FREE BIANALYTIC MAPS BETWEEN SPECTRAHEDRA AND SPECTRABALLS IN A GENERIC SETTING

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This paper, which would not exist without techniques he pioneered, is dedicated to Joe Ball on the occasion of his 70th birthday.

Abstract. Given a tuple $E = (E_1, \dots, E_g)$ of $d \times d$ matrices, the collection $\mathcal{B}_E$ of those tuples of matrices $X = (X_1, \dots, X_g)$ (of the same size) such that $\| \sum E_j \otimes X_j \| \le 1$ is a spectraball. Likewise, given a tuple $B = (B_1, \dots, B_g)$ of $e \times e$ matrices the collection $\mathcal{D}_B$ of tuples of matrices $X = (X_1, \dots, X_g)$ (of the same size) such that $I + \sum B_j \otimes X_j + \sum B_j^* \otimes X_j^* \succeq 0$ is a free spectrahedron. Assuming E and B are irreducible, plus an additional mild hypothesis, there is a free bianalytic map $p : \mathcal{B}_E \to \mathcal{D}_B$ normalized by $p(0) = 0$ and $p'(0) = I$ if and only if $\mathcal{B}_E = \mathcal{B}_B$ and B spans an algebra. Moreover p is unique, rational and has an elegant algebraic representation.

1. Introduction

In this article we continue our investigation of free bianalytic mappings between matrix convex domains. The results in this article stands on the bedrock of the noncommutative state space methods introduced to the operator theory community by Joe and his collaborators and they are inseparable from the profound influence of Joe's work in function theoretic operator theory and free analysis.

Fix g a positive integer. Given a positive integer n, let $M_n(\mathbb{C})^g$ denote the g-tuples $X = (X_1, \dots, X_g)$ of $n \times n$ matrices with entries from $\mathbb{C}$. Given $A \in M_d(\mathbb{C})^g$, the set $\mathcal{D}_A(1)$ consisting of $x \in \mathbb{C}^g$ such that

$$L_A(x) = I + \sum A_j x_j + \sum A_j^* x_j^* \succeq 0$$

Date: October 6, 2018.

2010 *Mathematics Subject Classification.* 47L25, 32H02, 13J30 (Primary); 14P10, 52A05, 46L07 (Secondary).

Key words and phrases. bianalytic map, birational map, linear matrix inequality (LMI), spectrahedron, convex set, Positivstellensatz, free analysis, real algebraic geometry.

[1]Research supported by the NSF grant DMS 1201498, and the Ford Motor Co.

[2]Supported by the Marsden Fund Council of the Royal Society of New Zealand. Partially supported by the Slovenian Research Agency grants J1-8132, N1-0057 and P1-0222.

[3]Research supported by NSF grants DMS-1361501 and DMS-1764231.

is a *spectrahedron*. Here $T \succeq 0$ indicates the selfadjoint matrix T is positive semidefinite. Spectrahedra are basic objects in a number of areas of mathematics; e.g. semidefinite programming, convex optimization and in real algebraic geometry [BPR13]. They also figure prominently in determinantal representations [Brä11, GK-VVW16, NT12, Vin93], the solution of the Lax conjecture [HV07], in the solution of the Kadison-Singer paving conjecture [MSS15], and in systems engineering [BGFB94, SIG96].

For $X \in M_n(\mathbb{C})^g$ and still with $A \in M_d(\mathbb{C})^g$, let

$$\Lambda_A(X) = \sum A_j \otimes X_j$$

and

$$L_A(X) = I + \Lambda_A(X) + \Lambda_A(X)^* = I + \sum A_j \otimes X_j + \sum A_j^* \otimes X_j^*.$$

The *free spectrahedron* determined by A is the sequence of sets $\mathcal{D}_A = (\mathcal{D}_A(n))$, where

$$\mathcal{D}_A(n) = \{X \in M_n(\mathbb{C})^g : L_A(X) \succeq 0\}.$$

Free spectrahedra arise naturally in applications such as systems engineering [dOHMP09] and in the theories of matrix convex sets, operator algebras, systems and spaces and completely positive maps [EW97, HKM17, Pau02]. They also provide tractable useful relaxations for spectrahedral inclusion problems that arise in semidefinite programming and engineering applications such as the matrix cube problem [B-TN02, HKMS+].

Given a tuple $E \in M_d(\mathbb{C})^g$, the set

$$\mathcal{B}_E = \{X : \|\Lambda_E(X)\| \le 1\}$$

is a *spectraball* [EHKM17, BMV18]. Spectraballs are special cases of free spectrahedra. Indeed, it is readily seen that

$$\mathcal{B}_E = \mathcal{D}_{\left(\begin{smallmatrix} 0 & E \\ 0 & 0 \end{smallmatrix}\right)}.$$

Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_n$. A *subset* Γ of $M(\mathbb{C})^g$ is a sequence $(\Gamma_n)_n$ where $\Gamma_n \subset M_n(\mathbb{C})^g$. (Sometimes we will write $\Gamma(n)$ in place of Γ_n.) The subset Γ is a *free set* if it is closed under direct sums and unitary similarity; that is, if $X \in \Gamma_n$ and $Y \in \Gamma_m$, then

$$X \oplus Y = \left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \dots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \right) \in \Gamma_{n+m};$$

and if U is an $n \times n$ unitary matrix, then

$$U^* X U = \big(U^* X_1 U, \dots, U^* X_g U\big) \in \Gamma_n.$$

We say the free set $\Gamma = (\Gamma_n)_n$ is *open* if each Γ_n is open. (Generally adjectives are applied levelwise to free sets unless noted otherwise.)

A *free function* $f : \Gamma \to M(\mathbb{C})$ is a sequence of functions $f_n : \Gamma_n \to M_n(\mathbb{C})$ that *respects intertwining*; that is, if $X \in \Gamma_n$, $Y \in \Gamma_n$, $T : \mathbb{C}^m \to \mathbb{C}^n$, and

$$XT = (X_1T, \dots, X_gT) = (TY_1, \dots, TY_g) = TY,$$

then $f_n(X)T = Tf_m(Y)$. Assuming Γ is an open free set, a free function $f : \Gamma \to M(\mathbb{C})$ is *analytic* if each f_n is analytic. Given free sets $\Gamma \subset M(\mathbb{C})^g$ and $\Delta \subset M(\mathbb{C})^h$, a *free mapping* $f : \Gamma \to \Delta$ consists of free maps $f^i : \Gamma \to M(\mathbb{C})$ such that $f(X) = \begin{pmatrix} f^1(X) & \dots & f^h(X) \end{pmatrix}$. In this case we write $f = \begin{pmatrix} f^1 & \dots & f^h \end{pmatrix}$. We refer the reader to [Voi04, KVV14] for a fuller discussion of free sets and functions.

