks. We refer to such a domain as a **matrix pencil ball**.

**Theorem 1.1.** *Let $A \in M_d(\mathbb{C})^g$ and suppose $L_A$ is a minimal defining pencil for $\mathcal{D}_A$.*

(1) *Assume $A$ has no reducing subspace. The free spectrahedron $\mathcal{D}_A$ is circular if and only if there is an orthogonal decomposition of $\mathbb{C}^d$ such that, with respect to this decomposition, the $A_s$ have the block decomposition*

$$A_s = \begin{pmatrix} 0 & A_s(1) & 0 & \cdots & 0 \\ 0 & 0 & A_s(2) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & A_s(k) \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{1.5}$$

*where the $A_s(j)$ are matrices of appropriate sizes and for each $j$ there exists at least one $s_j$ such that $A_{s_j}(j) \neq 0$.*

*In any case, $\mathcal{D}_A$ is circular if and only if the $A_s$ can be written as a direct sum of block superdiagonal matrices of the form (1.5).*

(2) *The free spectrahedron $\mathcal{D}_A$ is free circular if and only if there exist $s, t \in \mathbb{N}$ with $s + t = d$ and a tuple $F$ of $s \times t$ matrices such that $A$ is unitarily equivalent to*

$$E = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}. \tag{1.6}$$

*Proof.* Part (1) is proved in Section 2 by a geometric argument. In strong contrast, the proof of Part (2) – given in Section 3, see Theorem 3.6 and Corollary 3.7 – depends on a strengthening (Proposition 3.3) of the characterization [BMV+, Proposition 3.5] of free circular matrix convex sets (i.e., a version of the Effros-Winkler Theorem [EW97] for free circular matrix convex sets). We give a self-contained proof of the latter in Appendix A, see Theorem A.5. ∎

1.1.2. *Rotationally invariant free polynomials.* A free $d \times d$ matrix polynomial $p$ is **invariant under coordinate unitary conjugation** if for any $n$, and any $g$-tuple of unitaries $U = (U_1, \ldots, U_g) \in M_n(\mathbb{C})^g$ there exists a unitary $W$ such that for all $X \in M_n(\mathbb{C})^g$,

$$p(U_1^* X_1 U_1, \ldots, U_g^* X_g U_g) = W^* p(X) W. \tag{1.7}$$

Our main theorem on polynomials characterizes monic free matrix polynomials that are invariant under coordinate unitary conjugation.

**Theorem 1.2.** *If $p$ is a monic free matrix polynomial, then $p$ is invariant under coordinate unitary conjugation if and only if*

$$p(x) \overset{u}{\sim} p_1(x_1) \oplus \cdots \oplus p_g(x_g).$$

*That is, $p$ must be (up to unitary equivalence) a direct sum of univariate matrix polynomials.*

*Proof.* The proof appears in Section 4. ■

1.2. **Readers Guide.** In Section 2 we prove Theorem 1.1 (1) – the classification of all circular free spectrahedra; i.e., free spectrahedra that are closed under rotations by $e^{it}$. In Section 3 we characterize free circular spectrahedra thus finishing the proof of Theorem 1.1. Finally, in Section 4 we turn our attention to free matrix polynomials and prove Theorem 1.2. Appendix A contains a self-contained proof of the Ball-Marx-Vinnikov Theorem [BMV+] classifying free circular matrix convex sets. We prove a sharpened version by establishing an effective Hahn-Banach separation result for free circular domains, see Proposition A.4.

## 2. Circular Free Spectrahedra

A subset $\mathcal{D} \subseteq M(\mathbb{C})^g$ is **circular** if $Z \in \mathcal{D}$ implies $e^{it}Z \in \mathcal{D}$ for all $t \in \mathbb{R}$. In this section the first part of Theorem 1.1 characterizing circular free spectrahedra is established. The main idea of the proof is as follows. Assuming $\mathcal{D}_A$ is a circular free spectrahedron, for each $t \in \mathbb{R}$, the pencil $L_{e^{it}A}$ determines the same free spectrahedron as $L_A$, namely $\mathcal{D}_A = \mathcal{D}_{e^{it}A}$. We are thus in a position to apply the Gleichstellensatz (see e.g. [HKM13, Theorem 1.2] and [Zal+, Theorem 1.2]) characterizing when two free spectrahedra are the same.

**Remark 2.1.** It turns out if, for a $t$ such that $\frac{t}{\pi}$ is irrational, $e^{it}A$ is unitarily equivalent to $A$, then $\mathcal{D}_A$ is circular. This fact is a corollary of the proof of Theorem 1.1 (1) given below. For a direct proof, observe, if $e^{it}A = U^*AU$, then $e^{int}A = U^{*n}AU^n$ and thus, for a dense set of $t \in \mathbb{R}$, the tuple $e^{it}A$ is unitarily equivalent to $A$. A routine limiting argument completes the proof.

### 2.1. Set up for the Proof of Theorem 1.1 (1).

Suppose $A$ satisfies the hypotheses of the theorem, except for possibly the irreducibility condition. Here, a $g$-tuple $A \in M_d(\mathbb{C})^g$ is said to be **irreducible** if the $A_s$ have no common reducing subspace, i.e, if there is no proper subspace $M \subseteq M_d(\mathbb{C})$ such that $A_sM \subseteq M$ and $A_s^*M \subseteq M$ for each $1 \leq s \leq g$.

We first present the linear Gleichstellensatz adapted to our set up of free (non-symmetric) variables.

**Proposition 2.2.** *If $B \in M_e(\mathbb{C})^g$ satisfies $\mathcal{D}_A = \mathcal{D}_B$, where $A \in M_d(\mathbb{C})^g$ is minimal defining for $\mathcal{D}_A$, then $B$ is unitarily equivalent to $A \oplus J$ for some $g$-tuple $J$.*

*Proof.* The statement holds when working over the field of real numbers and evaluating at tuples of symmetric matrices by [HKM13, Theorem 1.2] and [Zal+, Theorem 1.2]. It is easy to see that the same proofs work over the field of complex numbers and evaluating at tuples of self-adjoint matrices. We now reduce the proposition to this case.

To each monic pencil $L_A(x)$ in free variables $x, x^*$ we can associate a monic pencil $\mathcal{L}_{(A^{\mathrm{re}},A^{\mathrm{im}})}(y,z)$ with self-adjoint coefficients in self-adjoint variables $y, z$ as follows. Let $A_j^{\mathrm{re}} = \frac{1}{2}(A_j + A_j^*)$ and $A_j^{\mathrm{im}} = \frac{1}{2i}(A_j - A_j^*)$ for $j = 1, \ldots, g$. Then

$$\mathcal{L}_{(A^{\mathrm{re}},A^{\mathrm{im}})}(y,z) = I - \sum_{j=1}^{g} A_j^{\mathrm{re}} y_j - \sum_{j=1}^{g} A_j^{\mathrm{im}} z_j.$$

Each $X \in \mathcal{D}_A$ yields a point $\frac{1}{2}\big(X + X^*, i(X - X^*)\big)$ in the free spectrahedron (in self-adjoint variables) $\mathscr{D}_{(A^{\mathrm{re}},A^{\mathrm{im}})}$. Conversely, given $(Y, Z) \in \mathscr{D}_{(A^{\mathrm{re}},A^{\mathrm{im}})}$ we have $Y - iZ \in \mathcal{D}_A$. Hence $\mathcal{D}_A = \mathcal{D}_B$ implies that $\mathscr{D}_{(A^{\mathrm{re}},A^{\mathrm{im}})} = \mathscr{D}_{(B^{\mathrm{re}},B^{\mathrm{im}})}$.

We claim that $\mathcal{L}_{(A^{\mathrm{re}},A^{\mathrm{im}})}(y,z)$ is a minimal defining pencil for $\mathscr{D}_{(A^{\mathrm{re}},A^{\mathrm{im}})}$. Indeed, as otherwise by the Gleichstellensatz ([HKM13, Theorem 1.2] or [Zal+, Theorem 1.2]), there will be a reducing subspace for $(A^{\mathrm{re}}, A^{\mathrm{im}})$ and a compression $\mathcal{L}_{(\tilde{A}^{\mathrm{re}},\tilde{A}^{\mathrm{im}})}(y,z)$ of $\mathcal{L}_{(A^{\mathrm{re}},A^{\mathrm{im}})}(y,z)$ to this subspace with $\mathscr{D}_{(A^{\mathrm{re}},A^{\mathrm{im}})} = \mathscr{D}_{(\tilde{A}^{\mathrm{re}},\tilde{A}^{\mathrm{im}})}$. But this in turn will yield a subpencil $L_{\tilde{A}}$ of $A$ with the same free spectrahedron as $A$, contradicting the minimality of $L_A$.

Hence, again by the Gleichstellensatz, $(A^{\mathrm{re}}, A^{\mathrm{im}})$ is (unitarily equivalent to) a subpencil of $(B^{\mathrm{re}}, B^{\mathrm{im}})$. But then $A$ is a subpencil of B, as desired. ■

Since, for each $t$, $A$ and $e^{it}A$ are minimal defining tuples for the free spectrahedron $\mathcal{D}_A = \mathcal{D}_{e^{it}A}$, by Proposition 2.2, for each $t \in \mathbb{R}$ there is a unitary $U = U_t \in M_d(\mathbb{C})$ such that, for each $s = 1, \ldots, g$,

$$U_t^* A_s U_t = e^{it} A_s. \tag{2.1}$$

For a fixed $s$, equation (2.1) holds for each real $t$ so the spectrum of $A_s$ is a circular set for each $s$. Since each $A_s$ is finite dimensional, the spectrum of each $A_s$ is $\{0\}$ and each $A_s$ is nilpotent.

Fix a number $t$ relatively irrational with respect to $\pi$. For notational ease, abbreviate $U = U_t$ (for this $t$). Being unitary, the matrix $U$ can be (block) diagonalized as

$$U = W^* D W$$

where $D \in M_d(\mathbb{C})$ is diagonal and $W \in M_d(\mathbb{C})$ is unitary. Equation (2.1) shows

$$D^* W A_s W^* D = e^{it} W A_s W^*$$

Clearly, $L_{WAW^*}$ and $L_A$ define the same free spectrahedron. Thus, without loss of generality, $U$ may be taken to have the form

$$U = (\lambda_1 I_{m_1} \oplus \lambda_2 I_{m_2} \oplus \cdots \oplus \lambda_{k+1} I_{m_{k+1}}), \tag{2.2}$$

where the $\lambda_j$ are distinct unimodular numbers. Let $\mathcal{S}_j$ denote the corresponding eigenspace of $U$ and let the $I_{m_j}$ be identity matrices on these spaces.

Since $\mathbb{C}^d = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_{k+1}$ we can use this orthogonal sum to give a block decomposition

$$A_s = (A_s(j,\ell))_{j,\ell} \tag{2.3}$$

subordinate to the $\mathcal{S}_i$. Note that

$$\overline{\lambda_j}\lambda_\ell A_s(j,\ell) = e^{it} A_s(j,\ell) \tag{2.4}$$

so it follows that

$$\lambda_\ell = e^{it}\lambda_j \qquad \text{or} \qquad A_s(j,\ell) = 0 \quad \text{for all } s. \tag{2.5}$$

Equation (2.5) implies $A_s(j,j) = 0$ for each $s$ and $j$.

**Lemma 2.3.** *Let $U \in M_d(\mathbb{C})$ be a unitary with the form of equation (2.2) and let $A = (A_1,\dots,A_g) \in M_d(\mathbb{C})^g$ be a $g$-tuple of matrices with block decomposition $A_s = (A_s(j,\ell)_{j,\ell})$ as described in equation (2.3). Assume there is a $t \in \mathbb{R}$ relatively irrational with respect to $\pi$ such that $e^{it}A_s = U^*A_sU$ for all $s$.*

*Given $1 \le j, \hat{j}, \ell, \hat{\ell} \le k+1$, if $A_s(j,\ell) \neq 0$ and if $A_{\hat{s}}(j,\hat{\ell}) \neq 0$, then, by equation (2.5), $\ell = \hat{\ell}$. Likewise, if $A_s(j,\ell) \neq 0$ and if $A_{\hat{s}}(\hat{j},\ell) \neq 0$, then $j = \hat{j}$. Moreover, if $(j,\ell)$ is a nonzero location, then, for $\hat{j} \neq j$ and $\hat{\ell} \neq \ell$ and all $s$, the matrices $A_s(\hat{j},\ell)$ and $A_s(j,\hat{\ell})$ are both zero.*

*Proof.* Fix $1 \le j \le k+1$ and note from equation (2.5) that if $A_s(j,\ell)$ and $A_{\hat{s}}(j,\hat{\ell})$ are both not zero, then $\lambda_\ell = e^{it}\lambda_j$ and $\lambda_{\hat{\ell}} = e^{it}\lambda_j$. In particular, $\lambda_\ell = \lambda_{\hat{\ell}}$. Since the $\lambda_k$ are distinct it follows that $\ell = \hat{\ell}$. Similarly if $A_s(j,\ell)$ and $A_{\hat{s}}(\hat{j},\ell)$ are both not zero, then equation (2.5) shows $\lambda_j = \lambda_{\hat{j}}$, hence $j = \hat{j}$. ∎

Given a family of matrices $A = \{A_s\}_{s=1}^g$ with the block decomposition $A_s = (A_s(j,\ell))_{j,\ell}$, a sequence of pairs from the set $\{1,\dots,k+1\}$ of the form

$$\mathcal{C} = \{(j_0,j_1),(j_1,j_2),(j_2,j_3),\dots,(j_m,j_{m+1})\} \tag{2.6}$$

such that for each $1 \le r \le m$ there is an $s$ such that $A_s(j_r,j_{r+1}) \neq 0$ is an **admissible chain**. Call $j_0$ the left end of $\mathcal{C}$ and denote by $\mathcal{S}_{\mathcal{C}}$ the subspace

$$\mathcal{S}_{\mathcal{C}} = \mathcal{S}_{j_0} \oplus \mathcal{S}_{j_1} \oplus \mathcal{S}_{j_2} \oplus \cdots \oplus \mathcal{S}_{j_{m+1}}. \tag{2.7}$$

The family $A$ has a **block zero column** if there is an $\ell$ such that $A_s(j,\ell) = 0$ for all $s, j$.

