PROPER ANALYTIC FREE MAPS

J. WILLIAM HELTON[1], IGOR KLEP[2], AND SCOTT MCCULLOUGH[3]

ABSTRACT. This paper concerns *analytic free maps*. These maps are free analogs of classical analytic functions in several complex variables, and are defined in terms of non-commuting variables amongst which there are no relations - they are *free* variables. Analytic free maps include vector-valued polynomials in free (non-commuting) variables and form a canonical class of mappings from one non-commutative domain $\mathcal{D}$ in say g variables to another non-commutative domain $\tilde{\mathcal{D}}$ in $\tilde{g}$ variables.

As a natural extension of the usual notion, an analytic free map is proper if it maps the boundary of $\mathcal{D}$ into the boundary of $\tilde{\mathcal{D}}$. Assuming that both domains contain 0, we show that if $f : \mathcal{D} \to \tilde{\mathcal{D}}$ is a proper analytic free map, and $f(0) = 0$, then f is one-to-one. Moreover, if also $g = \tilde{g}$, then f is invertible and f^{-1} is also an analytic free map. These conclusions on the map f are the strongest possible without additional assumptions on the domains $\mathcal{D}$ and $\tilde{\mathcal{D}}$.

1. INTRODUCTION

The notion of an analytic, free or non-commutative, map arises naturally in free probability, the study of non-commutative (free) rational functions [BGM, Vo1, Vo2, SV, KVV], and systems theory [HBJP].

In this note rigidity results for such functions paralleling those for their classical commutative counterparts are established. The free setting leads to substantially stronger results. Namely, if f is a proper analytic free map from a non-commutative domain in g variables to another in $\tilde{g}$ variables, then f is injective and $\tilde{g} \geq g$. If in addition $\tilde{g} = g$, then f is onto and has an inverse which is itself a (proper) analytic free map. This injectivity conclusion contrasts markedly to the classical case where a (commutative) *proper* analytic function f from one domain in $\mathbb{C}^g$ to another in $\mathbb{C}^g$, need not be

injective, although it must be onto. For classical theory of some commutative proper analytic maps see [DAn].

The definitions as used in this paper are given in the following section. The main result of the paper is in Section 3. Analytic free analogs of classical (commutative) rigidity theorems is the theme of Section 4. The article concludes with examples in Section 5, all of which involve linear matrix inequalities (LMIs).

2. Free Maps

This section contains the background on non-commutative sets and on *free maps* at the level of generality needed for this paper. As we shall see, free maps which are continuous are also analytic in several senses, a fact which (mostly) justifies the terminology analytic free map in the introduction. Indeed one typically thinks of free maps as being analytic, but in a weak sense.

The discussion borrows heavily from the recent basic work of Voiculescu [Vo1, Vo2] and of Kalyuzhnyi-Verbovetskiĭ and Vinnikov [KVV], see also the references therein. These papers contain a power series approach to free maps and for more on this one can see Popescu [Po1, Po2], or also [HKMS, HKM1].

2.1. **Non-commutative Sets and Domains.** Fix a positive integer g. Given a positive integer n, let $M_n(\mathbb{C})^g$ denote g-tuples of $n \times n$ matrices. Of course, $M_n(\mathbb{C})^g$ is naturally identified with $M_n(\mathbb{C}) \otimes \mathbb{C}^g$.

A sequence $\mathcal{U} = (\mathcal{U}(n))_{n \in \mathbb{N}}$, where $\mathcal{U}(n) \subseteq M_n(\mathbb{C})^g$, is a **non-commutative set** if it is **closed with respect to simultaneous unitary similarity**; i.e., if $X \in \mathcal{U}(n)$ and U is an $n \times n$ unitary matrix, then

$$U^* X U = (U^* X_1 U, \dots, U^* X_g U) \in \mathcal{U}(n);$$

and if it is **closed with respect to direct sums**; i.e., if $X \in \mathcal{U}(n)$ and $Y \in \mathcal{U}(m)$ implies

$$X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathcal{U}(n+m).$$

Non-commutative sets differ from the fully matricial $\mathbb{C}^g$-sets of Voiculescu [Vo1, Section 6] in that the latter are closed with respect to simultaneous similarity, not just simultaneous *unitary* similarity. Remark 2.3 below briefly discusses the significance of this distinction for the results on proper analytic free maps in this paper.