In this note, we characterize the free bianalytic maps $p : \mathcal{B}_E \to \mathcal{D}_B$ under some mild conditions on $E \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^g$ and on p and its inverse q. These free functions take a highly algebraic form that we call *convexotonic.* A tuple $\Xi = (\Xi_1, \dots, \Xi_g) \in M_g(\mathbb{C})^g$ satisfying

$$\Xi_k \Xi_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} \Xi_s$$

for each $1 \le j, k \le g$ is *convexotonic.* Convexotonic tuples naturally arise from finite dimensional algebras. If $\{J_1, \dots, J_g\} \subset M_d(\mathbb{C})$ is linearly independent and spans an algebra, then there exists a uniquely determined tuple $\Psi \in M_g(\mathbb{C})^g$ such that

$$J_k J_j = \sum_{s=1}^{g} (\Psi_j)_{k,s} J_s \tag{1.1}$$

and Proposition 2.1 says Ψ is convexotonic.

Given a convexotonic tuple $\Xi \in M_g(\mathbb{C})^g$, the expressions $p = \begin{pmatrix} p^1 & \cdots & p^g \end{pmatrix}$ and $q = \begin{pmatrix} q^1 & \cdots & q^g \end{pmatrix}$ whose components have the form

$$p^i(x) = \sum_j x_j \,(I - \Lambda_\Xi(x))^{-1}_{j,i} \qquad \text{and} \qquad q^i(x) = \sum_j x_j \,(I + \Lambda_\Xi(x))^{-1}_{j,i}\,, \tag{1.2}$$

that is, in row form,

$$p(x) = x(I - \Lambda_\Xi(x))^{-1} \qquad \text{and} \qquad q = x(I + \Lambda_\Xi(x))^{-1}$$

are, by definition, *convexotonic.* The components of p (resp. q) are free functions with (free) domains consisting of those X for which $I - \Lambda_\Xi(X)$ (resp. $I + \Lambda_\Xi(X)$) is invertible. Hence p and q are free functions. It turns out (see [AHKM18, Proposition 6.2]) the mappings p and q are inverses of one another.

Before continuing, we would like to point out that the component functions p^i of the convexotonic map p of equation (1.2) are in fact free rational functions regular at 0. Accordingly we refer to p and q as *birational* or free birational maps. Free rational functions are most easily described and naturally understood in terms of realization theory as developed in the series of papers [BGM05, BGM06a, BGM06b] of Ball-Groenewald-Malakorn. Indeed, based on those articles and on the results of [KVV09, Theorem 3.1] and [Vol17, Theorem 3.5]) a *free rational function regular at* 0 can, for the purposes of this article, be defined with minimal overhead as an expression of the form

$$r(x) = c^*\big(I - \Lambda_S(x)\big)^{-1} b$$

where s is a positive integer, $S \in M_s(\mathbb{C})^g$ and $b, c \in \mathbb{C}^s$ are vectors. The expression r is known as a realization. Realizations are easy to manipulate and the theory of realizations is a powerful tool. The realization r is evaluated in the obvious fashion for a tuple $X \in M_n(\mathbb{C})^g$ as long as $I - \Lambda_S(X)$ is invertible. Free polynomials are free rational functions that are regular at 0 and free rational functions regular at 0 are stable with respect to the formal algebraic operations of addition, multiplication and inversion in the sense that if r is a free rational function regular at 0 and $r(0) \neq 0$, then its multiplicative inverse r^{-1} is also a free rational function regular at 0. Thus, expressing p^i as

$$p^i = \sum_{s=1}^{g} x_s e_s^* (I - \Lambda_\Xi(x))^{-1} e_i$$

shows it is a free rational function regular at 0.

To state our main theorem precisely we need a bit more terminology. A subset $\{u^1, \dots, u^{d+1}\}$ of $\mathbb{C}^d$ is a *hyperbasis* for $\mathbb{C}^d$ if each d element subset is a basis. The tuple $A \in M_d(\mathbb{C})^g$ is *sv-generic* if there exists $\alpha^1, \dots, \alpha^{d+1}$ and $\beta^1, \dots, \beta^d$ in $\mathbb{C}^g$ such that, for each $1 \le j \le d+1$, the matrix $I - \Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$ is positive semidefinite, has a one-dimensional kernel spanned by u^j and the set $\{u^1, \dots, u^{d+1}\}$ is a hyperbasis for $\mathbb{C}^d$; and, for each $1 \le k \le g$, the matrix $I - \Lambda_A(\beta^k)\Lambda_A(\beta^k)^*$ is positive semidefinite, has a one-dimensional kernel spanned by v^k and the set $\{v^1, \dots, v^d\}$ is a basis for $\mathbb{C}^d$. Generic tuples A satisfy this property, see [AHKM18, Remark 7.5]. Given a matrix-valued free analytic polynomial Q, the set

$$\mathcal{G}_Q = \{X \in M(\mathbb{C})^g : \|Q(X)\| < 1\} \subset M(\mathbb{C})^g$$

is a *free pseudoconvex* set.

Theorem 1.1. *Suppose $E \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^g$. If*

(i) *E is sv-generic and linearly independent;*
(ii) *B is sv-generic and $\mathcal{D}_B$ is bounded;*
(iii) *$p : \mathcal{B}_E \to \mathcal{D}_B$ is bianalytic with $p(0) = 0$ and $p'(0) = I$; and*
(iv) *p is defined on a pseudoconvex domain containing $\mathcal{B}_E$ and $q : \mathcal{D}_B \to \mathcal{B}_E$, the inverse of p, is defined on a pseudoconvex domain containing $\mathcal{D}_B$,*

then there exist $g \times g$ unitary matrices Z and M and a tuple $\Xi \in M_g(\mathbb{C})^g$ such that

(1) *$B = M^*ZEM$;*

(2) *for each* $1 \le j, k \le g$,

$$E_k Z E_j = \sum_s (\Xi_j)_{k,s} E_s; \tag{1.3}$$

(3) *the tuple* B *spans an algebra and*

$$B_k B_j = \sum_s (\Xi_j)_{k,s} B_s;$$

(4) Ξ *is convexotonic and* p *is the corresponding convexotonic map* $p = x(I - \Lambda_\Xi(x))^{-1}$.

Remark 1.2. Several remarks are in order.