**Lemma 2.4.** *Assume the setup and hypotheses of Lemma 2.3 with chain structure as described in equation (2.6).*

1. *If $\mathcal{C}$ is a chain as in (2.6), then the $j_k$ are distinct.*
2. *The family $A$ has a block zero column.*

*Proof.* Suppose $\mathcal{C}$ is a chain as in (2.6), but the $j_k$ are not distinct. Since $A_s(j,j) = 0$ for all $j$ and $s$, in this case we may assume that $m \ge 1$ and $j_{m+1} = j_0$ and $j_k \ne j_\ell$ for $0 \le k, \ell \le m$. By reindexing if needed, we may assume that

$$\mathcal{C} = \{(m,1),(1,2),(2,3),\dots,(m-1,m)\}$$

is an admissible chain. Summarizing, for each $1 \le j < m$ there exists an $s_j$ such that $A_{s_j}(j,j+1) \ne 0$ and there exists an $s_m$ such that $A_{s_m}(m,1) \ne 0$. Equation (2.5) implies $\lambda_j = \lambda_1 e^{(j-1)it}$ for each $1 \le j \le p$. Thus $\lambda_p$ must be both $\lambda_1 e^{-it}$ and $\lambda_1 e^{(p-1)it}$. Hence $pt$ is a multiple of $2\pi$ contradicting the choice of $t$ as relatively irrational with respect to $\pi$ and the proof of item (1) is complete.

Turning to item (2) and arguing by contradiction, suppose for each $\ell$ there exists a $j_\ell$ and an $s_\ell$ so that $A_{s_\ell}(j_\ell,\ell) \ne 0$. In this case, since, by Lemma 2.3, each column and row has exactly one nonzero entry and since all diagonal entries of $A_s$ are zero, there is an $m$ and distinct indices $j_0, j_1, \dots, j_m$ such that

$$\mathcal{C} = \{(j_0,j_m),(j_m,j_{m-1}),\dots,(j_2,j_1),(j_1,j_0)\}$$

is an admissible chain. An application of item (1) concludes the proof. ∎

The following lemma completes the set up for the proof of Theorem 1.1 (1).

**Lemma 2.5.** *Assume the set up and hypotheses of Lemma 2.4 and assume $\mathcal{C}$ is a maximal chain whose left end $j_0$ is a block zero column of the $A_s$. Then the following hold*

1. *$\mathcal{S}_{\mathcal{C}}$ (defined in equation (2.7)) is a common reducing subspace for each $A_s$.*
2. *The restriction of each $A_s$ to $\mathcal{S}_{\mathcal{C}}$ has the form of equation (1.5) with respect to the orthogonal decomposition of $\mathcal{S}$ as described by equation (2.7).*
3. *If $A$ is an irreducible family and $A_s(\ell,\hat{j}) = 0$ for all $1 \le \ell \le k+1$ and $1 \le s \le g$, then $\hat{j} = j_0$. In particular, the $A_s$ have exactly one block zero column. By reindexing if needed, $\{1,\dots,k+1\}$ is an admissible chain and $A_s(\ell,1) = 0$ for all $s, j$.*

*Proof.* Use the notations of equations (2.6) and (2.7). In particular, $A_s(j,k)$ maps $\mathcal{S}_k$ into $\mathcal{S}_j$.

By the definition of chain, for each $1 \le \ell \le m$, there is an $s_\ell$ such that $A_{s_\ell}(j_\ell, j_{\ell+1}) \ne 0$. From Lemma 2.3, for each $1 \le \ell \le m$, each $j \ne j_{\ell+1}$ and each $1 \le s \le g$ the matrix $A_s(j_\ell, j) = 0$. Hence, $A_s \mathcal{S}_{j_{\ell+1}} \subseteq \mathcal{S}_{j_\ell}$. On the other hand, $A_s(j,j_0)\mathcal{S}_j = 0$ by the choice of $j_0$. It follows that $A_s \mathcal{S}_{\mathcal{C}} \subseteq \mathcal{S}_{\mathcal{C}}$. Thus $\mathcal{S}_{\mathcal{C}}$ is a common *invariant* subspace for the $A_s$. On the other hand, since, for each $0 \le \ell \le m$ the location $(j_\ell, j_{\ell+1})$ is a nonzero location, Lemma 2.3 shows $A_s(j_\ell, q) = 0$ for all $q \notin \{j_0,\dots,j_{m+1}\}$ and all $1 \le s \le g$. Finally, the existence of a $q \notin \{j_0,\dots,j_{m+1}\}$ and an $s$ such that $A_s(j_{m+1},q) \ne 0$ contradicts the maximality of the chain $\mathcal{C}$. Hence $A_s(j_{ml+1},q) = 0$ for all such $q$ and all $s$. It follows that $\mathcal{S}^\perp$ is also a common invariant subspace for the $A_s$. Hence $\mathcal{S}$ is reducing.

Items (2) and (3) follow immediately from item (1) and the definition of irreducible. ∎

2.2. **Proof of Theorem 1.1 (1).** The initial set up of the proof shows that, up to unitary equivalence, the $A_s$ are nilpotent matrices and that there exists a $t$ relatively irrational with respect to $\pi$ and a unitary $U$ with the form of equation (2.2) such that

$$U^* A_s U = e^{it} A_s \quad \text{for all } s.$$

Relative to the block decomposition for $U$, write $A_s = (A_s(j,\ell)_{j,\ell})$ (as described in equation (2.3)). Applying Lemma 2.3 shows that if $A_s(j,\ell) \neq 0$ and $A_{\hat{s}}(j,\hat{\ell}) \neq 0$, then $\ell = \hat{\ell}$ and if $A_s(j,\ell) \neq 0$ and $A_{\hat{s}}(\hat{j},\ell) \neq 0$, then $j = \hat{j}$.

Applying Lemma 2.4 shows that there is some $j_0$ such that $A_s(\ell, j_0) = 0$ for all $s$ and $\ell$. It follows that the $A_s$ have a maximal admissible chain $\mathcal{C}$ of the form

$$\mathcal{C} = \{(j_0, j_1), (j_1, j_2), (j_2, j_3), \dots, (j_m, j_{m+1})\}$$

whose left end $j_0$ is a block zero column of the $A_s$.

Applying Lemma 2.5 (1) shows that $\mathcal{S}_{\mathcal{C}}$ (as defined in equation (2.7)) is a common reducing subspace for the $A_s$. Lemma 2.5 (2) and (3) show that the $A_s$ have the form of equation (1.5) and complete the proof. ■

2.3. **Examples.** Here are two classical examples of circular free spectrahedra.

**Example 2.6.** The **Bi-disk** is a circular free spectrahedron given as the positivity set of

$$L_A(z) = \begin{pmatrix} 1 & z_1 \\ z_1^* & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & z_2 \\ z_2^* & 1 \end{pmatrix} \tag{2.8}$$

**Example 2.7.** The **Ball** is a circular free spectrahedron given as the positivity set of

$$L_A(z) = \begin{pmatrix} 1 & z_1 & z_2 & \cdots & z_g \\ z_1^* & 1 & 0 & \cdots & 0 \\ z_2^* & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_g^* & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{2.9}$$

## 3. A Free Circular Free Spectrahedron is a Matrix Pencil Ball

This section contains the proof of Theorem 1.1 (2). Throughout, $A \in M_d(\mathbb{C})^g$ is a fixed tuple of $d \times d$ matrices and it is assumed that the free spectrahedron $\mathcal{D}_A$ is free circular. We will state precisely and prove in Theorem 3.6 below there is an $N$ (at most $d^3$) and a tuple $F \in M_N(\mathbb{C})^g$ such that, $\mathcal{D}_A = \mathcal{D}_E$, where

$$E = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

A separate argument, given as Corollary 3.7, shows in fact, if $A$ is minimal, then $A$ is unitarily equivalent to $E$. Thus, in any case $E$ can be chosen to be of size $d$.

3.1. **Free Circular Matrix Convex Sets.** In this section we describe free circular matrix convex sets. The set $\Gamma \subseteq M(\mathbb{C})^g$ is **matrix convex** [EW97] if it is **closed under direct sums** in the sense that if $X \in \Gamma(n)$ and $Y \in \Gamma(m)$, then the tuple $X \oplus Y$ whose $j$-th entry is

$$X_j \oplus Y_j = \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}$$

is in $\Gamma(n+m)$; and is **closed under isometric conjugation** in the sense that if $X \in \Gamma(n)$ and $V$ is an $n \times m$ isometric matrix, then

$$V^*XV = \begin{pmatrix} V^*X_1V, & \dots, & V^*X_gV \end{pmatrix} \in \Gamma(m).$$

In the case $0 \in \Gamma(1)$, if $\Gamma$ is closed under direct sums and isometric conjugation, then it is closed under contractive conjugation (replacing $V$ isometric with $V$ contractive) [HM04]. It is not hard to show, if $\Gamma$ is matrix convex, then each $\Gamma(n)$ is convex in the conventional sense.

The Effros-Winkler matricial Hahn-Banach separation theorem [EW97] says if $\Gamma$ is closed (meaning each $\Gamma(n)$ is closed), matrix convex, $0 \in \Gamma(1)$, and if $Y \notin M_n(\mathbb{C})^g \setminus \Gamma(n)$, then there exists a tuple $A \in M_n(\mathbb{C})^g$ such that $L_A(X) \succeq 0$ for $X \in \Gamma$, but $L_A(Y) \not\succeq 0$. In this sense $\mathcal{D}_A$ is the free analog of a separating hyperplane and a closed matrix convex set is an intersection of free spectrahedra.

Proposition 3.3 below is the analog of the Effros-Winkler separation theorem for free circular matrix convex sets. It is an effective version of [BMV+, Proposition 3.5].

**Lemma 3.1.** *Suppose $\mathcal{D} \subseteq M(\mathbb{C})^g$ contains $0$ and is closed with respect to direct sums. If for each pair of positive integers $s, t$, each $Y \in \mathcal{D}(t)$ and each pair of $t \times s$ isometries $V_1, V_2$ (so $t \geq s$), $V_2^* X V_1 \in \mathcal{D}(s)$, then for each pair $m, n$ of positive integers, each $X \in \mathcal{D}(n)$ and each pair $C_1, C_2$ of $m \times n$ contractions, $C_2^* X C_1 \in \mathcal{D}(m)$.*

*Proof.* Let positive integers $m, n$, a tuple $X \in \mathcal{D}(n)$ and a pair of $m \times n$ contractions $C_1, C_2$ be given. Let $D_j = (I - C_j^* C_j)^{\frac{1}{2}}$. With this choice of $D_j$, the $(m+n) \times m$ matrices

$$V_j = \begin{pmatrix} C_j & D_j \end{pmatrix}$$

are isometries. Since $\mathcal{D}$ is closed with respect to direct sums and contains $0$, it follows that $X \oplus 0 \in \mathcal{D}(n+m)$. Since $\mathcal{D}$ is closed with respect to multiplying on the left by the adjoint of an isometry and the right by an isometry (of the same sizes),

$$V_2^* \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} V_1 = C_2^* X C_1 \in \mathcal{D}(m). \qquad \blacksquare$$

Following [BMV+] we call a graded set $\mathcal{C} = (\mathcal{C}(n))_{n \in \mathbb{N}}$ **matrix balanced** if for each pair $m, n$ of positive integers, each $X \in \mathcal{C}(n)$ and pair of $n \times m$ contractions $C_1, C_2$, the matrix $C_2^* X C_1 \in \mathcal{C}(m)$. Observe, if $\mathcal{C}$ is matrix balanced and closed with respect to direct sums, then it is matrix convex and in particular each $\mathcal{D}_A(n)$ is convex in the ordinary sense.