The non-commutative set $\mathcal{U}$ is a **non-commutative domain** if each $\mathcal{U}(n)$ is open and connected. Of course the sequence $M(\mathbb{C})^g = (M_n(\mathbb{C})^g)$ is itself a non-commutative

domain. Given $\varepsilon > 0$, the set $\mathcal{N}_\varepsilon = (\mathcal{N}_\varepsilon(n))$ given by

$$(2.1) \qquad \mathcal{N}_\varepsilon(n) = \left\{ X \in M_n(\mathbb{C})^g : \sum X_j X_j^* < \varepsilon^2 \right\}$$

is a non-commutative domain which we call the **non-commutative ε-neighborhood of 0 in $\mathbb{C}^g$**. The non-commutative set $\mathcal{U}$ is **bounded** if there is a $C \in \mathbb{R}$ such that

$$(2.2) \qquad C^2 - \sum X_j X_j^* \succ 0$$

for every n and $X \in \mathcal{U}(n)$. Equivalently, for some $\lambda \in \mathbb{R}$, we have $\mathcal{U} \subseteq \mathcal{N}_\lambda$. Note that this condition is stronger than asking that each $\mathcal{U}(n)$ is bounded.

Let $\mathbb{C}\langle x_1, \ldots, x_g \rangle$ denote the $\mathbb{C}$-algebra freely generated by g non-commuting letters $x = (x_1, \ldots, x_g)$. Its elements are linear combinations of words in x and are called **polynomials**. Given an $r \times r$ matrix-valued polynomial $p \in M_r(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \ldots, x_g \rangle$ with $p(0) = 0$, let $\mathcal{D}(n)$ denote the connected component of

$$\{ X \in M_n(\mathbb{C})^g : I + p(X) + p(X)^* \succ 0 \}$$

containing the origin. The sequence $\mathcal{D} = (\mathcal{D}(n))$ is a non-commutative domain which is semi-algebraic in nature. Note that $\mathcal{D}$ contains an $\varepsilon > 0$ neighborhood of 0, and that the choice

$$p = \frac{1}{\varepsilon} \begin{pmatrix} & & x_1 \\ 0_{g \times g} & & \vdots \\ & & x_g \\ 0_{1 \times g} & 0_{1 \times 1} \end{pmatrix}$$

gives $\mathcal{D} = \mathcal{N}_\varepsilon$. Further examples of natural non-commutative domains can be generated by considering non-commutative polynomials in both the variables $x = (x_1, \ldots, x_g)$ and their formal adjoints, $x^* = (x_1^*, \ldots, x_g^*)$. The case of domains determined by linear matrix inequalities appears in Section 5.

2.2. Free Mappings.
Let $\mathcal{U}$ denote a non-commutative subset of $M(\mathbb{C})^g$ and let $\tilde{g}$ be a positive integer. A **free map** f from $\mathcal{U}$ into $M(\mathbb{C})^{\tilde{g}}$ is a sequence of functions $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ which **respects intertwining maps**; i.e., if $X \in \mathcal{U}(n)$, $Y \in \mathcal{U}(m)$, $\Gamma : \mathbb{C}^m \to \mathbb{C}^n$, and

$$X\Gamma = (X_1\Gamma, \ldots, X_g\Gamma) = (\Gamma Y_1, \ldots, \Gamma Y_g) = \Gamma Y,$$

then $f[n](X)(I \otimes \Gamma) = (I \otimes \Gamma)f[m](Y)$. Note if $X \in \mathcal{U}(n)$ it is natural to write simply $f(X)$ instead of the more cumbersome $f[n](X)$ and likewise $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$. In a similar fashion, we will often write $f(X)\Gamma = \Gamma f(Y)$, instead of $f[n](X)(I_n \otimes \Gamma) = (I_m \otimes \Gamma)f[m](Y)$.

Remark 2.1. Each $f[n]$ can be represented as

$$f[n] = \begin{pmatrix} f[n]_1 \\ \vdots \\ f[n]_{\tilde{g}} \end{pmatrix}$$

where $f[n]_j : \mathcal{U}(n) \to M_n(\mathbb{C})$. Of course, for each j, the sequence $(f[n]_j)$ is a free map $f_j : \mathcal{U} \to M(\mathbb{C})$ with $f_j[n] = f[n]_j$. In particular, if $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$, $X \in \mathcal{U}(n)$, and $v = \sum e_j \otimes v_j$, then

$$f(X)^* v = \sum f_j(X)^* v_j.$$

Let $\mathcal{U}$ be a given non-commutative subset of $M(\mathbb{C})^g$ and suppose $f = (f[n])$ is a sequence of functions $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$. The sequence f **respects direct sums** if, for each n, m and $X \in \mathcal{U}(n)$ and $Y \in \mathcal{U}(m)$,

$$f(X \oplus Y) = f(X) \oplus f(Y).$$

Similarly, f **respects similarity** if for each n and $X, Y \in \mathcal{U}(n)$ and invertible $n \times n$ matrix S such that $XS = SY$,

$$f(X)S = Sf(Y).$$

The following proposition gives an alternate characterization of free maps.