(i) A free spectrahedron $\mathcal{D}$ is *sv-generic* if there exists an sv-generic tuple A such that $\mathcal{D} = \mathcal{D}_A$. The article [AHKM18] contains a version of Theorem 1.1 for bianalytic mappings between sv-generic free spectrahedra (actually a weaker, but more complicated to formulate, condition from [AHKM18] that we call eig-generic would also suffice here). The sv-generic free spectrahedra are in fact generic among free spectrahedra in the sense of algebraic geometry. However, spectraballs, within the class of free spectrahedra, are *never* sv-generic in view of Lemma 4.1. Hence, Theorem 1.1 extends Theorem [AHKM18, Theorem 1.8], to the important special case of maps from spectraballs to free spectrahedra.

(ii) Let

$$F_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad F_2 = F_1^2 = \begin{pmatrix} 0 & E_2 \\ 0 & 0 \end{pmatrix}, \tag{1.4}$$

where the tuple E is given in equation (4.1). The tuple F is nilpotent. Thus, by Lemma 4.1, it is not sv-generic and the results of [AHKM18] do not apply to bianalytic maps $r : \mathcal{D}_F \to \mathcal{D}_B$ with $r(0) = 0$ and $r'(0) = I$. Moreover, $\mathcal{D}_F$ is not a spectraball by Proposition 4.3 and thus Theorem 1.1 does not directly apply either. However, as we show, there is an sv-generic tuple E and a bianalytic map $p : \mathcal{B}_E \to \mathcal{D}_F$ with $p(0) = 0$ and $p'(0) = I$ (see Proposition 4.3). On the other hand, Theorem 1.1 does apply to bianalytic maps $f : \mathcal{B}_E \to \mathcal{D}_B$. By composing f with p^{-1}, Propositions 4.4 and 4.6 classify the choices for B and all bianalytic maps between $\mathcal{D}_F$ and $\mathcal{D}_B$. In particular, these maps are convexotonic.

(iii) It is easy to check that item (1) implies $\mathcal{B}_E = \mathcal{B}_B$.

(iv) Since E is assumed linearly independent Ξ is uniquely determined by equation (1.3). Further, by Proposition 2.1, Ξ is convexotonic.

(v) Items (2) and (3) are equivalent.

(vi) Note that, while p is only assumed to be bianalytic, the conclusion is that p is birational, a phenomena encountered frequently in rigidity theory in several complex variables, cf. [For93].

(vii) A key ingredient in the proof of Theorem 1.1 is a suitable Positivstellensätze. Namely, p maps $\mathcal{B}_E$ into $\mathcal{D}_B$ if and only if $L_B(p(X)) \succeq 0$ for all $X \in \mathcal{B}_E$, and this equivalence feeds naturally into Positivstellensätze, a pillar of real algebraic geometry. The one used here (from [AHKM18]) is related to that of [AM14], which was developed in full generality in [BMV18].

(viii) An easy argument shows, for $A \in M_d(\mathbb{C})^g$, if $\mathcal{D}_A$ is bounded, then A (really $\{A_1, \ldots, A_g\}$) is linearly independent [HKM13, Proposition 2.6(2)]. The converse fails in general; e.g., if each A_j is positive semidefinite. On the other hand, $E \in M_d(\mathbb{C})^g$ is linearly independent if and only if $\mathcal{B}_E$ is bounded [HKM13, Proposition 2.6(1)].

There is a natural converse to Theorem 1.1. Let $\operatorname{int}(\mathcal{D}_A)$ and $\operatorname{int}(\mathcal{B}_A)$ denote the interiors of $\mathcal{D}_A$ and $\mathcal{B}_A$ respectively. Recall a mapping between metric spaces is *proper* if the inverse image of compact sets are compact. Thus, for open sets $\mathcal{U} \subset M(\mathbb{C})^g$ and $\mathcal{V} \subset M(\mathbb{C})^h$, a free mapping $f : \mathcal{U} \to \mathcal{V}$ is proper if each $f_n : \mathcal{U}_n \to \mathcal{V}_n$ is proper.

Proposition 1.3. *Suppose $J \in M_d(\mathbb{C})^g$ is linearly independent, spans an algebra, Ξ is the resulting convexotonic tuple,*

$$J_k J_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} J_s,$$

and q is the convexotonic (birational) map,

$$q(x) = x(I + \Lambda_\Xi(x))^{-1}.$$

Then

(1) *The domain of q contains $\mathcal{D}_J$.*
(2) *q is a bianalytic map between $\operatorname{int}(\mathcal{D}_J)$ and $\operatorname{int}(\mathcal{B}_J)$; that is p, the (convexotonic) inverse of q, maps $\operatorname{int}(\mathcal{B}_J)$ into $\operatorname{int}(\mathcal{D}_J)$. In particular, q is proper.*
(3) *q maps the boundary of $\mathcal{D}_J$ into the boundary of $\mathcal{B}_J$;*
(4) *if, in addition, $\mathcal{D}_J$ is bounded, then q is a bianalytic map between $\mathcal{D}_J$ and $\mathcal{B}_J$. In particular, the domain of p contains $\mathcal{B}_J$.*

In case J does not span an algebra, we have the following corollary of Proposition 1.3.

Corollary 1.4. *Let $A \in M_d(\mathbb{C})^g$ and assume A is linearly independent (e.g. $\mathcal{D}_A$ is bounded). Let $\mathcal{A}$ denote the algebra spanned by the tuple A. If $C_1, \ldots, C_h \in M_d(\mathbb{C})$ and the tuple $J = (J_1, \ldots, J_{g+h}) = (A_1, \ldots, A_g, C_1, \ldots, C_h)$ is linearly independent and spans $\mathcal{A}$, then there is a rational map f with $f(0) = 0$ and $f'(0) = I$ such that*

(1) *f is an injective proper map from $\operatorname{int}(\mathcal{D}_A)$ into $\operatorname{int}(\mathcal{B}_A)$; and*

(2) *f maps the boundary of $\mathcal{D}_A$ into boundary of $\mathcal{B}_A$.*

Further, the tuple $\Xi \in M_{g+h}(\mathbb{C})^{g+h}$, uniquely determined by

$$J_k J_j = \sum_{s=1}^{h} (\Xi_j)_{k,s} J_s,$$

is convexotonic and

$$f(x) = \begin{pmatrix} x_1 & \cdots & x_g & 0 & \cdots & 0 \end{pmatrix} \left(I + \sum_{j=1}^{g} \Xi_j x_j \right)^{-1}.$$

For further results, not already cited, on free bianalytic and proper free analytic maps see [HKMS09, HKM11a, HKM11b, Pop10, KŠ17, MS08] and the references therein.

The remainder of the article is organized as follows. Proposition 1.3 and Corollary 1.4 are established in Section 2. Theorem 1.1 is proved in Section 3. The article concludes with several examples; see Section 4.

2. Proof of Proposition 1.3

This section gives the proof of Proposition 1.3. Implicit in the statement of that result, and used in the proof of Theorem 1.1, is the connection between finite dimensional algebras and convexotonic tuples described in the following proposition.