**Proposition 3.2.** *A subset $\mathcal{D}$ of $M(\mathbb{C})^g$ is closed with respect to direct sums and matrix balanced if and only if it is matrix convex, free circular and contains* 0.

*Proof.* Choosing $C_2 = Z^*$ and $C_1 = I$ shows if $\mathcal{D}$ is matrix balanced, then it is free circular. Choosing $C_1 = C_2$ shows matrix balanced implies matrix convex. Choosing either $C_1$ or $C_2$ equal zero shows $0 \in \mathcal{D}$. Hence, $\mathcal{D}$ matrix balanced implies matrix convex, free circular and $0 \in \mathcal{D}$.

In view of Lemma 3.1, it suffices to prove the converse under the added assumption that the $C_j$ are isometries. In this case, there exists an $n \times n$ unitary matrix $W$ such that $WC_1 = C_2$. Letting $Z = W^*$ gives $ZX \in \mathcal{D}(n)$ by the free circular hypothesis. Thus $C_1^*(ZX)C_1 \in \mathcal{D}(m)$ by the matrix convex assumption. Finally, as $C_1^* Z = C_2^*$ the result follows. ∎

Given $\epsilon > 0$, the **free $\epsilon$-neighborhood of** 0, denoted $\mathcal{N}_\epsilon$, is the graded set $(\mathcal{N}_\epsilon(n))_{n=1}^\infty$ where

$$\mathcal{N}_\epsilon(n) = \{X \in M_n(\mathbb{C})^g : \sum \|X_j\| < \epsilon\}.$$

**Proposition 3.3.** *Let $\mathcal{C} = (\mathcal{C}(n))$ denote a free circular matrix convex subset of the graded set $M(\mathbb{C})^g$ that contains a free $\epsilon$-neighborhood of* 0*. If $X^{\mathrm{b}} \in M_n(\mathbb{C})^g$ is in the boundary of $\mathcal{C}(n)$, then there is a tuple $Q \in M_n(\mathbb{C})^g$ such that $\|\Lambda_Q(Y)\| \leq 1$ for all $m$ and $Y \in \mathcal{C}(m)$ and such that $\|\Lambda_Q(X^{\mathrm{b}})\| = 1$.*

*Proof.* By Proposition 3.2 and [BMV+, Proposition 3.5], $\mathcal{C} = \mathcal{B}_F$ for some operator tuple $F$ acting on a Hilbert space $H$. Here $\mathcal{B}_F$ is the operator pencil ball determined by $F$, i.e.,

$$\mathcal{B}_F = \Big\{X \in M(\mathbb{C})^g : \big\| \sum_j F_j \otimes X_j \big\| \leq 1\Big\}.$$

Let $\Lambda_F(x) = \sum_{j=1}^g F_j x_j$ denote the homogeneous operator pencil determined by $F$. Since $X^{\mathrm{b}}$ is in the boundary of $\mathcal{B}_F$, we see that $\|\Lambda_F(X^{\mathrm{b}})\| = 1$. Hence, there exists a sequence of unit vectors $\gamma_k \in H \otimes \mathbb{C}^n$ such that $(\|\Lambda_F(X^{\mathrm{b}})\gamma_k\|)_k$ tends to 1. Fix $k$. Write $\gamma_k = \sum_{j=1}^n \gamma_{k,j} \otimes e_j$. Let $\Gamma_k$ denote an $n$ dimensional subspace of $H$ containing the span of $\{\gamma_{k,1}, \dots, \gamma_{k,n}\}$ (if the dimension of $H$ is less than $n$, then there is nothing to prove) and let $G^k = V^* F V \in M_n(\mathbb{C})^g$, where $V : \mathbb{C}^n \to \Gamma_k$ is an isometry. It follows that $(\|\Lambda_{G^k}(X^{\mathrm{b}})\|)_k$ tends to 1. By compactness, $(G^k)$ has a subsequence which converges in norm to some $G \in M_n(\mathbb{C})^g$. It follows that $\|\Lambda_G(X^{\mathrm{b}})\| = 1$ and $\Lambda_G$ is at most one in norm on $\mathcal{C}$. ∎

The authors of [BMV+] obtain [BMV+, Proposition 3.5] as a consequence of Ruan's representation theorem for operator spaces (see [ER00, Theorem 2.3.5] or [Pau02, Chapter 13]). We give an elementary self-contained proof of Proposition 3.3 in Appendix A.

3.2. **Criteria for Membership in a Free Spectrahedron.** This section contains three simple lemmas preliminary to the proof of Theorem 1.1 (2).

**Lemma 3.4.** *A tuple $X \in M_n(\mathbb{C})^g$ lies in $\mathcal{D}_A(n)$ if and only if for every subspace $M$ of $\mathbb{C}^n$ of dimension $e \le d$, the tuple $V^*XV$ lies in $\mathcal{D}_A(e)$, where $V : M \to \mathbb{C}^n$ is the inclusion map.*

*Proof.* To prove the non-trivial direction, let a vector $v \in \mathbb{C}^d \otimes \mathbb{C}^n$ be given. Write $v = \sum_{j=1}^d e_j \otimes v_j$, where $\{e_1, \dots, e_d\}$ is an orthonormal basis for $\mathbb{C}^d$. Let $M$ denote the span of $\{v_1, \dots, v_d\}$. Thus $M$ has dimension $e \le d$. Let $V$ denote the inclusion of $M$ into $\mathbb{C}^n$. Since $V^*XV \in \mathcal{D}_A(e)$ by assumption,

$$\langle L_A(X)v, v\rangle = \langle L_A(V^*XV)v, v\rangle \ge 0$$

and the desired conclusion follows. ■

Before proceeding we address a technical point related to the Kronecker product that occurs in the following lemma. Note that for any $B_1, B_2 \in M_\ell(\mathbb{C})$ and $Z \in M_\nu(\mathbb{C})$ we have the identity

$$(B_1 \oplus B_2) \otimes Z = (B_1 \otimes Z) \oplus (B_2 \oplus Z). \tag{3.1}$$

On the other hand, while $Z \otimes (B_1 \oplus B_2) \ne (Z \otimes B_1) \oplus (Z \otimes B_2)$, the fact that these two expressions are unitarily equivalent suffices for our arguments. In fact, there is a permutation matrix, often called the canonical shuffle, $\Pi_{\ell,\nu} \in M_{\nu\ell}(\mathbb{C})$ such that $B \otimes Z = \Pi_{\ell,\nu}^*(Z \otimes B)\Pi_{\ell,\nu}$ for any matrices $B \in M_\ell(\mathbb{C})$ and $Z \in M_\nu(\mathbb{C})$. We write $B \otimes Z \overset{\text{c.s.}}{\sim} Z \otimes B$.

**Lemma 3.5.** *Suppose $\mathcal{D}_A$ is matrix balanced, closed with respect to direct sums and $\Lambda = \Lambda_F$ is a homogeneous linear pencil.*

(i) *If $\|\Lambda(X)\| > 1$ for all $X \in M_d(\mathbb{C})^g \setminus \mathcal{D}_A(d)$, then, $\|\Lambda(Y)\| > 1$ for each $1 \le e \le d$ and $Y \in M_e(\mathbb{C})^g \setminus \mathcal{D}_A(e)$.*
(ii) *If $\|\Lambda(X)\| > 1$ for all $X \in M_d(\mathbb{C})^g \setminus \mathcal{D}_A(d)$, then $\|\Lambda(X)\| > 1$ for all $X \notin \mathcal{D}_A$.*
(iii) *If $\|\Lambda(X)\| \le 1$ for all $X \in \mathcal{D}_A$ and $\|\Lambda(X)\| = 1$ for all $X \in \partial\mathcal{D}_A(d)$, then $\mathcal{D}_A = \mathcal{D}_E$, where $E = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.*

*Proof.* To prove item (i), suppose $1 \le e \le d$ and $Y \in M_e(\mathbb{C})^g \setminus \mathcal{D}_A(e)$. Thus $L_A(Y) \not\succeq 0$. Let $0$ denote the tuple of zeros in $M_{d-e}(\mathbb{C})^g$ and let $X = Y \oplus 0$. Now $X \notin \mathcal{D}_A(d)$ since $L_A(X) \overset{\text{c.s.}}{\sim} L_A(Y) \oplus I \not\succeq 0$. By hypothesis, $\|\Lambda(X)\| > 1$. But $\Lambda(X) \overset{\text{c.s.}}{\sim} \Lambda(Y) \oplus 0$. Hence, $\|\Lambda(Y)\| > 1$.

By item (i), to prove item (ii) it may be assumed that $\|\Lambda(X)\| > 1$ for all $1 \le e \le d$ and $X \in M_e(\mathbb{C})^g \setminus \mathcal{D}_A(e)$. Let $n$ and $Y \in M_n(\mathbb{C})^g \setminus \mathcal{D}_A(n)$ be given. By Lemma 3.4, there is a subspace $M$ of dimension $e \le d$ such that, $X = V^*YV \notin \mathcal{D}_A(e)$, where $V$ is the inclusion of $M$ into $\mathbb{C}^d$. Hence, by assumption, $\|\Lambda(X)\| > 1$. Hence there is a unit vector

$v \in \mathbb{C}^N \otimes M \subseteq \mathbb{C}^N \otimes \mathbb{C}^n$, where $N$ is the size of the pencil $\Lambda$, such that $\|\Lambda(X)v\| > 1$. Consequently,

$$1 < \|\Lambda(X)v\| = \|(I \otimes V)^* \Lambda(Y)(I \otimes V)v\| \leq \|\Lambda(Y)\| \, \|v\| = \|\Lambda(Y)\|.$$

To prove item (iii), first note that the hypotheses immediately imply $\mathcal{D}_A \subseteq \mathcal{D}_E$. To prove the reverse inclusion, observe, if $X \notin \mathcal{D}_A(d)$, then there is an $0 < r < 1$ such that $rX \in \partial\mathcal{D}_A(d)$ (since 0 is in $\mathcal{D}_A(d)$ and $\mathcal{D}_A(d)$ is convex) and hence $\|\Lambda(X)\| = \frac{1}{r} > 1$. Thus, if $X \notin \mathcal{D}_A(d)$, then $\|\Lambda(X)\| > 1$. It follows from item (ii) that $X \notin \mathcal{D}_A$ implies $X \notin \mathcal{D}_E$. Hence $\mathcal{D}_E \subseteq \mathcal{D}_A$ and the proof is complete. ∎

3.3. **Free Circular Free Spectrahedra.** The final part of Theorem 1.1, stated in a somewhat different form below as Corollary 3.7, is proved in this subsection.

**Theorem 3.6** (Theorem 1.1 (2))**.** *If $\mathcal{D}_A$ is a free circular spectrahedron, then there exists a homogeneous linear pencil $\Lambda$ such that $\|\Lambda(X)\| \leq 1$ if and only if $X \in \mathcal{D}_A$. Moreover, $\Lambda$ is the direct sum of at most $d^2$ homogeneous linear pencils of size (at most) $d$.*

**Corollary 3.7.** *Suppose $A \in M_d(\mathbb{C})^g$. If $L_A$ is a minimal defining pencil for $\mathcal{D}_A$ and $\mathcal{D}_A$ is free circular, then there exists positive integers $s, t$ such that $s + t = d$ and a $g$-tuple $F$ of $s \times t$ matrices with entries from $\mathbb{C}$ such that,*

$$A \overset{u}{\sim} \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}.$$

*Proof of Corollary 3.7.* By Theorem 3.6, there exist positive integers $m, n$ and a tuple $G$ of $m \times n$ matrices such that $\mathcal{D}_A = \mathcal{D}_B$, where

$$B = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}.$$

In particular, the size of $B$ is $(m + n) \times (m + n)$. Next observe, without loss of generality, it may be assumed that $\ker(G) = \{0\} = \ker(G^*)$.

