A NOTE ON VALUES OF NONCOMMUTATIVE POLYNOMIALS

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Abstract. We find a class of algebras $\mathcal{A}$ satisfying the following property: for every nontrivial noncommutative polynomial $f(X_1, \ldots, X_n)$, the linear span of all its values $f(a_1, \ldots, a_n)$, $a_i \in \mathcal{A}$, equals $\mathcal{A}$. This class includes the algebras of all bounded and all compact operators on an infinite dimensional Hilbert space.

1. Introduction

Starting with Helton's seminal paper [Hel] there has been considerable interest over the last years in values of noncommuting polynomials on matrix algebras. In one of the papers in this area the second author and Schweighofer [KS] showed that Connes' embedding conjecture is equivalent to a certain algebraic assertion which involves the trace of polynomial values on matrices. This has motivated us [BK] to consider the linear span of values of a noncommutative polynomial f on the matrix algebra $M_d(\mathbb{F})$; here, $\mathbb{F}$ is a field with $\text{char}(\mathbb{F}) = 0$. It turns out [BK, Theorem 4.5] that this span can be either:

(1) $\{0\}$;
(2) the set of all scalar matrices;
(3) the set of all trace zero matrices; or
(4) the whole algebra $M_d(\mathbb{F})$.

From the precise statement of this theorem it also follows that if $2d > \deg f$, then (1) and (2) do not occur and (3) occurs only when f is a sum of commutators.

What to except in infinite dimensional analogues of $M_d(\mathbb{F})$? More specifically, let $\mathcal{H}$ be infinite dimensional Hilbert space, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the algebras of all bounded and compact linear operators on $\mathcal{H}$, respectively. What is the linear span of polynomial values in $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$? A very special (but decisive, as we shall see) case of this question was settled by Halmos [Hal] and Pearcy and Topping [PT] (see also Anderson [And]) a long time ago: every operator in $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$, respectively, is a sum of commutators. That is, the linear span of values of the polynomial $X_1X_2 - X_2X_1$ on $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ is all of $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$, respectively. We will prove that the same is true for *every* nonconstant polynomial. This result will be derived as a corollary of our main theorem which presents a class of algebras with the property that the span of values of "almost" every polynomial is equal to the whole algebra.

2. Results

By $\mathbb{F}\langle \bar{X} \rangle$ we denote the free algebra over a field $\mathbb{F}$ generated by $\bar{X} = \{X_1, X_2, \ldots\}$, i.e., the algebra of all noncommutative polynomials in X_i. Let $f = f(X_1, \ldots, X_n) \in \mathbb{F}\langle \bar{X} \rangle$. We say that f is *homogeneous* in the variable X_i if all monomials of f have

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the same degree in X_i. If this degree is 1, then we say that f is *linear* in X_1. If f is linear in every variable X_i, $1 \le i \le n$, then we say that f is *multilinear*.

Let $\mathcal{A}$ be an algebra over $\mathbb{F}$. By $f(\mathcal{A})$ we denote the set of all values $f(a_1, \ldots, a_n)$ with $a_i \in \mathcal{A}$, $i = 1, \ldots, n$. Recall that $f = f(X_1, \ldots, X_n) \in \mathbb{F}\langle \bar{X} \rangle$ is said to be an *identity* of $\mathcal{A}$ if $f(\mathcal{A}) = \{0\}$. If $f(\mathcal{A})$ is contained in the center of $\mathcal{A}$, but f is not an identity of $\mathcal{A}$, then f is said to be a *central polynomial* of $\mathcal{A}$. By $\operatorname{span} f(\mathcal{A})$ we denote the linear span of $f(\mathcal{A})$. We are interested in the question when does $\operatorname{span} f(\mathcal{A}) = \mathcal{A}$ hold.

For the proof of our main theorem three rather elementary lemmas will be needed. The first and the simplest one is a slightly simplified version of [BK, Lemma 2.2]. Its proof is based on the standard Vandermonde argument.

Lemma 2.1. *Let $\mathcal{V}$ be a vector space over an infinite field $\mathbb{F}$, and let $\mathcal{U}$ be a subspace. Suppose that $c_0, c_1, \ldots, c_n \in \mathcal{V}$ are such that $\sum_{i=0}^{n} \lambda^i c_i \in \mathcal{U}$ for all $\lambda \in \mathbb{F}$. Then each $c_i \in \mathcal{U}$.*

Recall that a vector subspace $\mathcal{L}$ of $\mathcal{A}$ is said to be a *Lie ideal* of $\mathcal{A}$ if $[\ell, a] \in \mathcal{L}$ for all $\ell \in \mathcal{L}$ and $a \in \mathcal{A}$; here, $[u, v] = uv - vu$. For a recent treatise of Lie ideals from an algebraic as well as functional analytic viewpoint we refer the reader to [BKS].

Our second lemma is a special case of [BK, Theorem 2.3].