Proposition 2.1. *Suppose $G \in M_{d\times e}(\mathbb{C})^g$ and $\{G_1, \dots, G_g\}$ is linearly independent, $C \in M_{e\times d}(\mathbb{C})$ and $\Psi \in M_g(\mathbb{C})^g$. If*

$$G_\ell C G_j = \sum_{s=1}^{g} (\Psi_j)_{\ell,s} G_s, \tag{2.1}$$

then the tuple Ψ is convexotonic. In particular, if $J \in M_d(\mathbb{C})^g$ is linearly independent and spans an algebra, then the tuple Ψ uniquely determined by equation (2.1) is convexotonic.

Proof. For notational ease let $T = CG \in M_e(\mathbb{C})^g$. The hypothesis implies T spans an algebra (but not that T is linearly independent). Routine calculations give

$$[G_\ell T_j]T_k = \sum_{t=1}^{g} (\Psi_j)_{\ell,t} G_t\, T_k = \sum_{s,t=1} (\Psi_j)_{\ell,t} (\Psi_k)_{t,s} G_s = \sum_s (\Psi_j\, \Psi_k)_{\ell,s} G_s.$$

On the other hand

$$G_\ell [T_j T_k] = G_\ell C [G_j T_k] = \sum_t G_\ell (\Psi_k)_{j,t} T_t = \sum_{s,t} (\Psi_t)_{\ell,s} (\Psi_k)_{j,t} G_s.$$

By independence of G,

$$(\Psi_j \Psi_k)_{\ell,s} = \sum_t (\Psi_k)_{j,t} (\Psi_t)_{\ell,s}$$

and therefore

$$\Psi_j \Psi_k = \sum_t (\Psi_k)_{j,t} \Psi_t$$

and the proof is complete. $\square$

Lemma 2.2. *Suppose $F \in M_d(\mathbb{C})^g$. If $I + \Lambda_F(X) + \Lambda_F(X)^* \succeq 0$, then $I + \Lambda_F(X)$ is invertible.*

Proof. Arguing the contrapositive, suppose $I + \Lambda_F(X)$ is not invertible. In this case there is a unit vector γ such that

$$\Lambda_F(X)\gamma = -\gamma.$$

Hence,

$$\langle (I + \Lambda_F(X) + \Lambda_F(X)^*)\gamma, \gamma\rangle = \langle \Lambda_F(X)^*\gamma, \gamma\rangle = \langle \gamma, \Lambda_F(X)\gamma\rangle = -1. \qquad \square$$

Lemma 2.3. *Let $T \in M_d(\mathbb{C})$. Then*

(a) *$I + T + T^* \succeq 0$ if and only if $I + T$ is invertible and $\|(I+T)^{-1}T\| \le 1$;*
(b) *$I + T + T^* \succ 0$ if and only if $I + T$ is invertible and $\|(I+T)^{-1}T\| < 1$.*

Similarly if $I - T$ is invertible, then $\|T\| \le 1$ if and only if $I + R + R^* \succ 0$, where $R = T(I-T)^{-1}$.

Proof. (a) We have the following chain of equivalences:

$$\begin{aligned}
\|(I+T)^{-1}T\| \le 1 \quad &\iff \quad I - \big((I+T)^{-1}T\big)\big((I+T)^{-1}T\big)^* \succeq 0 \\
&\iff \quad I - (I+T)^{-1}TT^*(I+T)^{-*} \succeq 0 \\
&\iff \quad (I+T)(I+T)^* - TT^* \succeq 0 \\
&\iff \quad I + T + T^* \succeq 0.
\end{aligned}$$

The proof of (b) is the same. $\square$

Proposition 2.4. *For $F \in M_d(\mathbb{C})^g$ we have*

$$\mathcal{D}_F = \{X \colon \|(1 + \Lambda_F(X))^{-1}\Lambda_F(X)\| \le 1\}.$$

Proof. Immediate from Lemma 2.3. $\square$

Proof of Proposition 1.3. Let q denote the convexotonic map associated to the convexotonic tuple Ξ in the statement of the proposition,

$$q(x) = \begin{pmatrix} x_1 & \cdots & x_g \end{pmatrix} (I + \Lambda_\Xi(x))^{-1} = x\,(I + \Lambda_\Xi(x))^{-1}.$$

Compute

$$\begin{aligned}\Lambda_J(q(x))\,\Lambda_J(x) &= \sum_{s,k=1}^{g} q^s(x)x_kJ_sJ_k = \sum_{j=1}^{g}\sum_{s=1}^{g} q^s(x)\left[\sum_{k=1}^{g} x_k(\Xi_k)_{s,j}\right]J_j\\ &= \sum_{j=1}^{g}\sum_{s=1}^{g} q^s(x)(\Lambda_\Xi(x))_{s,j}J_j = \sum_{j=1}^{g}\sum_{t=1}^{g} x_t\left[\sum_{s=1}^{g}(I+\Lambda_\Xi(x))^{-1}_{t,s}(\Lambda_\Xi(x))_{s,j}\right]J_j\\ &= \sum_{j=1}^{g}\sum_{t=1}^{g} x_t[(I+\Lambda_\Xi(x))^{-1}\Lambda_\Xi(x)]_{t,j}J_j.\end{aligned}$$

Hence,

$$\Lambda_J(q(x))\,(I+\Lambda_J(x)) = \sum_{j=1}^{g}\sum_{t=1}^{g} x_t[(I+\Lambda_\Xi(x))^{-1}(I+\Lambda_\Xi(x))]_{t,j}J_j = \Lambda_J(x).$$

Thus, as free (matrix-valued) rational functions regular at 0,

$$\Lambda_J(q(x)) = (I+\Lambda_J(x))^{-1}\,\Lambda_J(x) =: F(x). \tag{2.2}$$

Since J is linearly independent, given $1 \le k \le g$, there is a linear functional λ such that $\lambda(J_j) = 0$ for $j \ne k$ and $\lambda(J_k) = 1$. Applying λ to equation (2.2), gives

$$q^k(x) = \lambda(F(x)). \tag{2.3}$$

Since $\lambda(F(x))$ is a free rational function whose domain contains

$$\mathscr{D} = \{X : I + \Lambda_J(X) \text{ is invertible}\},$$

the same is true for q^k. (As a technical matter, each side of equation (2.3) is a rational expression. Since they are defined and agree on a neighborhood of 0, they determine the same free rational function. It is the domain of this rational function that contains $\mathscr{D}$. See [Vol17], and also [KVV09], for full details.) By Lemma 2.2, $\mathscr{D}$ contains $\mathcal{D}_J$ (as $X \in \mathcal{D}_J$ implies $I + \Lambda_J(X)$ is invertible). Hence the domain of the free rational mapping q contains $\mathcal{D}_J$. By Lemma 2.3 and equation (2.2), q maps the interior of $\mathcal{D}_J$ into the interior of $\mathcal{B}_J$ and the boundary of $\mathcal{D}_J$ into the boundary of $\mathcal{B}_J$.