There is a reducing subspace $\mathcal{E} \subseteq \mathbb{C}^m \oplus \mathbb{C}^n$ such that, letting $E$ denote the restriction of $B$ to $\mathcal{E}$, the monic linear pencil $L_E$ is minimal defining for $\mathcal{D}_A$ (cf. Proposition 2.2). Hence by loc. cit. $A$ and $E$ are unitarily equivalent. Let $\mathcal{G}$ denote the projection of $\mathcal{E}$ onto the first coordinate and $\mathcal{G}_*$ denote the projection onto the second coordinate. Thus $\mathcal{E} \subseteq \mathcal{G} \oplus \mathcal{G}_*$. On the other hand, since $\mathcal{E}$ is reducing for $E$,

$$G_j^* G_j \mathcal{G}_* = B_j^* B_j \mathcal{E} = \begin{pmatrix} 0 & 0 \\ 0 & G_j^* G_j \end{pmatrix} \mathcal{E} \subseteq \mathcal{E}.$$

Hence each $G_j^*$ maps $\mathcal{G}_*$ into $\mathcal{G}_*$ and $\sum_{j=1}^{g} G_j^* G_j \mathcal{G}_* \subseteq \mathcal{G}_*$. On the other hand, since $\sum G_j^* G_j$ does not have a kernel, it follows that the span of the subspaces $G_j^* \mathcal{G}_*$ is precisely $\mathcal{G}_*$. Thus

$\mathcal{G}_* \subseteq \mathcal{E}$. Likewise $\mathcal{G} \subseteq \mathcal{E}$. Hence $\mathcal{E} = \mathcal{G} \oplus \mathcal{G}_*$ and thus,

$$E = W^*BW = \begin{pmatrix} 0 & VGV_* \\ 0 & 0 \end{pmatrix},$$

where $W$ is the inclusion of $\mathcal{E}$ into $\mathbb{C}^{m+n}$ and $V$ and $V_*$ are the inclusions of $\mathcal{G}$ and $\mathcal{G}_*$ into $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. ∎

The proof of Theorem 3.6 rests on two preliminary lemmas. Given a vector $v = \sum_{k=1}^d e_k \otimes v_k \in \mathbb{C}^d \otimes \mathbb{C}^n$ and matrix $\eta \in M_d(\mathbb{C})$, let

$$[\eta, v] = \sum_{s=1}^d e_s \otimes \big( \sum_{k=1}^d \eta_{s,k} v_k \big) \in \mathbb{C}^d \otimes \mathbb{C}^n = \mathbb{C}^{nd}.$$

A pair $(X, v) \in M_n(\mathbb{C})^g \times (\mathbb{C}^{nd} \setminus \{0\})$ is in the **detailed boundary** of $\mathcal{D}_A(n)$ if $X \in \mathcal{D}_A$ and $L_A(X)v = 0$.

**Lemma 3.8.** *Fix positive integers $n, N$ and suppose $(X^j, v^j) \in M_n(\mathbb{C})^g \otimes \mathbb{C}^{nd}$ are in the detailed boundary of $\mathcal{D}_A(n)$ for $1 \le j \le N$. Write, $v^j \in \mathbb{C}^{nd} = \mathbb{C}^d \otimes \mathbb{C}^n$ as*

$$v^j = \sum_{k=1}^d e_k \otimes v_k^j. \tag{3.2}$$

*Let $\mathcal{P}$ denote the subspace of $M_d(\mathbb{C})$ consisting of those matrices $c$ such that $[c, v^j] = 0$ for all $1 \le j \le N$. (In this context, we identify $M_d(\mathbb{C})$ with $\mathbb{C}^{d^2}$ or equivalently endow $M_d(\mathbb{C})$ with the Hilbert-Schmidt norm.) There exists a homogeneous linear pencil $\Lambda$ of size $d$, an $\ell$ and a nonzero matrix $\eta \in \mathcal{P}^\perp$ such that $\|\Lambda(Z)\| \le 1$ for all $Z \in \mathcal{D}_A$, and such that $[\eta, v^\ell] \neq 0$ and if $[\eta, v^j] \neq 0$, then $\|\Lambda(X^j)\| = 1$.*

*Proof.* Let $\{\epsilon_j\}_{j=1}^N$ be the standard orthonormal basis for $\mathbb{C}^N$ and let $\mathcal{E}^j = \epsilon_j \epsilon_j^* \in \mathbb{C}^{N \times N}$. Let $Y = \sum_{j=1}^N X^j \otimes \mathcal{E}^j$. Since $Y$ is unitarily equivalent to $\oplus_{j=1}^N X^j$, it follows that $Y \in \mathcal{D}(nN)$. Let $v = \sum_{j=1}^N v^j \otimes \epsilon^j \in \mathbb{C}^{ndN}$. Thus, $v = \sum_{k=1}^d e_k \otimes v_k$, where, for $1 \le k \le d$,

$$v_k = v_k^j \otimes \epsilon_j \in \mathbb{C}^{nN}.$$

Let $M$ denote the span of $\{v_k : 1 \le k \le d\}$ as a subspace of $\mathbb{C}^{nN}$ and let $m$ denote the dimension of $M$. In particular, $m \le d$. Let $V$ denote the inclusion of $M$ into $\mathbb{C}^{nN}$ and let $Z = V^*YV$. Note that $Z \in \mathcal{D}_A(m)$ since $\mathcal{D}_A$ is matrix convex and $V$ is an isometry. Observe that

$$\begin{aligned} \Lambda_A(Y) &= \sum_{k=1}^g A_k \otimes Y_k = \sum_{k=1}^g A_k \otimes \big( \sum_{j=1}^N (X_k^j \otimes \mathcal{E}^j) \big) \\ &= \sum_{j=1}^N \big( \sum_{k=1}^g A_k \otimes X_k^j \big) \otimes \mathcal{E}^j = \sum_{j=1}^N \Lambda_A(X^j) \otimes \mathcal{E}^j. \end{aligned} \tag{3.3}$$

It follows from equation (3.3), that

$$\langle L_A(Z)v, v\rangle = \langle L_A(Y)v, v\rangle = \sum_{j=1}^{N} \langle L_A(X^j)v^j, v^j\rangle = 0.$$

Thus $Z$ boundary of $\mathcal{D}_A(m)$. By Proposition 3.3, there is a homogeneous linear pencil $\Lambda$ of size $m \le d$ (and without loss of generality we take $\Lambda$ of size $d$) such that $\|\Lambda(X)\| \le 1$ for all $X \in \mathcal{D}_A$ and $\|\Lambda(Z)\| = 1$. Thus, there is a unit vector $\gamma \in \mathbb{C}^d \otimes M$ such that $\|\Lambda(Z)\gamma\| = 1$. It follows that $\gamma$ is in the span of $\{e_s \otimes v_k : 1 \le s, k \le d\}$; i.e, $\gamma \in \mathbb{C}^d \otimes M$. In particular, there is a $\mu \in M_d(\mathbb{C})$ such that $\gamma = \sum_{s=1}^{d} e_s \otimes (\sum_{k=1}^{d} \mu_{s,k} v_k) = [\mu, v]$. Let $\gamma^j = [\mu, v^j]$. Thus $\gamma = \sum_{j=1}^{N} \gamma^j \otimes \epsilon^j$ and $\gamma^j \ne 0$ if and only if $[\mu, v^j] \ne 0$. Estimate, using equation (3.3),

$$\begin{aligned} 1 = \|\Lambda(Z)\gamma\|^2 &= \|\Lambda(V^*YV)\gamma\|^2 = \|(I \otimes V^*)\Lambda(Y)\gamma\|^2 \\ &\le \|\Lambda(Y)\gamma\|^2 = \sum_{j=1}^{N} \|\Lambda(X^j)\gamma^j\|^2 \le \sum_{j=1}^{N} \|\gamma^j\|^2 = 1. \end{aligned}$$

It follows that $\|\Lambda(X^j)\gamma^j\| = \|\gamma^j\|$ for all $1 \le j \le N$. Moreover, there exists an $\ell$ such that $\|\gamma^\ell\| \ne 0$. Equivalently, $[\mu, v^\ell] \ne 0$. Furthermore, $\|\Lambda(X^\ell)\| = 1$ for each such $\ell$. To complete the proof, let $\eta$ denote the projection of $\mu$ onto $\mathcal{P}^\perp$. Since $[\eta, v^j] = [\mu, v^j] = \gamma^j$, it follows that $[\eta, \gamma^j] \ne 0$ implies $\|\Lambda(X^j)\| = 1$. Finally, $[\eta, v^\ell] \ne 0$. ∎

**Lemma 3.9.** *Fix a positive integer $n$ and suppose $(X^j, v^j)$ is a sequence from the detailed boundary of $\mathcal{D}_A(n)$. Write, $v^j \in \mathbb{C}^d \otimes \mathbb{C}^n$ as in (3.2). Let $\mathcal{P}$ denote the subspace of $M_d(\mathbb{C})$ consisting of those matrices $c$ such that $[c, v^j] = 0$ for all $j$.*

*There exists a homogeneous linear pencil $\Lambda$ of size $d$ and a nonzero matrix $\eta \in \mathcal{P}^\perp$ such that $\|\Lambda(Z)\| \le 1$ for all $Z \in \mathcal{D}_A$ and such that if $[\eta, v^j] \ne 0$, then $\|\Lambda(X^j)\| = 1$. In particular, there is a $j$ such that $[\eta, v^j] \ne 0$.*

*Proof.* For positive integers $N$, let $\mathcal{P}_N$ denote the subspace of $M_d(\mathbb{C})$ consisting of those matrices $c$ such that $[c, v^j] = 0$ for $1 \le j \le N$. Hence, $\mathcal{P}_1 \supseteq \mathcal{P}_2 \supseteq \cdots$ and $\mathcal{P} = \cap_{N=1}^{\infty} \mathcal{P}_N$.

By Lemma 3.8, for each $N$ there exists a homogeneous linear pencil $\Lambda^N$ of size $d$ and a unit vector (matrix of Hilbert-Schmidt norm one) $\eta^N \in \mathcal{P}^\perp$ such that $\|\Lambda^N(X)\| \le 1$ for all $X \in \mathcal{D}_A$ and, if $1 \le j \le N$ and $[\eta^N, v^j] \ne 0$, then $\|\Lambda^N(X^j)\| = 1$. Write,

$$\Lambda^N(x) = \sum_{j=1}^{g} \Lambda_j^N x_j.$$

Since $\mathcal{D}_A$ contains a free neighborhood of 0, there is a uniform bound on the norms of the matrices $\{\Lambda_j^N : j, N\}$. It follows that there are subsequences $(\Lambda^{N_\ell})_\ell$ and $(\eta^{N_\ell})_\ell$ converging to some $\Lambda$ and $\eta$ respectively. In particular, $\|\Lambda(X)\| \le 1$ for all $X \in \mathcal{D}_A$. Since $\eta^{N_\ell} \in \mathcal{P}_M$ for $N_\ell \ge M$ and since $\mathcal{P}_M$ is a (closed) subspace of $M_d(\mathbb{C})$, it follows that $\eta \in P_M$ and consequently $\eta \in \mathcal{P}^\perp$ is a unit vector. Hence there is a $j$ such that $[\eta, v^j] \ne 0$. Thus

$[\eta^{N_\ell}, v^j] \neq 0$ for large enough $\ell$. For such $\ell$ it follows, from Lemma 3.8, that $\|\Lambda^{N_\ell}(X^j)\| = 1$ and hence $\|\Lambda(X^j)\| = 1$. ∎

*Proof of Theorem 3.6.* Let $J_0$ denote a countable set and choose a dense subset $\{X^j : j \in J_0\}$ of the boundary of $\mathcal{D}_A(d)$ indexed by $J_0$. For each $j \in J_0$ there is a unit vector $v^j$ such that $(X^j, v^j)$ is in the detailed boundary of $\mathcal{D}_A(d)$. Write $v^j = \sum_{k=1}^g e_j \otimes v_k^j$. Let $\mathcal{P}_0$ denote those vectors $c \in M_d(\mathbb{C})$ such that $[c, v^j] = 0$ for all $j \in J_0$. By Lemma 3.9, there exists a linear pencil $\Lambda^1$ of size $d$ and a unit vector $\eta^1 \in \mathcal{P}_0^\perp$ such that $\|\Lambda(Z)\| \leq 1$ for all $m$ and $Z \in \mathcal{D}_A(m)$ and $\|\Lambda(X^j)\| = 1$ for each $j \in J_0$ such that $[\eta^1, v^j] \neq 0$. Moreover, there is a $j_0 \in J_0$ such that $[\eta^1, v^{j_0}] \neq 0$. Let $J_1$ denote those indices $j \in J_0$ such that $[\eta^1, v^j] = 0$. Thus, $\|\Lambda^1(X^j)\| = 1$ for $j \notin J_1$ and $J_1$ is a proper subset of $J_0$ since $j_0 \in J_0$, but $j_0 \notin J_1$. If $J_1$ is empty, the proof is nearly complete. Otherwise, let $\mathcal{P}_1$ denote the subspace of vectors $c \in M_d(\mathbb{C})$ such that $[c, v^j] = 0$ for all $j \in J_1$. Observe that $\eta^1 \in \mathcal{P}_1$, but $\eta^1 \notin \mathcal{P}_0$ since $[\eta^1, v^{j_0}] \neq 0$. Therefore $\mathcal{P}_0$ is a proper subspace of $\mathcal{P}_1$. For the collection $\{(X^j, v^j) : j \in J_1\}$ there exists a homogeneous linear pencil $\Lambda^2$ of size $d$ and unit vector $\eta^2 \in \mathcal{P}_1^\perp$ such that if $j \in J_2$ and $[\eta_2, v^j] \neq 0$, then $\|\Lambda^2(X^j)\| = 1$ and, letting $J_2$ denote those $j \in J_1$ such that $[\eta^2, v^j] = 0$, the subspace $\mathcal{P}_2$ consisting of those $c \in M_d(\mathbb{C})$ such that $[c, v^j] = 0$ for all $j \in J_2$ properly contains $\mathcal{P}_1$. Recursively define $\mathcal{P}_N$ and observe $M_d(\mathbb{C}) \supseteq \mathcal{P}_N$. Since $M_d(\mathbb{C})$ is finite dimensional this process terminates after $\rho \leq d^2$ steps and produces

(i) a chain of subspaces $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_\rho = M_d(\mathbb{C})$ of $M_d(\mathbb{C})$;
(ii) a chain of subsets $J_0 \supsetneq J_1 \supsetneq \cdots \supsetneq J_\rho = \varnothing$;
(iii) homogeneous linear pencils $\Lambda^r$ for $1 \leq r \leq \rho$ of size $d$ such that $\|\Lambda^r(X)\| \leq 1$ for $X \in \mathcal{D}_A$ and $\|\Lambda(X^j)\| = 1$ for each $j \in J_{r-1} \setminus J_r$.