Lemma 2.2. *Let $\mathcal{A}$ be an algebra over an infinite field $\mathbb{F}$, and let $f \in \mathbb{F}\langle \bar{X} \rangle$. Then* $\operatorname{span} f(\mathcal{A})$ *is a Lie ideal of $\mathcal{A}$.*

Every vector subspace of the center of $\mathcal{A}$ is obviously a Lie ideal of $\mathcal{A}$. Lie ideals that are not contained in the center are called *noncentral*. The third lemma follows from an old result of Herstein [Her, Theorem 1.2].

Lemma 2.3. *Let $\mathcal{S}$ be a simple algebra over a field $\mathbb{F}$ with* $\operatorname{char}(\mathbb{F}) \neq 2$. *If $\mathcal{M}$ is both a noncentral Lie ideal of $\mathcal{S}$ and a subalgebra of $\mathcal{S}$, then $\mathcal{M} = \mathcal{S}$.*

We are now in a position to prove our main result.

Theorem 2.4. *Let $\mathcal{S}$ and $\mathcal{B}$ be algebras over a field $\mathbb{F}$ with* $\operatorname{char}(\mathbb{F}) = 0$, *and let $\mathcal{A} = \mathcal{S} \otimes \mathcal{B}$. Suppose that $\mathcal{S}$ is simple, and suppose that $\mathcal{B}$ satisfies*

- (a) *every element in $\mathcal{B}$ is a sum of commutators; and*
- (b) *for each $n \ge 1$, every element in $\mathcal{B}$ is a linear combination of elements b^n, $b \in \mathcal{B}$.*

If $f \in \mathbb{F}\langle \bar{X} \rangle$ is neither an identity nor a central polynomial of $\mathcal{S}$, then

$$\operatorname{span} f(\mathcal{A}) = \mathcal{A}.$$

(*In case $\mathcal{A}$ is nonunital, only polynomials f with zero constant term are considered.*)

Proof. Let $f = f(X_1, \ldots, X_n)$. Let us write $f = g_i + h_i$ where g_i is a sum of all monomials of f in which X_i appears and h_i is a sum of all monomials of f in which X_i does not appear. Thus, $h_i = h_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ and hence

$$h_i(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)$$

for all $a_i \in \mathcal{A}$. Therefore $\operatorname{span} h_i(\mathcal{A}) \subseteq \operatorname{span} f(\mathcal{A})$, which clearly implies $\operatorname{span} g_i(\mathcal{A}) \subseteq \operatorname{span} f(\mathcal{A})$. At least one of g_i and h_i is neither an identity nor a central polynomial of $\mathcal{S}$. Therefore there is no loss of generality in assuming that either X_i appears in every monomial of f or f does not involve X_i at all. Since f cannot be a constant polynomial and hence it must involve some of the X_i's, we may assume, again without loss of generality, that each monomial of f involves all X_i, $i = 1, \ldots, n$.

Next we claim that there is no loss of generality in assuming that f is homogeneous in X_1. Write $f = f_1 + \ldots + f_m$, where f_i is the sum of all monomials of f

that have degree i in X_1. Note that

$$f(\lambda a_1, a_2, \dots, a_n) = \sum_{i=1}^{m} \lambda^i f_i(a_1, \dots, a_n) \in \operatorname{span} f(\mathcal{A})$$

for all $\lambda \in \mathbb{F}$ and all $a_i \in \mathcal{A}$, so $f_i(a_1, \dots, a_n) \in \operatorname{span} f(\mathcal{A})$ by Lemma 2.1. Thus, $\operatorname{span} f_i(\mathcal{A}) \subseteq \operatorname{span} f(\mathcal{A})$. At least one f_i is neither an identity nor a central polynomial of $\mathcal{S}$. Therefore it suffices to prove the theorem for f_i. This proves our claim.

Let us now show that there is no loss of generality in assuming that f is linear in X_1. If $\deg_{X_1} f > 1$, we apply the multilinearization process to f, i.e., we introduce a new polynomial $\Delta_{1,n+1} f = f'(X_1, \dots, X_n, X_{n+1})$:

$$f' = f\left(X_1 + X_{n+1}, X_2, \dots, X_n\right) - f\left(X_1, X_2, \dots, X_n\right) - f\left(X_{n+1}, X_2, \dots, X_n\right).$$