Similarly,

$$(I-\Lambda_J(x))^{-1}\,\Lambda_J(x) = \Lambda_J(p(x)), \tag{2.4}$$

where $p(x) = x(I-\Lambda_\Xi(x))^{-1}$. Arguing as above shows the domain of p contains the set

$$\mathscr{E} = \{X : I - \Lambda_J(X) \text{ is invertible}\},$$

which in turn contains $\operatorname{int}(\mathcal{B}_J)$ (since $\|\Lambda_J(X)\| < 1$ allows for an application of Lemma 2.3). By Lemma 2.3 and equation (2.4), p maps the interior of $\mathcal{B}_J$ into the interior of

$\mathcal{D}_J$. Hence q is bianalytic between these interiors. Further, if X is in the boundary of $\mathcal{B}_J$, then for $t \in \mathbb{C}$ and $|t| < 1$, we have $p(tX) \in \operatorname{int}(\mathcal{D}_J)$ and

$$\Lambda_J(p(tX)) = (I - \Lambda_J(tX))^{-1}\, \Lambda_J(tX).$$

Assuming $\mathcal{D}_J$ is bounded, it follows that $I - \Lambda_J(X)$ is invertible and thus X is in the domain of p and $p(X)$ is in the boundary of $\mathcal{D}_J$. □

Proof of Corollary 1.4. Letting $z = (z_1, \dots, z_{g+h})$ denote a $g+h$ tuple of freely noncommuting indeterminants, and Ξ the convexotonic $g+h$ tuple as described in the corollary, by Proposition 1.3 the birational mapping

$$q(z) = z(I + \Lambda_\Xi(z))^{-1}$$

is a bianalytic (hence injective and proper) mapping between $\operatorname{int}(\mathcal{D}_J)$ and $\operatorname{int}(\mathcal{B}_J)$ that also maps boundary to boundary. The mapping $\iota : \mathcal{D}_A \to \mathcal{D}_J$ defined by $\iota(x) = (x, 0)$ is proper from $\operatorname{int}(\mathcal{D}_A)$ to $\operatorname{int}(\mathcal{D}_J)$ and maps boundary to boundary. Hence, the composition

$$r(x) = p(\iota(x)) = \begin{pmatrix} x & 0 \end{pmatrix}\, (I - \Lambda_\Xi(x, 0))^{-1}$$

is a proper map from $\operatorname{int}(\mathcal{D}_A)$ into $\operatorname{int}(\mathcal{B}_J)$ that also maps boundary to boundary. □

3. Proof of Theorem 1.1

Given E, let

$$A = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

Thus $\mathcal{B}_E = \mathcal{D}_A$ and, among other things, by assumption there is a bianalytic map $p : \mathcal{D}_A \to \mathcal{D}_B$. It follows by the analytic Positivstellensätze [AHKM18, Theorem 1.9] applied to the matrix-valued free analytic function

$$G(z) = \Lambda_A(p(z))$$

that there exists a Hilbert space H, an isometry $\widetilde{C}$ on the range of $I_H \otimes A$ and an isometry $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^{2d}$ such that, with $\widetilde{R} = (\widetilde{C} - I)I_H \otimes A$,

$$L_B(p(x)) = I + G(z)^* + G(z) = \mathscr{W}^*(I - \Lambda_{\widetilde{R}}(x))^{-*} L_{I_H \otimes A}(x)(I - \Lambda_{\widetilde{R}}(x))^{-1}\mathscr{W}. \tag{3.1}$$

That the analytic Positivstellensätze requires $\mathcal{D}_A$ to be bounded and G to extend analytically to a pseudoconvex set containing $\mathcal{D}_A$ explains the need for the hypotheses that $\mathcal{D}_A = \mathcal{B}_E$ is bounded (equivalently E is linearly independent) and p extends analytically to a pseudoconvex set containing $\mathcal{D}_A$.

Since E is sv-generic, both $\ker(E) := \cap \ker(E_j) = \{0\}$ and $\ker(E^*) = \{0\}$. In particular,

$$\operatorname{rg}(A) := \operatorname{span}(\bigcup_{j=1}^{g} \operatorname{rg}(A_j)) = \mathbb{C}^d \oplus \{0\}.$$

In particular, $\dim(\operatorname{rg}(A)) = d$. Likewise $\dim(\operatorname{rg}(A^*)) = d$ too.

The next step involves a call to [AHKM18, Lemma 7.7]. That lemma is stated in terms of conditions referred to as eig-generic, weakly eig-generic, $*$-generic and weakly $*$-generic formally defined in [AHKM18, Definition 7.3]. It is readily seen that if a g-tuple F of $N \times N$ matrices is sv-generic, then it is both eig-generic and $*$-generic (and thus weakly eig-generic and weakly $*$-generic). In particular, $\operatorname{rg}(F) = \mathbb{C}^N = \operatorname{rg}(F^*)$. Thus, both E and B are both eig-generic and $*$-generic. That E is eig-generic implies A is weakly eig-generic; and that E is $*$-generic implies A is weakly $*$-generic.

By [AHKM18, Lemma 7.7(1)], $d = \dim(\operatorname{rg}(A^*)) \le \dim(\operatorname{rg}(B^*)) = e$. Applying [AHKM18, Lemma 7.7(1)] to $q : \mathcal{D}_B \to \mathcal{D}_A$ (so reversing the roles of A and B), it also follows that $e \le d$. Hence $\dim(\operatorname{rg}(A^*)) = d = e = \dim(\operatorname{rg}(B^*))$ and $\dim(\operatorname{rg}(A)) = d = e = \dim(\operatorname{rg}(B))$. Thus we may now invoke (the weakly version of) [AHKM18, Lemma 7.7(4)] that says there is a vector $\lambda \in H$ and a unitary $M : \operatorname{rg}(B^*) \to \operatorname{rg}(A^*)$ and an isometry $N : \operatorname{rg}(B^*) \cap \operatorname{rg}(B) \to \operatorname{rg}(A)$ such that $\mathscr{W} v = \lambda \otimes \iota M v$ for $v \in \operatorname{rg}(B^*)$ and $\widetilde{C}(\lambda \otimes \iota N v) = \lambda \otimes \iota M v$ for $v \in \operatorname{rg}(B^*) \cap \operatorname{rg}(B)$ (where we over use ι, letting it denote the inclusions $\operatorname{rg}(A^*) \subset \mathbb{C}^{2d}$ and $\operatorname{rg}(B^*) \subset \mathbb{C}^{2d}$). This general statement in our case specializes, because $\operatorname{rg}(B^*) = \mathbb{C}^d$, $\operatorname{rg}(B) = \mathbb{C}^d$ and $\dim(\operatorname{rg}(A)) = d$, to give

(i) $M : \mathbb{C}^d \to \operatorname{rg}(A^*)$ is unitary;
(ii) $N : \mathbb{C}^d \to \operatorname{rg}(A)$ is unitary;
(iii) $\mathscr{W} v = \lambda \otimes \iota M v$; and
(iv) $\widetilde{C}(\lambda \otimes \iota N v) = \lambda \otimes \iota M v$ for $v \in \mathbb{C}^d$.