Let $\Lambda = \oplus_{r=1}^\rho \Lambda^r$. Thus, by construction, $\|\Lambda(X)\| \leq 1$ for all $X \in \mathcal{D}$ and $\|\Lambda(X^j)\| = 1$ for all $j \in J_0$. By continuity $\|\Lambda(Y)\| = 1$ for all $Y$ in the boundary of $\mathcal{D}_A(d)$. An application of Lemma 3.5 (iii) completes the proof of the existence of $\Lambda$. The bound $d^3$ follows since $\Lambda$ is the direct sum of at most $d^2$ pencils each of size at most $d$. ∎

## 4. Free Polynomials Invariant under Coordinate Unitary Conjugation

The main result of this section is Theorem 4.1 characterizing monic free matrix polynomials that are invariant under coordinate unitary conjugation. The needed background on free polynomials and their evaluations are collected in the next subsection. Experts can skip straight to Subsection 4.2.

### 4.1. Words, Free Polynomials and Evaluations.

We write $\langle x, x^* \rangle$ for the monoid freely generated by $x = (x_1, \dots x_g)$ and $x^* = (x_1^*, \dots, x_g^*)$, i.e., $\langle x, x^* \rangle$ consists of **words** in the $2g$ noncommuting letters $x_1, \dots, x_g, x_1^*, \dots x_g^*$ (including the **empty word** $\varnothing$ which plays the role of the identity). Let $\mathbb{C}\langle x, x^* \rangle$ denote the associative $\mathbb{C}$-algebra freely generated by $x$

and $x^*$, i.e., the elements of $\mathbb{C}\langle x, x^*\rangle$ are polynomials in the freely noncommuting variables $x$ and $x^*$ with coefficients in $\mathbb{C}$. Its elements are called **free polynomials**. The **involution** $^*$ on $\mathbb{C}\langle x, x^*\rangle$ extends the complex conjugation on $\mathbb{C}$, satisfies $(x_i^*)^* = x_i$, reverses the order of words, and acts $\mathbb{R}$-linearly on polynomials. Polynomials fixed under this involution are **symmetric**. The length of the longest word in a free polynomial $f \in \mathbb{C}\langle x, x^*\rangle$ is the **degree** of $f$ and is denoted by $\deg(f)$ or $|f|$ if $f \in \langle x, x^*\rangle$. The set of all words of degree at most $k$ is $\langle x, x^*\rangle_k$, and $\mathbb{C}\langle x, x^*\rangle_k$ is the vector space of all free polynomials of degree at most $k$.

Fix positive integers $v$ and $\ell$. **Free matrix polynomials** - elements of $\mathbb{C}^{\ell\times v}\langle x, x^*\rangle = \mathbb{C}^{\ell\times v} \otimes \mathbb{C}\langle x, x^*\rangle$; i.e., $\ell \times v$ matrices with entries from $\mathbb{C}\langle x\rangle$ - will play a role in what follows. Elements of $\mathbb{C}^{\ell\times v}\langle x\rangle$ are represented as

$$p(x) = \sum_{w\in\langle x,x^*\rangle} B_w w(x) \in \mathbb{C}^{\ell\times v}\langle x, x^*\rangle \tag{4.1}$$

where the sum is finite, $B_w \in \mathbb{C}^{\ell\times v}$, and $w(x)$ runs over words in $x$ and $x^*$. The involution $^*$ extends to matrix polynomials by

$$p(x)^* = \sum_{w\in\langle x,x^*\rangle} B_w^* w(x)^* \in \mathbb{C}^{v\times \ell}\langle x, x^*\rangle.$$

If $v = \ell$ and $p(x)^* = p(x)$, we say $p$ is symmetric. Additionally if $p(0) = I$, we say $p$ is **monic**.

If $p \in \mathbb{C}\langle x, x^*\rangle$ is a free polynomial and $X \in M_n(\mathbb{C})^g$, then the evaluation $p(X) \in M_n(\mathbb{C})$ is defined in the natural way by replacing $x_i$ by $X_i$, $x_i^*$ by $X_i^*$ and sending the empty word to the appropriately sized identity matrix. Such evaluations produce (all) finite dimensional $*$-representations of the algebra of free polynomials. Polynomial evaluations extend to matrix polynomials by evaluating entrywise. That is, if $p$ is as in (4.1), then

$$p(X) = \sum_{w\in\langle x,x^*\rangle} B_w \otimes w(X) \in C^{\ell\times v} \otimes M_n(\mathbb{C}).$$

Note that if $p \in M_d(\mathbb{C})\langle x, x^*\rangle$ is symmetric and $X \in M_n(\mathbb{C})^g$, then $p(X) \in M_d(\mathbb{C})\otimes M_n(\mathbb{C}) = M_{dn}(\mathbb{C})$ is a self-adjoint matrix.

4.2. **Invariant Polynomials.** In this subsection we prove Theorem 1.2 stated below in a self contained fashion for the reader's convenience. Write $A \overset{u}{\sim} B$ to indicate the matrices $A$ and $B$ are unitarily equivalent.

**Theorem 4.1.** *Suppose $p$ is a monic free $d \times d$ matrix polynomial. For each $n$ and for each $g$-tuple of unitaries $U = (U_1, \dots, U_g) \in M_n(\mathbb{C})^g$ there exists a unitary $W$ such that for all $X \in M_n(\mathbb{C})^g$,*

$$p(U_1^* X_1 U_1, \dots, U_g^* X_g U_g) = W^* p(X_1, \dots, X_g) W$$

*if and only if*

$$p(x) \overset{u}{\sim} p_1(x_1) \oplus \cdots \oplus p_g(x_g); \tag{4.2}$$

*i.e., $p$ is (up to unitary equivalence) a direct sum of univariate matrix polynomials.*

The following lemma is needed in the proof of Theorem 4.1.

**Lemma 4.2.** *Suppose $p(x) = \sum_{i=1}^{g} p_i(x_i)$ is a free $d \times d$ matrix polynomial, where*

$$p_i(x_i)p_j(x_j) = p_i(x_i)^* p_j(x_j) = p_i(x_i)p_j(x_j)^* = 0 \tag{4.3}$$

*whenever $i \neq j$. Then there exists a unitary $U$ such that*

$$U^* p(x) U = \hat{p}_1(x_1) \oplus \cdots \oplus \hat{p}_g(x_g). \tag{4.4}$$

*for some free matrix polynomials $\hat{p}_j$ each in the variables $x_j, x_j^*$ alone.*

*Proof.* Suppose (4.3) holds whenever $i \neq j$. Using the notation $w_i(x_i)$ to denote words in $x_i$ and $x_i^*$, write

$$p_i(x_i) = \sum_{w_i} A_{w_i} w_i(x_i).$$

Then

$$p_i(x_i)p_j(x_j) = \sum_{w_i, w_j} A_{w_i} A_{w_j} w_i(x_i) w_j(x_j) = 0. \tag{4.5}$$

Note that, if $w_i(x_i), v_i(x_i), w_j(x_j), v_j(x_j)$ are words in $x_i$ and $x_j$, respectively, then we have $w_i(x_i)w_j(x_j) = v_i(x_i)v_j(x_j)$ if and only if $w_i(x_i) = v_i(x_i)$ and $w_j(x_j) = v_j(x_j)$. This implies that each monomial appears on the right hand side of (4.5) exactly once. It follows that $A_{w_i} A_{w_j} = 0$ for all $w_i, w_j$ whenever $i \neq j$. Similarly,

$$p_i(x_i)^* p_j(x_j) = \sum_{w_i, w_j} A_{w_i}^* A_{w_j} w_i(x_i)^* w_j(x_j) = 0. \tag{4.6}$$

Since each monomial appears on the right hand side of (4.6) exactly once, it follows that $A_{w_i}^* A_{w_j} = 0$ for all $w_i, w_j$ whenever $i \neq j$. Furthermore,

$$p_i(x_i)p_j(x_j)^* = \sum_{w_i, w_j} A_{w_i} A_{w_j}^* w_i(x_i) w_j(x_j)^* = 0. \tag{4.7}$$

It follows that $A_{w_i} A_{w_j}^* = 0$ for all $w_i, w_j$ whenever $i \neq j$.

Let $\mathcal{A}_j$ denote the finite dimensional (non-unital) $C^*$-algebra generated by

$$\{A_{w_j} : w_j \text{ is a word in } x_j, x_j^*\}.$$

Then

$$\mathcal{A}_j \mathcal{A}_\ell = \{0\} \quad \text{for } j \neq \ell. \tag{4.8}$$

Decompose $\mathbb{C}^d$ as a direct sum of invariant (hence reducing) subspaces for $\mathcal{A}_1$, say

$$\mathbb{C}^d = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m \oplus \mathcal{S}_{m+1},$$

where $\mathcal{A}_1$ acts irreducibly on $\mathcal{S}_j$ for $j \leq m$ and $\mathcal{A}_1(\mathcal{S}_{m+1}) = 0$. From (4.8) it follows that $\mathcal{A}_k$ for $k \geq 2$ vanishes on $\mathcal{S}_1, \dots, \mathcal{S}_m$. In particular, $\mathcal{S}_{m+1} = (\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_m)^\perp$ is invariant under $\mathcal{A}_k$ for $k \geq 2$. Thus $p(x) = \hat{p}_1(x_1) \oplus q(\hat{x})$ where $q$ is a free matrix polynomial depending only on $\hat{x} = (x_2, \dots x_g)$ and $\hat{x}^* = (x_2^*, \dots, x_g^*)$. We can repeat the above consideration – decomposing $\mathcal{S}_{m+1}$ into a direct sum of reducing subspaces for $\mathcal{A}_2$, etc. Tracking down all these decompositions yields the desired block form (4.4). ∎

*Proof of Theorem 4.1.* Let $x = (x_1, \dots, x_g)$ be a $g$-tuple of noncommuting letters and suppose $p$ is a monic free $d \times d$ matrix polynomial that is invariant under coordinate unitary conjugation. Here $p$ is given by $p(x) = \sum_w B_w w(x)$. Call a monomial a (noncommutative) cross term if it contains a product of the form $x_i x_j$ or $x_i x_j^*$ or $x_i^* x_j$ or $x_i^* x_j^*$ where $i \neq j$. Our immediate goal is to show that $p(x)$ does not have any cross terms.

To this end, let $\mathcal{C}_x$ be the set of all cross term monomials and define the free matrix polynomials $p^{\mathrm{ncr}}$ and $p^{\mathrm{cr}}$ by

$$p^{\mathrm{ncr}}(x) = \sum_{w(x) \notin \mathcal{C}_x} B_w w(x), \qquad p^{\mathrm{cr}}(x) = \sum_{w(x) \in \mathcal{C}_x} B_w w(x). \tag{4.9}$$

Here $p^{\mathrm{ncr}}(0) = I_d$ and $p^{\mathrm{cr}}(0) = 0_d$. With this notation,

$$p(x) = p^{\mathrm{ncr}}(x) + p^{\mathrm{cr}}(x). \tag{4.10}$$

To show $p$ has no cross terms we will show $p(x) = p^{\mathrm{ncr}}(x)$.