This reduces the degree in X_1 by one. Clearly, $\operatorname{span} f'(\mathcal{A}) \subseteq \operatorname{span} f(\mathcal{A})$. Observe that f can be retrieved from f' by resubstituting $X_{n+1} \mapsto X_1$, more exactly

$$(2^{\deg_{X_1} f} - 2) f = f'(X_1, \dots, X_n, X_1).$$

Hence f' is not an identity nor a central polynomial of $\mathcal{S}$. Note however that f' is not necessarily homogeneous in X_1, but for all its homogeneous components f'_i we have $\operatorname{span} f'_i(\mathcal{A}) \subseteq \operatorname{span} f'(\mathcal{A})$; one can check this by using Lemma 2.1, like in the previous paragraph. At least one of these components, say f'_j, is not an identity nor a central polynomial of $\mathcal{S}$. Thus we restrict our attention to f'_j. If necessary, we continue applying $\Delta_{1,\_}$, and after a finite number of steps we obtain a polynomial Δf linear in X_1, which is neither an identity nor a central polynomial of $\mathcal{S}$, and satisfies $\operatorname{span} \Delta f(\mathcal{A}) \subseteq \operatorname{span} f(\mathcal{A})$. Hence we may assume f is linear in X_1.

Repeating the same argument with respect to other variables we finally see that without loss of generality we may assume that f is multilinear.

Set $\mathcal{L} = \operatorname{span} f(\mathcal{A})$ and $\mathcal{M} = \{m \in \mathcal{S} \,|\, m \otimes \mathcal{B} \subseteq \mathcal{L}\}$. By Lemma 2.2, $\mathcal{L}$ is a Lie ideal of $\mathcal{A}$. Therefore $[m, s] \otimes b^2 = [m \otimes b, s \otimes b] \in \mathcal{L}$ for all $m \in \mathcal{M}$, $b \in \mathcal{B}$, $s \in \mathcal{S}$. Using (b) it follows that $[m, s] \in \mathcal{M}$. Therefore $\mathcal{M}$ is a Lie ideal of $\mathcal{S}$. Pick $s_1, \dots, s_n \in \mathcal{S}$ such that $s_0 = f(s_1, \dots, s_n)$ does not lie in the center of $\mathcal{S}$. For every $b \in \mathcal{B}$ we have

$$s_0 \otimes b^n = f(s_1 \otimes b, s_2 \otimes b, \dots, s_n \otimes b) \in \mathcal{L}.$$

In view of (b) this yields $s_0 \in \mathcal{M}$. Accordingly, $\mathcal{M}$ is a noncentral Lie ideal of $\mathcal{S}$. Next, given $m \in \mathcal{M}$ and $b, b' \in \mathcal{B}$, we have

$$m^2 \otimes [b, b'] = [m \otimes b, m \otimes b'] \in \mathcal{L}.$$

By (a), this gives $m^2 \in \mathcal{M}$. From

$$m_1 m_2 = \frac{1}{2}([m_1, m_2] + (m_1 + m_2)^2 - m_1^2 - m_2^2)$$

it now follows that $\mathcal{M}$ is a subalgebra of $\mathcal{S}$. Using Lemma 2.3 we now conclude that $\mathcal{M} = \mathcal{S}$, i.e., $\mathcal{A} = \mathcal{S} \otimes \mathcal{B} \subseteq \mathcal{L} \subseteq \mathcal{A}$. ■

From the identity

$$n!\, b = \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \left((b+i)^n - i^n\right)$$

it is immediate that (b) is fulfilled if $\mathcal{B}$ has a unity. In this case the proof can be actually slightly simplified by avoiding use of powers of elements in $\mathcal{B}$. Further, every C^*-algebra $\mathcal{B}$ satisfies (b). Indeed, every element in $\mathcal{B}$ is a linear combination of positive elements, and for positive elements we can define nth roots.

Corollary 2.5. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then*

$$\operatorname{span} f(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$$

for every nonconstant polynomial $f \in \mathbb{C}\langle \bar{X} \rangle$.

Proof. It is well known that there does not exist a nonzero polynomial that is an identity of $M_d(\mathbb{C})$ for every $d \geq 1$, cf. [Row, Lemma 1.4.3]. Therefore there exists $d \geq 1$ such that $[f, X_{n+1}]$ is not an identity of $M_d(\mathbb{C})$. This means that f is neither an identity nor a central polynomial of $M_d(\mathbb{C})$. Since $\mathcal{H}$ is infinite dimensional, we have $\mathcal{B}(\mathcal{H}) \cong M_d(\mathcal{B}(\mathcal{H})) \cong M_d(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H})$. Now we are in a position to use Theorem 2.4. Indeed, $M_d(\mathbb{C})$ is a simple algebra, and the algebra $\mathcal{B}(\mathcal{H})$ satisfies (a) by [Hal], and satisfies (b) since it is unital. ■

Corollary 2.6. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then*

$$\operatorname{span} f(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$$

for every nonzero polynomial $f \in \mathbb{C}\langle \bar{X} \rangle$ with zero constant term.

Proof. The proof is essentially the same as that of Corollary 2.5. The only difference occurs in verifying whether $\mathcal{K}(\mathcal{H})$ satisfies the conditions of Theorem 2.4. For this we just note that (a) is shown in [PT], and (b) follows by the remark preceding the statement of Corollary 2.5. ■

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