It follows that there is a unitary mapping $Z : \operatorname{rg}(A) \to \operatorname{rg}(A^*)$ such that, for $w \in \operatorname{rg}(A)$,

$$\widetilde{C}(\lambda \otimes w) = \lambda \otimes \iota Z w.$$

Let $[\lambda] = \mathbb{C}\lambda$, the one-dimensional subspace of H spanned by the unit vector λ.

Let

$$C = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix}.$$

In particular C is isometric on the range of A. Let $R = (C - I)A$. For $1 \le j \le d$, and $\gamma \in \mathbb{C}^{2d}$,

$$\begin{aligned} \widetilde{R}_j(\lambda \otimes \gamma) &= (\widetilde{C} - I)[I_H \otimes A_j](\lambda \otimes \gamma) \\ &= (\widetilde{C} - I)(\lambda \otimes A_j \gamma) \\ &= \lambda \otimes (\iota Z - I) A_j \gamma \\ &= \lambda \otimes (C - I) A_j \gamma \\ &= \lambda \otimes R_j \gamma. \end{aligned}$$

Thus $[\lambda]\otimes\mathbb{C}^{2d}$ is invariant for the tuple $\widetilde{R}$ and further

$$\widetilde{R}_j(\lambda\otimes I)=\lambda\otimes R_j.$$

It follows that $[\lambda]\otimes\mathbb{C}^{2d}$ is invariant for the mapping $(I-\Lambda_{\widetilde{R}}(x))^{-1}$ and moreover,

$$(I-\Lambda_{\widetilde{R}}(x))^{-1}(\lambda\otimes I)=\lambda\otimes(I-\Lambda_R(x))^{-1}.$$

Finally, since $\mathscr{W}$ maps into $[\lambda]\otimes\mathbb{C}^{2d}$ and $\mathscr{W}\gamma=\lambda\otimes\iota M\gamma$,

$$W(x):=(I-\Lambda_{\widetilde{R}}(x))^{-1}\mathscr{W}=\lambda\otimes(I-\Lambda_R(x))^{-1}\iota M.$$

Since also $[\lambda]\otimes\mathbb{C}^{2d}$ is invariant for $L_{I_H\otimes A}(x)$,

$$L_{I_H\otimes A}(x)W(x)=\lambda\otimes L_A(x)(I-\Lambda_R(x))^{-1}\iota M.$$

Returning to equation (3.1) and using $\lambda^*\lambda=1$,

$$\begin{aligned} L_B(p(x))&=(\lambda^*\otimes(\iota M)^*(I-\Lambda_R(x))^{-*}(\lambda\otimes L_A(x)(I-\Lambda_R(x))^{-1}\iota M\\ &=M^*\iota^*(I-\Lambda_R(x))^{-*}L_A(x)(I-\Lambda_R(x))^{-1}\iota M.\end{aligned}\tag{3.2}$$

Comparing the coefficients of the x_j terms in equation (3.2) gives

$$B=M^*\iota^*CA\iota M.$$

Since $M:\mathbb{C}^d\to\operatorname{rg}(A^*)$ is unitary,

$$\iota M=\begin{pmatrix}0\\U\end{pmatrix}:\mathbb{C}^d\to\mathbb{C}^{2d}=\mathbb{C}^d\oplus\mathbb{C}^d=\operatorname{rg}(A)\oplus\operatorname{rg}(A^*)\tag{3.3}$$

for a unitary mapping $U:\mathbb{C}^d\to\mathbb{C}^d$. Thus,

$$B=U^*ZEU.$$

Since

$$R=(C-I)A=\begin{pmatrix}0&-E\\0&ZE\end{pmatrix}.$$

it follows that

$$F(x):=(I-\Lambda_R(x))^{-1}=\begin{pmatrix}I&-\Lambda_E(x)(I-\Lambda_{ZE}(x))^{-1}\\0&(I-\Lambda_{ZE}(x))^{-1}\end{pmatrix}.$$

Consequently,

$$\begin{aligned}&F(x)^*L_A(x)F(x)\\ &=\begin{pmatrix}I&0\\-(I-\Lambda_{ZE}(x))^{-*}\Lambda_E(x)^*&(I-\Lambda_{ZE}(x))^{-*}\end{pmatrix}\begin{pmatrix}I&\Lambda_E(x)\\\Lambda_E(x)^*&I\end{pmatrix}\begin{pmatrix}I&-\Lambda_E(x)(I-\Lambda_{ZE}(x))^{-1}\\0&(I-\Lambda_{ZE}(x))^{-1}\end{pmatrix}\\ &\qquad=\begin{pmatrix}I&0\\0&(I-\Lambda_E(x))^{-*}(I-\Lambda_E(x)^*\Lambda_E(x))(I-\Lambda_E(x))^{-1}\end{pmatrix}.\end{aligned}$$

Hence, from equations (3.2) and (3.3),

$$L_B(p(x)) = U^*(I - \Lambda_E(x))^{-*}(I - \Lambda_E(x)^*\Lambda_E(x))(I - \Lambda_E(x))^{-1}U.$$

Further, letting

$$\widetilde{B} = CA = \begin{pmatrix} 0 & 0 \\ 0 & ZE \end{pmatrix},$$

we have

$$L_{\widetilde{B}}(p(x)) = \mathcal{W}^* F(x)^* L_A(x) F(x) \mathcal{W}$$

where

$$\mathcal{W} = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}.$$

By [AHKM18, Theorem 6.7], p is a convexotonic mapping determined by the (uniquely determined) convexotonic tuple Ξ satisfying

$$A_k(C - I)A_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} A_s.$$

Equivalently,

$$E_k Z E_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} E_s. \tag{3.4}$$

Finally to prove item (3), multiply equation (3.4) by Z on the left and use $B = ZE$ to obtain,

$$B_k B_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} B_s.$$

4. Examples

In this section we take up some examples that motivate Theorem 1.1 and Corollary 1.4. First we show that a spectraball, as a member of the class of free spectrahedra, is never sv-generic.