Define $\tilde{x}_1, \dots, \tilde{x}_g$ by

$$\tilde{x}_1 = x_1 \oplus 0 \oplus \cdots \oplus 0, \quad \tilde{x}_2 = 0 \oplus x_2 \oplus \cdots \oplus 0, \dots, \quad \tilde{x}_g = 0 \oplus \cdots \oplus 0 \oplus x_g,$$

Choose permutation matrices $U_i$ so that $U_i^* \tilde{x}_i U_i = x_i \oplus 0 \oplus \cdots \oplus 0$ for all $i$.

Recall the canonical shuffle discussed before Lemma 3.5. We use it again here dealing with polynomials. Namely, if $f$ is a $d \times d$ free matrix polynomial then the notation $f(x) \overset{\text{c.s.}}{\sim} h(x)$ means that for all $n$ and for all $X \in M_n(\mathbb{C})^g$ there exists a matrix $\hat{\Pi}_n$ that is a product of direct sums of canonical shuffles such that $\hat{\Pi}_n^* f(X) \hat{\Pi}_n = h(X)$.

Consider $p(\tilde{x}) = p(\tilde{x}_1, \dots, \tilde{x}_g)$. Since $\tilde{x}_i \tilde{x}_j = 0 = \tilde{x}_i \tilde{x}_j^* = \tilde{x}_i^* \tilde{x}_j$ whenever $i \neq j$ we see that

$$q(x) = p(\tilde{x}) \overset{\text{c.s.}}{\sim} q_1(x_1) \oplus \cdots \oplus q_g(x_g)$$

where the $q_i$ are monic matrix polynomials each depending only on $x_i$ and $x_i^*$. Furthermore,

$$p(U_1^* \tilde{x}_1 U_1, \dots, U_g^* \tilde{x}_g U_g) \overset{u}{\sim} p(x) \oplus p(0) \oplus \cdots \oplus p(0) = p(x) \oplus I_{d(g-1)} = \sum_w (B_w \oplus 0_{d(g-1)}) w(x).$$

Fix $n$ and consider the evaluations $p(X) \oplus I_{nd(g-1)}$ and $q(X)$ on $g$-tuples of matrices $X \in M_n(\mathbb{C})^g$. By assumption there exists a unitary $V_n$ depending only on our permutation matrices $U_i$ and on $n$ such that

$$V_n^* q(X) V_n = p((U_1 \otimes I_n)^* \tilde{X}_1 (U_1 \otimes I_n), \dots, (U_g \otimes I_n)^* \tilde{X}_g (U_g \otimes I_n)) = p(X) \oplus I_{nd(g-1)} \tag{4.11}$$

for all $X \in M_n(\mathbb{C})^g$.

Define the $n \times n$ matrix $\mathcal{X}_k^n$ by $\mathcal{X}_k^n = (\mathcal{X}_{k,ij}^n)_{ij}$ for $1 \leq k \leq g$. Here the $\mathcal{X}_{k,ij}^n$ are commuting variables and $\mathcal{X}_k^n$ is called a generic matrix. Define the $g$-tuple of $n \times n$ matrices $\mathcal{X}^n$ by $\mathcal{X}^n = (\mathcal{X}_1^n, \dots, \mathcal{X}_g^n)$. We say a word in the commuting letters $\{\mathcal{X}_{k,ij}^n\}_{i,j,k}$ and $\{(\mathcal{X}_{k,ij}^n)^*\}_{i,j,k}$ is a commutative cross term if it contains a product of the form $\mathcal{X}_{k,ij}^n \mathcal{X}_{\ell,rs}^n$ or $\mathcal{X}_{k,ij}^n (\mathcal{X}_{\ell,rs}^n)^*$ with $k \neq \ell$. Then (4.11) is equivalent to

$$V_n^* q(\mathcal{X}^n) V_n = p(\mathcal{X}^n) \oplus I_{nd(g-1)}. \tag{4.12}$$

We next show that the entries of $p(\mathcal{X}^n)$ have no commutative cross terms.

Since $q(x)$ contains no cross terms it follows that for all $n$ the entries of $q(\mathcal{X}^n)$ contain no commutative cross terms. Furthermore, the entries of $V_n^* q(\mathcal{X}^n) V_n$ are linear combinations of the entries of $q(\mathcal{X}^n)$ so it follows that for all $n$ the entries of $V_n^* q(\mathcal{X}^n) V_n$ contain no commutative cross terms. Using (4.12) we conclude that for all $n$ the entries of $p(\mathcal{X}^n) \oplus I_{nd(g-1)}$, and hence the entries of of $p(\mathcal{X}^n)$, contain no commutative cross terms.

Since the entries of $p(\mathcal{X}^n)$ have no commutative cross terms we know from equation (4.10) that the entries of $p^{\text{ncr}}(\mathcal{X}^n) + p^{\text{cr}}(\mathcal{X}^n)$ have no commutative cross terms. If a monomial $w(x)$ is not a cross term, then none of the entries of $w(\mathcal{X}^n)$ are commutative cross terms. Therefore none of the entries of $p^{\text{ncr}}(\mathcal{X}^n)$ are commutative cross terms. Since $p(\mathcal{X}^n)$ has no commutative cross terms this implies that none the entries of $p^{\text{cr}}$ cannot be commutative cross terms. We conclude $p^{\text{cr}}(\mathcal{X}^n) = 0_{nd \times nd}$ and therefore

$$p^{\text{ncr}}(\mathcal{X}^n) = p(\mathcal{X}^n). \tag{4.13}$$

Equation (4.13) holds for all $n$, so we obtain that for all $n$ and for all $g$-tuples of $n \times n$ matrices $X$ we have the equality

$$p^{\text{ncr}}(X) = p(X). \tag{4.14}$$

Since equation (4.14) holds for all $n$, we conclude

$$p^{\text{ncr}}(x) = p(x). \tag{4.15}$$

Therefore, $p$ has no cross terms, as claimed.

Now $p$ can be written $p(x) = I + \sum_{i=1}^g p_i(x_i)$ where $p_i(0) = 0$. Additionally, since $p$ is invariant under coordinate unitary conjugation it follows that $p^2$ defined by

$$p^2(x) = I + 2\sum_i p_i(x) + \sum_{i,j} p_i(x_i) p_j(x_j) \tag{4.16}$$

is also invariant under coordinate unitary conjugation and hence $p^2$ cannot have any cross terms. Thus equation (4.16) implies that $p_i(x_i) p_j(x_j) = 0$ whenever $i \neq j$.

Additionally, since $p$ is invariant under coordinate unitary conjugation, given any unitaries $U_i \in M_n(\mathbb{C})$ there exists some unitary $U \in M_{nd}(\mathbb{C})$ depending only on the $U_i$ such that for any $X \in M_n(\mathbb{C})^g$ we have

$$p(X_1, \dots X_g) = U^* p(U_1^* X_1 U_1, \dots, U_g^* X_g U_g) U. \tag{4.17}$$

It immediately follows that

$$p(X_1, \dots X_g)^* = U^* p(U_1^* X_1 U_1, \dots, U_g^* X_g U_g)^* U. \tag{4.18}$$

These two equations imply

$$\begin{aligned} p(x)^* p(x) &= I + \textstyle\sum_i p_i(x) + \sum_i p_i(x)^* + \sum_{i,j} p_i(x_i)^* p_j(x_j), \\ p(x) p(x)^* &= I + \textstyle\sum_i p_i(x) + \sum_i p_i(x)^* + \sum_{i,j} p_i(x_i) p_j(x_j)^* \end{aligned} \tag{4.19}$$

are also invariant under coordinate unitary conjugation and therefore have no cross terms. Therefore $p_i(x_i) p_j(x_j)^* = p_i(x_i)^* p_j(x_j) = 0$ whenever $i \neq j$.

It follows from Lemma 4.2 that there exists a unitary $V \in M_d(\mathbb{C})$ such that

$$V^* p(x) V = \hat{p}_1(x_1) \oplus \cdots \oplus \hat{p}_g(x_g),$$

where the $\hat{p}_i$ are monic free matrix polynomials in the variable $x_i$. Thus, if $p$ is invariant under coordinate unitary conjugation, then equation (4.2) holds.

The converse is straightforward. If (4.2) holds, then evidently $p$ is invariant under coordinate unitary conjugation. ∎

**Remark 4.3.** We say a free spectrahedron $\mathcal{D}$ is invariant under coordinate unitary conjugation if $X \in \mathcal{D}$ implies $(U_1^* X_1 U_1, \dots, U_g^* X_g U_g) \in \mathcal{D}$ for all $X \in M_n(\mathbb{C})^g$ and all unitaries $U_1, \dots, U_g \in M_n(\mathbb{C})$. Suppose the symmetric monic linear pencil $L_A$ is minimal in defining a free spectrahedron $\mathcal{D}_A$. It follows from Theorem 4.1 and [HKM13, Theorem 1.2] that $\mathcal{D}_A$ is invariant under coordinate unitary conjugation if and only if there is a unitary $U$ so that

$$U^* L_A(x) U = \bigoplus_{j=1}^{g} \left( I - A_j x_j - A_j x_j^* \right).$$

## Appendix A. Free Circular Matrix Convex Sets are Operator Pencil Balls

In this section we characterize free circular subsets of $M(\mathbb{C})^g$. A subset $D \subseteq M(\mathbb{C})^g$ is **free circular** if $UX \in \mathcal{D}$ for each $n$, each $X \in \mathcal{D}(n)$ and each $n \times n$ unitary matrix $U$.

### A.1. Properties of Free Circular Sets.

A free set $\mathcal{D} \subseteq M(\mathbb{C})^g$ is an **operator pencil ball** if there exists a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ and a $g$-tuple $A \in \mathcal{B}(\mathcal{H})^g$ such that $X \in \mathcal{D}$ if and only if

$$\|\Lambda_A(X)\| \leq 1.$$

(Observe that the formulas (1.1) – (1.4) naturally extend to tuples of operators $A$.) In particular, an operator pencil ball can be described as the positivity set of the symmetric operator pencil

$$\begin{pmatrix} I & \Lambda_A(x) \\ \Lambda_A(x)^* & I \end{pmatrix}.$$

If $\mathcal{H}$ is finite dimensional (so $\mathcal{B}(\mathcal{H})^g \cong M_d(\mathbb{C})^g$), the set $\mathcal{D}$ is a **matrix pencil ball**.

The main result of this section is Theorem A.5. It shows that a free circular matrix convex free set containing a neighborhood of 0 is an operator pencil ball, and is thus a free circular analog of the Effros-Winkler matricial Hahn-Banach separation theorem [EW97, HM12].

**Lemma A.1.** Suppose $\mathcal{C}$ is matrix balanced, closed with respect to direct sums and contains 0 in its interior and $Q \in M_d(\mathbb{C})^g$. If $\|\Lambda_Q(X)\| \le 1$ for $X \in \mathcal{C}$, then $\|\Lambda_Q(X)\| < 1$ for $X$ in the interior of $\mathcal{C}$. Conversely, if $\|\Lambda_Q(X)\| < 1$ for $X$ in the interior of $\mathcal{C}$, then $\|\Lambda_Q(X)\| \le 1$ for $X \in \mathcal{C}$.

A.2. **States and Representations of Separating Linear Functionals.** Let $M_\ell(\mathbb{C})_{\rm sa}$ denote self-adjoint elements of $M_\ell(\mathbb{C})$ and suppose $\mathcal{S}$ is a subspace of $M_\ell(\mathbb{C})_{\rm sa}$. An **affine linear mapping** $f : \mathcal{S} \to \mathbb{R}$ is a function of the form $f(x) = a_f + \lambda_f(x)$, where $\lambda_f : \mathcal{S} \to \mathbb{R}$ is linear over $\mathbb{R}$ and $a_f \in \mathbb{R}$. The following lemma is a version of [EW97, Lemma 5.2].

**Lemma A.2.** *Suppose $\mathcal{F}$ is a convex set of affine linear mappings $f : \mathcal{S} \to \mathbb{R}$ and $\mathcal{T} \subseteq \mathcal{S}$ is compact and convex. If for each $f \in \mathcal{F}$ there is a $T \in \mathcal{T}$ such that $f(T) \ge 0$, then there is a $\mathfrak{T} \in \mathcal{T}$ such that $f(\mathfrak{T}) \ge 0$ for every $f \in \mathcal{F}$.*

*Proof.* Each $f \in \mathcal{F}$ is continuous, a fact we will use freely. For $f \in \mathcal{F}$, let

$$B_f = \{T \in \mathcal{T} : f(T) \ge 0\} \subseteq \mathcal{T}.$$

By hypothesis each $B_f$ is non-empty and it suffices to prove that

$$\bigcap_{f \in \mathcal{F}} B_f \ne \varnothing.$$

Since each $B_f$ is compact, it suffices to prove that the collection $\{B_f : f \in \mathcal{F}\}$ has the finite intersection property. Accordingly, let $f_1, \dots, f_m \in \mathcal{F}$ be given. Arguing by contradiction, suppose $\bigcap_{j=1}^m B_{f_j} = \varnothing$. Define $F : \mathcal{S} \to \mathbb{R}^m$ by

$$F(T) = (f_1(T), \dots, f_m(T)).$$

Then $F(\mathcal{T})$ is both convex and compact because $\mathcal{T}$ is both convex and compact since $F$ is continuous. Moreover, $F(\mathcal{T})$ does not intersect

$$\mathbb{R}^m_{\ge 0} = \{x = (x_1, \dots, x_m) : x_j \ge 0 \text{ for each } j\}.$$

Hence there is a linear functional $\lambda : \mathbb{R}^m \to \mathbb{R}$ such that $\lambda(F(\mathcal{T})) < 0$ and $\lambda(\mathbb{R}^m_{\ge 0}) \ge 0$. There exists $\lambda_j$ such that $\lambda(x) = \sum \lambda_j x_j$. Since $\lambda(\mathbb{R}^m_{\ge 0}) \ge 0$ it follows that each $\lambda_j \ge 0$ and, since $\lambda \neq 0$, there is a $k$ such that $\lambda_k > 0$. Without loss of generality, it may be assumed that $\sum \lambda_j = 1$. Let

$$f = \sum \lambda_j f_j.$$

Since $\mathcal{F}$ is convex, it follows that $f \in \mathcal{F}$. On the other hand, $f(T) = \lambda(F(T))$. Hence if $T \in \mathcal{T}$, then $f(T) < 0$. Thus, for this $f$ there does not exist a $T \in \mathcal{T}$ such that $f(T) \geq 0$, a contradiction which completes the proof. ∎

**Lemma A.3.** *Let $\mathcal{C} = (\mathcal{C}(n))$ denote a matrix balanced subset of the graded set $M(\mathbb{C})^g$ that is closed with respect to direct sums. Let $n$ and an $\mathbb{C}$-linear functional $\mathcal{L} : M_n(\mathbb{C})^g \to \mathbb{C}$ be given. If $\operatorname{Re}(\mathcal{L}(X)) \leq 1$ for each $X \in \mathcal{C}(n)$, then there exits positive semidefinite $n \times n$ matrices $\mathfrak{T}_1$ and $\mathfrak{T}_2$ each of trace norm one such that for each $m$, each $Y \in \mathcal{C}(m)$, and each pair $C = (C_1, C_2)$ of $m \times n$ matrices*

$$2\operatorname{Re}(\mathcal{L}(C_2^* Y C_1)) \leq \operatorname{tr}(C_1 \mathfrak{T}_1 C_1^*) + \operatorname{tr}(C_2 \mathfrak{T}_2 C_2^*).$$

*Proof.* Let $\ell = 2n$ and let

$$\mathcal{T} = \{T = T_1 \oplus T_2 : T_j \in M_n(\mathbb{C})_{\text{sa}},\ T_j \succeq 0, \text{ and } \operatorname{tr}(T_j) = 1\}.$$

In particular $\mathcal{T}$ is a compact convex subset of the $\ell \times \ell$ matrices.

Given a positive integer $m$, a tuple $Y$ in $\mathcal{C}(m)$ and $m \times n$ contraction matrices $C_1, C_2$, define $f_{Y,C} : M_n(\mathbb{C})_{\text{sa}} \oplus \mathbb{M}_n(\mathbb{C})_{\text{sa}} \to \mathbb{R}$ by

$$f_{Y,C}(T_1 \oplus T_2) = \sum_{j=1}^{2} \operatorname{tr}(C_j T_j C_j^*) - 2\operatorname{Re}(\mathcal{L}(C_2^* Y C_1)).$$

Now we show that the collection

$$\mathcal{F} = \{f_{Y,C} : Y \in \mathcal{C}(m),\ C = (C_1, C_2) \text{ where } C_1, C_2 \in \mathbb{C}^{m \times n} \text{ are contractions and } m, n \in \mathbb{N}\}$$

is a convex set. Start with a positive integer $s$, nonnegative numbers $\lambda_1, \ldots, \lambda_s$ with $\sum \lambda_j = 1$, and with $(Y_j, C_{j,1}, C_{j,2})$ for $j = 1, \ldots, s$ where $Y_j \in \mathcal{C}(m_j)$ and $C_{j,p}$ are $m_j \times n$ contraction matrices. Let $Z = \oplus Y_j$ and let $F_p$ denote the (block) column matrix with entries $\sqrt{\lambda_j} C_{j,p}$. Then $Z \in \mathcal{C}(m)$ where $m = \sum m_j$ and

$$F_p^* F_p = \sum \lambda_j C_{j,p}^* C_{j,p} \preceq \sum \lambda_j I = I.$$

Hence each $F_p$ is a contraction. By definition,

$$\sum \lambda_j C_{j,2}^* Y_j C_1 = F_2^* Z F_1, \qquad \sum \lambda_j \operatorname{tr}(C_{j,p} T_p C_{j,p}^*) = \operatorname{tr}(F_p T_p F_p^*).$$

Therefore

$$\sum \lambda_j f_{Y_j, C_j}(T) = f_{Z,F}(T)$$

so $\mathcal{F}$ is convex.

Observe, for any $X \in \mathcal{C}$ and pair of matrices $C_2$ and $C_1$ (of the appropriate sizes) $\operatorname{Re}\mathcal{L}(C_2^* X C_1) \leq \|C_2\|\,\|C_1\|$. Now let $f_{Y,C} \in \mathcal{F}$ be given. Choose unit vectors $\gamma_j$ such that

$$\|C_j\gamma_j\| = \|C_j\|,$$

let $T_j = \gamma_j^*\gamma_j$ and finally $T = T_1 \oplus T_2$. With these notations,

$$2\operatorname{Re}(\mathcal{L}(C_2^* Y C_1) \leq 2\|C_1\|\,\|C_2\| \leq \|C_1\|^2 + \|C_2^2\| = \operatorname{tr}(C_1\gamma_1\gamma_1^* C_1) + \operatorname{tr}(C_2\gamma_2\gamma_2^* C_2^*)$$

and thus,

$$f_{Y,C}(T) = \sum_{j=1}^{2} \operatorname{tr}(C_j\gamma_j\gamma_j^* C_j^*) - 2\operatorname{Re}(\mathcal{L}(C_2^* Y C_1)) \geq 0.$$

Consequently, for each $f_{Y,C}$ there is a $T \in \mathcal{T}$ such that $f_{Y,C}(T) \geq 0$. From Lemma A.2, there is a $\mathfrak{T} \in \mathcal{T}$ such that $f_{Y,C}(\mathfrak{T}) \geq 0$ for every $Y$ and $C$. ∎

A.3. **An Effros-Winkler Theorem for Free Circular Matrix Convex Sets.** In this section we present the effective version of [BMV+, Proposition 3.5], i.e., Proposition 3.3, restated here for the convenience of the reader as Proposition A.4.

**Proposition A.4.** *Let $\mathcal{C} = (\mathcal{C}(n))$ denote a matrix balanced subset of the graded set $M(\mathbb{C})^g$ that contains a free $\epsilon$-neighborhood of $0$ and is closed with respect to direct sums. If $X^{\mathrm{b}} \in M_n(\mathbb{C})^g$ is in the boundary of $\mathcal{C}(n)$, then there is a tuple $Q \in M_n(\mathbb{C})^g$ such that $\|\Lambda_Q(Y)\| \leq 1$ for all $m$ and $Y \in \mathcal{C}(m)$ and such that $\|\Lambda_Q(X^{\mathrm{b}})\| = 1$. Furthermore, if $Y$ is in the interior of $\mathcal{C}$, then $\|\Lambda_Q(Y)\| < 1$.*

*Proof.* By the usual Hahn-Banach separation theorem and the assumption that $\mathcal{C}(n)$ contains an $\epsilon$-neighborhood of 0, there is a linear functional $\mathcal{L} : M_n(\mathbb{C})^g \to \mathbb{C}$ such that $\operatorname{Re}(\mathcal{L}(X^{\mathrm{b}})) = 1 \geq \operatorname{Re}(\mathcal{L}(\mathcal{C}(n)))$.

From Lemma A.3 there exists positive semidefinite $n \times n$ matrices $T_1$ and $T_2$ of trace norm one such that $\sum_{p=1}^{2} \operatorname{tr}(C_p T_p C_p^*) - 2\operatorname{Re}(\mathcal{L}(C_2^* Y C_1)) \geq 0$ for each $m$, each pair of $m \times n$ contractions $C_1, C_2$, and each $Y \in \mathcal{C}(m)$. Hence, by homogeneity, for each $m$, each pair of $m \times n$ matrices $C_2, C_1$, and each $Y \in \mathcal{C}(m)$,

$$\sum_{p=1}^{2} \operatorname{tr}(C_p T_p C_p^*) - 2\operatorname{Re}(\mathcal{L}(C_2^* Y C_1)) \geq 0 \tag{A.1}$$

Note this inequality is sharp in the sense,

$$\sum_{p=1}^{2} \operatorname{tr}(T_p) - 2\operatorname{Re}(\mathcal{L}(X^{\mathrm{b}})) = 0. \tag{A.2}$$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_g\}$ denote the standard orthonormal basis for $\mathbb{C}^g$. Thus, if $M$ is an $n \times n$ matrix, then $M \otimes \mathbf{e}_\ell = (M_1, \dots, M_g) \in M_n(\mathbb{C})^g$ is the $g$-tuple with $M_j = 0$ for $j \neq \ell$ and $M_\ell = M$. Given $1 \leq \ell \leq g$, define a bilinear form on $\mathbb{C}^n$ by

$$\mathcal{B}_\ell(c, d) = \mathcal{L}(cd^* \otimes \mathbf{e}_\ell)$$

for $c, d \in \mathbb{C}^n$. There is a unique $n \times n$ matrix $B_\ell$ such that $\mathcal{B}_\ell(c, d) = \langle B_\ell c, d\rangle$.

Let $\Lambda_B$ denote the linear polynomial $\Lambda_B(x) = \sum_1^g B_j x_j$. Fix a positive integer $m$ and let $\{e_1, \dots, e_m\}$ denote the standard orthonormal basis for $\mathbb{C}^m$. Let $Y = (Y_1, \dots, Y_g) \in \mathcal{C}(m)$ be given and consider $\Lambda_B(Y)$. Given vectors $\gamma_p = \sum_{j=1}^m \gamma_{p,j} \otimes e_j$, for $p = 1, 2$, contained in $\mathbb{C}^n \otimes \mathbb{C}^m$, compute

$$\begin{aligned}\langle \Lambda_B(Y)\gamma_2, \gamma_1\rangle &= \sum_{i,j}\sum_\ell \langle B_\ell \gamma_{2,j}, \gamma_{1,i}\rangle\langle Y_\ell e_j, e_i\rangle = \sum_{i,j}\sum_\ell \mathcal{L}\big(\gamma_{2,j}\gamma_{1,i}^* \otimes \mathbf{e}_\ell\big)\langle Y_\ell e_j, e_i\rangle \\ &= \mathcal{L}\Big(\sum_\ell \sum_{i,j} \gamma_{2,i}\langle Y_\ell e_j, e_i\rangle \gamma_{1,j}^* \otimes \mathbf{e}_\ell\Big) = \mathcal{L}\Big(\sum_\ell \Gamma_2 Y_\ell \Gamma_1^* \otimes \mathbf{e}_\ell\Big) = \mathcal{L}(\Gamma_2 Y \Gamma_1^*),\end{aligned}$$

where $\Gamma_p$ is the matrix with $j$-th column $\gamma_{p,j}$. Using equation (A.1),

$$\begin{aligned}2\,\mathrm{Re}(\mathcal{L}(\Gamma_2 Y \Gamma_1^*)) &\leq \mathrm{tr}(\Gamma_1^* T_1 \Gamma_1) + \mathrm{tr}(\Gamma_2^* T_2 \Gamma_2) = \sum_{p=1}^2 \sum_{j=1}^m \langle T_p \gamma_{p,j}, \gamma_{p,j}\rangle \\ &= \sum_{p=1}^2 \langle (T_p \otimes I) \sum_j \gamma_{p,j} \otimes e_j, \sum_k \gamma_{p,k} \otimes e_k\rangle = \sum_{p=1}^2 \langle (T_p \otimes I)\gamma_p, \gamma_p\rangle.\end{aligned}$$

Thus,

$$\Phi(Y) = \begin{pmatrix} T_1 \otimes I & -\Lambda_B(Y) \\ -\Lambda_B(Y)^* & T_2 \otimes I \end{pmatrix} \succeq 0 \tag{A.3}$$

for every $m$ and $Y \in \mathcal{C}(m)$.