Lemma 4.1. *Suppose $B \in M_d(\mathbb{C})^g$.*

(a) If B is sv-generic, then $\ker(B) := \cap_{j=1}^{g} \ker(B_j) = \{0\}$.
(b) If B is nilpotent, then $\ker(B) \neq \{0\}$.
(c) If B is nilpotent, then $\mathcal{D} = \mathcal{D}_B$ is not sv-generic.
(d) If $\mathcal{D}$ is a spectraball, then $\mathcal{D}$ is not sv-generic.

Remark 4.2. In fact item (a), and thus items (c) and (d), remain true with sv-generic replaced by eig-generic [AHKM18, Definition 7.3].

Proof. If $\alpha \in \mathbb{C}^g$, $u \in \mathbb{C}^d$ and $[I - \Lambda_B(\alpha)^*\Lambda_B(\alpha)]u = 0$, then $u \in \mathrm{rg}(B^*) = \ker(B)^\perp$. Hence, if B is sv-generic, then there exists a basis $\{u^1, \dots, u^d\}$ of $\mathbb{C}^d$ such that each $u^j \in \mathrm{rg}(B^*)$. Thus $\mathbb{C}^d = \mathrm{rg}(B^*) = \ker(B)^\perp$ and therefore $\ker(B) = \{0\}$.

Now suppose B is nilpotent. Thus there is an N such that if β is a word whose length exceeds N, then $B^\beta = 0$. Hence there is a word α (potentially empty) such that $B^\alpha \neq 0$, but $B_j B^\alpha = 0$ for $1 \le j \le g$. It follows that $\{0\} \neq \mathrm{rg}(B^\alpha) \subset \ker(B)$, proving item (b).

To prove item (c), suppose B is nilpotent and let $\mathcal{D} = \mathcal{D}_B$. Let $M \in M_m(\mathbb{C})^g$ be a *minimal defining tuple* for $\mathcal{D}$, meaning $\mathcal{D} = \mathcal{D}_M$ and if $C \in M_s(\mathbb{C})^g$ and $\mathcal{D} = \mathcal{D}_C$, then $s \ge m$. By [EHKM17, Proposition 2.2], there is a tuple J such that B is unitarily equivalent to $M \oplus J$. Since B is nilpotent, so is M. Hence $\ker(M) \neq \{0\}$ by item (b). Now suppose C is any other tuple so that $\mathcal{D}_C = \mathcal{D}_B$. Another application of [EHKM17, Proposition 2.2] gives a tuple N such that C is unitarily equivalent to $M \oplus N$. Hence $\ker(C) \neq \{0\}$ and by item (a), C is not sv-generic. Thus $\mathcal{D} = \mathcal{D}_B$ is not sv-generic.

Finally suppose $\mathcal{D}$ is a spectraball. Hence there is a positive integer e and tuple $E \in M_e(\mathbb{C})^g$ such that $\mathcal{D} = \mathcal{B}_E$. Since $\mathcal{D} = \mathcal{B}_E = \mathcal{D}_A$, where

$$A = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} \in M_{2e},$$

and A is nilpotent, item (c) implies $\mathcal{D}$ is not sv-generic. $\square$

4.1. **A spectrahedron defined by a nilpotent tuple.** A spectrahedron defined by a nilpotent tuple cannot have sv-generic coefficients according to Lemma 4.1, but we give an example here of how one can overcome this problem by mapping to a spectraball.

Let

$$E_1 = I_2 \qquad \text{and} \qquad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

and let F denote the 2-tuple of 3×3 matrices given in equation (1.4). Note that $(1,1) \in \mathbb{C}^2$ is in $\mathcal{D}_F$, but $-(1,1) \notin \mathcal{D}_F$. Thus $\mathcal{D}_F$ is not rotationally invariant and hence not a spectraball. Hence Theorem 1.1 can not be applied to bianalytic mappings $\varphi : \mathcal{D}_F \to \mathcal{D}_B$. Since F is nilpotent, $\mathcal{D}_F$ is not sv-generic (Lemma 4.1) and therefore Theorem [AHKM18, Theorem 1.8] can not be applied to bianalytic mappings $\varphi : \mathcal{D}_F \to \mathcal{D}_B$ (even assuming B is sv-generic). On the other hand, F does span an algebra and thus Corollary 1.4 applies. A straightforward calculation shows that the origin-preserving birational map $q : \mathcal{D}_F \to \mathcal{B}_F$ of Proposition 1.3 is given by $q(x_1, x_2) = (x_1, x_2 + x_1^2)$. Evidently $\mathcal{B}_F = \mathcal{B}_E$. The following proposition summarizes the discussion above.

Proposition 4.3. *The mapping*

$$q(x_1, x_2) = (x_1, x_2 + x_1^2)$$

is a bianalytic map from $\mathcal{D}_F$ onto $\mathcal{B}_E$. Further, E is sv-generic, but $\mathcal{D}_F$ is neither a spectraball nor sv-generic.

According to Proposition 4.3, to classify bianalytic maps $f : \mathcal{D}_F \to \mathcal{D}_B$ it suffices to determine the bianalytic maps $h : \mathcal{B}_E \to \mathcal{D}_B$. Such maps are the subject of the next subsection.

4.2. **Bianalytic mappings of $\mathcal{B}_E$ to a free spectrahedron $\mathcal{D}_B$.** Theorem 1.1 applies in the case that B is sv-generic or has size 2.

Proposition 4.4. *Suppose $B \in M_e(\mathbb{C})^2$ and either $e = 2$ or B is sv-generic. If $f : \mathcal{B}_E \to \mathcal{D}_B$ is bianalytic, then $e = 2$ and there is a unimodular α and 2×2 unitary M such that $B = \alpha M^* E M$ and further f is the birational map*

$$f(x) = \begin{pmatrix} x_1(1 - \alpha x_1)^{-1} & (1 - \alpha x_1)^{-1} x_2 (1 - \alpha x_1)^{-1} \end{pmatrix}.$$

Remark 4.5. The mapping f is a variant (obtained by the linear change of variable (x_1, x_2) maps to $\alpha(x_1, x_2)$) of those appearing in $g = 2$ type IV algebra (see [AHKM18, Section 8.3] or Subsubsection 4.3.4 below).