Since $\mathcal{C}$ contains the $\epsilon$-neighborhood of 0, it contains $\pm\frac{\epsilon}{2}\mathbf{e}_j \in \mathbb{C}^g$. Hence, for each $j$,

$$0 \preceq \Phi(\pm\frac{\epsilon}{2}\mathbf{e}_j) = \begin{pmatrix} T_1 & \pm\Lambda_B(\frac{\epsilon}{2}\mathbf{e}_j) \\ \pm\Lambda_B(\frac{\epsilon}{2}\mathbf{e}_j)^* & T_2 \end{pmatrix} = \begin{pmatrix} T_1 & \pm\frac{\epsilon}{2}B_j \\ \pm\frac{\epsilon}{2}B_j^* & T_2 \end{pmatrix}.$$

Thus, while the $T_p$ need not be invertible, it can be assumed (passing to subspaces of smaller dimension if necessary) that they are invertible. Finally, multiplying left and right by $\oplus T_p^{-\frac{1}{2}}$ produces the linear polynomial $\Lambda_Q(x) = \sum_j Q_j x_j$ (with $Q_j = T_1^{-\frac{1}{2}} B_j T_2^{-\frac{1}{2}}$) such that, with $\Psi$ denoting the monic symmetric linear pencil

$$\Psi(x) = \begin{pmatrix} I & -\Lambda_Q(x) \\ -\Lambda_Q(x)^* & I \end{pmatrix},$$

$\Psi(Y) \succeq 0$ if and only if $\Phi(Y) \succeq 0$. In particular, $\Psi$ is positive definite on $\mathcal{C}$. Equivalently, $\|\Lambda_Q(Y)\| \leq 1$ for all $Y \in \mathcal{C}$.

On the other hand, computing as above, (A.2) becomes

$$\begin{aligned}\langle \Phi(X^{\mathrm{b}})e \oplus e, e \oplus e\rangle &= \sum_{p=1}^{2} \operatorname{tr}(T_p) - 2\operatorname{Re}(\langle \Lambda_B(X^{\mathrm{b}})e, e\rangle) \\ &= 2 - 2\sum_{\ell}\sum_{j,k} \operatorname{Re}(\langle B_\ell e_j, e_k\rangle \langle X_\ell^{\mathrm{b}} e_j, e_k\rangle) \\ &= 2 - 2\sum_{\ell,j,k} \operatorname{Re}(\mathcal{L}(e_j e_k^* \otimes \mathbf{e}_\ell) \langle X_\ell^{\mathrm{b}} e_j, e_k\rangle) \\ &= 2 - 2\operatorname{Re}\Big(\mathcal{L}[\sum_{\ell,j,k} \langle X_\ell^{\mathrm{b}} e_j, e_k\rangle (e_j e_k^* \otimes \mathbf{e}_\ell)]\Big) = 2 - 2\operatorname{Re}(\mathcal{L}(X^{\mathrm{b}})) = 0,\end{aligned}$$

where $e = \sum e_j \otimes e_j$. Since $X^{\mathrm{b}}$ is in $\mathcal{C}(n)$, it follows that $\Phi(X^{\mathrm{b}}) \succeq 0$. Thus $\Phi(X^{\mathrm{b}})(e \oplus e) = 0$, and since $(T_p \otimes I)e \neq 0$, it follows that $\Psi(X^{\mathrm{b}})$ is singular too. In particular, $\|\Lambda_Q(X^{\mathrm{b}})\| = 1$.

Finally, suppose $Y \in \mathcal{C}$ and $\|\Lambda_Q(Y)\| = 1$. If $t > 1$, then $\|\Lambda_Q(tY)\| > 1$ and hence $tY \notin \mathcal{C}$. Thus $Y$ is in the boundary of $\mathcal{C}$. Hence if $Y$ is in the interior of $\mathcal{C}$, then $\|\Lambda_Q(Y)\| < 1$. ∎

**Theorem A.5** (cf. [BMV+, Proposition 3.5])**.** *If $\mathcal{C} \subseteq M(\mathbb{C})^g$ is a closed matrix balanced, closed with respect to direct sums and $\mathcal{C}$ contains a free $\epsilon$-neighborhood of $0$, then $\mathcal{C}$ is an operator pencil ball.*

**Lemma A.6.** *If $\mathcal{C} \subseteq M(\mathbb{C})^g$ is a matrix convex set and if $\mathcal{C}(1)$ contains $0$ in its interior, then there exists a constant $\kappa$ such that if $Q \in M(\mathbb{C})^g$ and $\|\Lambda_Q(X)\| \leq 1$ for all $X \in \mathcal{C}$, then $\|Q_j\| \leq \kappa$ for each $1 \leq j \leq g$.*

*Proof.* Let $\{\mathbf{e}_j : 1 \leq j \leq g\}$ denote the standard basis for $\mathbb{C}^g$. By hypothesis, there is an $\epsilon > 0$ such that the tuple $\epsilon \mathbf{e}_j \in \mathcal{D}_Q(1)$. Hence, $1 \geq \|\Lambda(\epsilon \mathbf{e}_j)\| = \epsilon \|Q_j\|$. Choosing $\kappa = \frac{1}{\epsilon}$ completes the proof. ∎

*Proof of Theorem A.5.* For a fixed $n$, choose a countable set $K(n) \subseteq \partial C(n)$ with $\overline{K(n)} = \partial C(n)$. By assumption $\mathcal{C}$ contains a free $\epsilon$-neighborhood of $0$, so Proposition 3.3 implies that for each $X \in K(n)$ there exists a tuple $Q_X \in M_n(\mathbb{C})^g$ such that $\|\Lambda_{Q_X}(Y)\| \leq 1$ for all $m$ and $Y \in \mathcal{C}(m)$ and such that $\|\Lambda_{Q_X}(X)\| = 1$.

Set $K = \bigcup_n K(n)$ and define $Q = \bigoplus_{X \in K} Q_X$. Since $K(n)$ is countable for each $n$, it follows that $K$ is also countable. Furthermore, $Q$ is a bounded operator by Lemma A.6. We will show $\mathcal{C} = \{X \in M(\mathbb{C})^g : \|\Lambda_Q(X)\| \leq 1\}$.

By construction, $\|\Lambda_{Q_X}(Y)\| \leq 1$ for all $Y \in \mathcal{C}$. Hence $\mathcal{C} \subseteq \{X \in M(\mathbb{C})^g : \|\Lambda_Q(X)\| \leq 1\}$. Moreover, if $X \in K$, then $\|\Lambda_{Q_X}(X)\| = 1$. Since $K$ is dense in $\partial\mathcal{C}$ and $\Lambda_Q$ is continuous, $\|\Lambda_Q(X)\| = 1$ for all $X \in \partial\mathcal{C}$.

Finally, suppose $Y \notin \mathcal{C}$. Since $\mathcal{C}$ contains a free $\epsilon$-neighborhood of $0$ there exists some $t \in (0,1)$ such that $tY \in \partial\mathcal{C}$. It follows that $\|\Lambda_Q(tY)\| = 1$ and hence $\|\Lambda_Q(Y)\| = \frac{1}{t} > 1$.

Thus $\mathcal{C} \supseteq \{X \subseteq M(\mathbb{C})^g : \|\Lambda_Q(X)\| \leq 1\}$ and therefore, $\mathcal{C}$ is the operator pencil ball $\{X : \|\Lambda_Q(X)\| \leq 1\}$. ■

## References


[AM14] J. Agler, J. McCarthy: *Global holomorphic functions in several non-commuting variables*, Canad. J. Math. 67 (2015) 241–285. 2

[Arv08] W.B. Arveson: *The noncommutative Choquet boundary*, J. Amer. Math. Soc. 21 (2008) 1065–1084. 2

[BMV+] J.A. Ball, G. Marx, V. Vinnikov: *Interpolation and transfer-function realization for the noncommutative Schur-Agler class*, preprint http://arxiv.org/abs/1602.00762 2, 3, 4, 9, 10, 23, 25

[BPR13] G. Blekherman, P.A. Parrilo, R.R. Thomas (editors): *Semidefinite optimization and convex algebraic geometry*, MOS-SIAM Series on Optimization 13, SIAM, 2013. 2

[BGFB94] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan: *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics 15, SIAM, 1994. 2

[BKU78] R. Braun, W. Kaup, H. Upmeier: *On the automorphisms of circular and Reinhardt domains in complex Banach spaces*, Manuscripta Math. 25 (1978) 97–133. 2

[DDSS+] K.R. Davidson, A. Dor-On, O. Shalit, B. Solel: *Dilations, inclusions of matrix convex sets, and completely positive maps*, preprint http://arxiv.org/abs/1601.07993 2

[dOHMP09] M. de Oliveira, J.W. Helton, S. McCullough, M. Putinar: *Engineering systems and free semi-algebraic geometry*, in: Emerging applications of algebraic geometry (edited by M. Putinar, S. Sullivant), 17–61, Springer-Verlag, 2009. 2

[Eff09] E.G. Effros: *A matrix convexity approach to some celebrated quantum inequalities*, Proc. Nat. Acad. Sci. 106 (2009) 1006–1008. 2

[ER00] E.G. Effros, Z.-J. Ruan: *Operator Spaces*, London Mathematical Society Mono- graphs New Series 23, Oxford Science Publications, 2000. 10

[EW97] E.G. Effros, S. Winkler: *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, J. Funct. Anal. 144 (1997) 117–152. 2, 3, 9, 21

[Far12] D.R. Farenick: *Extremal matrix states on operator systems*, J. London Math. Soc. 61 (2000) 885–892. 2

[FP12] D. Farenick, V.I. Paulsen: *Operator system quotients of matrix algebras and their tensor products*, Math. Scand. 111 (2012) 210–243. 2

[HKM12] J.W. Helton, I. Klep, S. McCullough: *The convex Positivstellensatz in a free algebra*, Adv. Math. 231 (2012) 516–534. 2

[HKM13] J.W. Helton, I. Klep, S. McCullough: *The matricial relaxation of a linear matrix inequality*, Math. Program. 138 (2013) 401–445. 4, 5, 20

[HKM+] J.W. Helton, I. Klep, S. McCullough: *The Tracial Hahn-Banach Theorem, Polar Duals, Matrix Convex Sets, and Projections of Free Spectrahedra*, to appear in J. Eur. Math. Soc., http://arxiv.org/abs/1407.8198. 2

[HM12] J.W. Helton, S. McCullough: *Every free basic convex semi-algebraic set has an LMI representation*, Ann. of Math. (2) 176 (2012) 979–1013. 21

[HM04] J.W. Helton, S. McCullough: *A Positivstellensatz for noncommutative polynomials*, Trans. Amer. Math. Soc. **356** (2004) 3721–3737. 9

[KVV14] D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov: *Foundations of Free Noncommutative Function Theory,* Mathematical Surveys and Monographs 199, AMS, 2014. 2

[KPTT13] A.S. Kavruk, V.I. Paulsen, I.G. Todorov, M. Tomforde: *Quotients, exactness, and nuclearity in the operator system category*, Adv. Math. 235 (2013) 321–360. 2

[KŠ+] I. Klep, Š. Špenko: *Free function theory through matrix invariants*, to appear in Canad. J. Math. 2

[Kra01] S.G. Krantz: *Function Theory of Several Complex Variables*, AMS, 2001. 2

[Pau02] V. Paulsen: *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics 78, Cambridge University Press, 2002. 10

[Pop08] G. Popescu: *Noncommutative Berezin transforms and multivariable operator model theory*, J. Func. Anal. 254 (2008) 1003–1057. 2

[SIG97] R.E. Skelton, T. Iwasaki, K.M. Grigoriadis: *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, 1997. 2

[Tay72] J.L. Taylor: *A general framework for a multi-operator functional calculus*, Adv. Math. 9 (1972) 183–252. 2

[Voi10] D.-V. Voiculescu: *Free analysis questions II: The Grassmannian completion and the series expansions at the origin*, J. reine angew. Math. 645 (2010) 155–236. 2

[WW99] C. Webster, S. Winkler: *The Krein-Milman theorem in operator convexity*, Trans. Amer. Math. Soc. 351 (1999) 307–322. 2

[Wit84] G. Wittstock: *On matrix order and convexity*, in: Functional analysis: surveys and recent results, III (Paderborn, 1983), 175–188, North-Holland Math. Stud. 90, 1984. 2

[Zal+] A. Zalar: *Operator Positivstellensätze for noncommutative polynomials positive on matrix convex sets*, preprint http://arxiv.org/abs/1602.00765. 2, 4, 5

Eric Evert, Department of Mathematics, University of California, San Diego

*E-mail address*: eevert@ucsd.edu

J. William Helton, Department of Mathematics, University of California, San Diego

*E-mail address*: helton@math.ucsd.edu

Igor Klep, Department of Mathematics, The University of Auckland, New Zealand

*E-mail address*: igor.klep@auckland.ac.nz

Scott McCullough, Department of Mathematics, University of Florida, Gainesville

*E-mail address*: sam@math.ufl.edu

NOT FOR PUBLICATION

## Contents

# Index