Proof of Proposition 4.4. In this case the Z in Theorem 1.1 is a unimodular multiple of the identity. Indeed, by (3.4),

$$Z = E_1 Z E_1 = (\Xi_1)_{1,1} I + (\Xi_1)_{1,2} E_2$$

and since Z is unitary, it follows that $(\Xi_1)_{1,2} = 0$ and $Z = \alpha I$. It is now easy to verify that $\Xi = \alpha E$. Hence the corresponding convexotonic map is

$$\begin{aligned} f(x) &= x(I - \Lambda_\Xi(x))^{-1} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 - \alpha x_1 & -\alpha x_2 \\ 0 & 1 - \alpha x_1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} x_1(1 - \alpha x_1)^{-1} & (1 - \alpha x_1)^{-1} x_2 (1 - \alpha x_1)^{-1} \end{pmatrix}, \end{aligned}$$

as desired. □

Composing the f from Proposition 4.4 with the original $q = (x_1, x_2 + x_1^2)$, the bianalytic map between $\mathcal{D}_F$ and $\mathcal{B}_E$, gives the mapping from the original domain $\mathcal{D}_F$ to $\mathcal{D}_B$,

$$f \circ q = \begin{pmatrix} x_1(1 - \alpha x_1)^{-1} & (1 - \alpha x_1)^{-1} \left[x_2 + x_1^2\right] (1 - \alpha x_1)^{-1} \end{pmatrix}.$$

By [AHKM18, Theorem 1.8], if G, H and K are all sv-generic and $r : \mathcal{D}_G \to \mathcal{D}_H$ and $s : \mathcal{D}_H \to \mathcal{D}_K$ are bianalytic (and extend to be analytic on pseudoconvex domains containing $\mathcal{D}_G$ and $\mathcal{D}_H$ respectively), then r, s and $r \circ s$ are convexotonic. However, generally one does not expect an arbitrary composition of convexotonic maps to be convexotonic. (See [AHKM18, Subsection 8.4].) Thus, it is of interest to note that, even though our F is not sv-generic, the map $f \circ q$ is convexotonic.

Proposition 4.6. *The map $f \circ q$ is convexotonic corresponding to the tuple $\Xi = \big(\alpha I_2 + E_2, \alpha E_2\big)$.*

Proof. Here is an outline of the computation that proves the proposition.

$$
\begin{aligned}
x(I - \Lambda_\Xi(x))^{-1} &= x \begin{pmatrix} 1-\alpha x_1 & -(x_1+\alpha x_2) \\ 0 & 1-\alpha x_1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} (1-\alpha x_1)^{-1} & (1-\alpha x_1)^{-1}(x_1+\alpha x_2)(1-\alpha x_1)^{-1} \\ 0 & (1-\alpha x_1)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} (x_1(1-\alpha x_1)^{-1} & x_1(1-\alpha x_1)^{-1}(x_1+\alpha x_2)(1-\alpha x_1)^{-1} + x_2(1-\alpha x_1)^{-1} \end{pmatrix}.
\end{aligned}
$$

Analyzing the second entry above gives

$$
\begin{aligned}
&x_1(1-\alpha x_1)^{-1}(x_1+\alpha x_2)(1-\alpha x_1)^{-1} + x_2(1-\alpha x_1)^{-1} \\
&\quad = (1-\alpha x_1)^{-1}[x_1^2 + \alpha x_1 x_2 + (1-\alpha x_1)x_2](1-\alpha x_1)^{-1} \\
&\quad = (1-\alpha x_1)^{-1}[x_2 + x_1^2](1-\alpha x_1)^{-1},
\end{aligned}
$$

as desired. ∎

4.3. **Two dimensional algebras with $g = 2$.** In this section we consider, in view of Corollary 1.4, the four indecomposable algebras $\mathcal{A}$ of dimension two. In each case we choose a tuple $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ and compute the resulting convexotonic map $G : \mathcal{D}_\mathcal{R} \to \mathcal{B}_\mathcal{R}$. We adopt the names for these algebras used in [AHKM18].

4.3.1. *$g = 2$ type I algebra.* Let $\mathcal{R} = F$, where F is given by (1.4). In this case we already saw $q(x_1, x_2) = (x_1, x_2 + x_1^2)$. In this case $\mathcal{D}_F$ and $\mathcal{B}_F$ are both bounded. While the tuple F is not sv-generic, the tuple E of equation (4.1) is and moreover $\mathcal{B}_F = \mathcal{B}_E$. Hence Theorem 1.1 does indeed apply (by replacing F by E).

4.3.2. *$g = 2$ type II algebra.* Let

$$
\mathcal{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

We have

$$
(I + \Lambda_\mathcal{R}(x))^{-1}\Lambda_\mathcal{R}(x) = \begin{pmatrix} (1+x_1)^{-1}x_1 & (1+x_1)^{-1}x_2 \\ 0 & 0 \end{pmatrix}.
$$

Hence $q = ((1+x_1)^{-1}x_1 \quad (1+x_1)^{-1}x_2)$ is a birational map from $\operatorname{int}(\mathcal{D}_\mathcal{R})$ to the spectraball $\operatorname{int}(\mathcal{B}_\mathcal{R})$ that also maps the boundary of $\mathcal{D}_\mathcal{R}$ into the boundary of $\mathcal{B}_\mathcal{R}$. On the other hand, if X_1 is skew selfadjoint, then $(X_1, 0) \in \mathcal{D}_{\mathcal{R}}$, so that $\mathcal{D}_{\mathcal{R}}$ is not bounded and, for instance, the tuple

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

is in $\mathcal{B}_{\mathcal{R}}$ but not the range of q. In this example, $\mathcal{R}$ has a (common nontrivial) cokernel and is thus not sv-generic. Hence Theorem 1.1 does not apply.

4.3.3. *$g = 2$ type III algebra.* This case, in which

$$\mathcal{R}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

is very similar to the $g = 2$ type II case.

4.3.4. *$g = 2$ type IV algebra.* Let $\mathcal{R} = E$, where E is defined in equation (4.1), and observe

$$(I + \Lambda_{\mathcal{R}}(x))^{-1} \Lambda_{\mathcal{R}}(x) = \begin{pmatrix} (1+x_1)^{-1}x_1 & (1+x_1)^{-1}x_2(1+x_1)^{-1} \\ 0 & (1+x_1)^{-1}x_1 \end{pmatrix}.$$

In this case,

$$q(x) = \begin{pmatrix} x_1(1+x_1)^{-1} & (1+x_1)^{-1}x_2(1+x_1)^{-1} \end{pmatrix}$$

is bianalytic from $\mathrm{int}(\mathcal{D}_{\mathcal{R}})$ to $\mathrm{int}(\mathcal{B}_E)$ and maps boundary into boundary, but does not map boundary onto boundary. In this case $\mathcal{B}_{\mathcal{R}}$ is bounded and sv-generic and hence Theorem 1.1 does apply (with appropriate assumptions on $\mathcal{D}_B$ and $p : \mathcal{D}_{\mathcal{R}} \to \mathcal{D}_B$).

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