Proposition 2.2. *Suppose $\mathcal{U}$ is a non-commutative subset of $M(\mathbb{C})^g$. A sequence $f = (f[n])$ of functions $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ is a free map if and only if it respects direct sums and similarity.*

Proof. Observe $f(X)\Gamma = \Gamma f(Y)$ if and only if

$$\begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix} \begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix} \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix}.$$

Thus if f respects direct sums and similarity, then f respects intertwining.

On the other hand, if f respects intertwining then, by choosing Γ to be an appropriate projection, it is easily seen that f respects direct sums too. ∎

Remark 2.3. Let $\mathcal{U}$ be a non-commutative domain in $M(\mathbb{C})^g$ and suppose $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map. If $X \in \mathcal{U}$ is similar to Y with $Y = S^{-1}XS$, then we can define $f(Y) = S^{-1}f(X)S$. In this way f naturally extends to a free map on $\mathcal{H}(\mathcal{U}) \subseteq M(\mathbb{C})^g$ defined by

$$\mathcal{H}(\mathcal{U})(n) = \{Y \in M_n(\mathbb{C})^g : \text{ there is an } X \in \mathcal{U}(n) \text{ such that } Y \text{ is similar to } X\}.$$

Thus if $\mathcal{U}$ is a domain of holomorphy, then $\mathcal{H}(\mathcal{U}) = \mathcal{U}$.

On the other hand, because our results on proper analytic free maps to come depend strongly upon the non-commutative set $\mathcal{U}$ itself, the distinction between non-commutative sets and fully matricial sets as in [Vo1] is important. See also [HM, HKM2].

We close this subsection with the following simple observation.

Proposition 2.4. *If $\mathcal{U}$ is a non-commutative subset of $M(\mathbb{C})^g$ and $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map, then the range of f, equal to the sequence $f(\mathcal{U}) = \big(f(\mathcal{U}(n))\big)$, is itself a non-commutative subset of $M(\mathbb{C})^{\tilde{g}}$.*

2.3. **A Continuous Free Map is Analytic.** Let $\mathcal{U} \subseteq M(\mathbb{C})^g$ be a non-commutative set. A free map $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is **continuous** if each $f[n] : \mathcal{U}(n) \to M_n(\mathbb{C})^{\tilde{g}}$ is continuous. Likewise, if $\mathcal{U}$ is a non-commutative domain, then f is called **analytic** if each $f[n]$ is analytic. This implies the existence of directional derivatives for all directions at each point in the domain, and this is the property we shall use later below.

Proposition 2.5. *Suppose $\mathcal{U}$ is a non-commutative domain in $M(\mathbb{C})^g$.*

(1) *A continuous free map $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is analytic.*
(2) *If $X \in \mathcal{U}(n)$, and $H \in M_n(\mathbb{C})^g$ has sufficiently small norm, then*

$$f \begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & f'(X)[H] \\ 0 & f(X) \end{pmatrix}.$$

The proof invokes the following lemma which also plays an important role in the next subsection.

Lemma 2.6. *Suppose $\mathcal{U} \subseteq M(\mathbb{C})^g$ is a non-commutative set and $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map. Suppose $X \in \mathcal{U}(n)$, $Y \in \mathcal{U}(m)$, and Γ is an $n \times m$ matrix. Let*

$$(2.3) \qquad\qquad C_j = X_j \Gamma - \Gamma Y_j, \quad Z_j = \begin{pmatrix} X_j & C_j \\ 0 & Y_j \end{pmatrix}.$$

If $Z = (Z_1, \ldots, Z_g) \in \mathcal{U}(n+m)$, then

$$(2.4) \qquad\qquad f_j(Z) = \begin{pmatrix} f_j(X) & f_j(X)\Gamma - \Gamma f_j(Y) \\ 0 & f_j(Y) \end{pmatrix}$$

This formula generalizes to larger block matrices.

Proof. With

$$S = \begin{pmatrix} I & \Gamma \\ 0 & I \end{pmatrix}$$

we have

$$\tilde{Z}_j = \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} = SZ_jS^{-1}.$$

Thus, writing $f = (f_1, \ldots, f_{\tilde{g}})^T$ and using the fact that f respects intertwining maps, for each j,

$$f_j(Z) = Sf_j(\tilde{Z})S^{-1} = \begin{pmatrix} f_j(X) & f_j(X)\Gamma - \Gamma f_j(Y) \\ 0 & f_j(Y) \end{pmatrix}. \qquad \blacksquare$$

Proof of Proposition 2.5. Fix n and $X \in \mathcal{U}(n)$. Because $\mathcal{U}(2n)$ is open and $X \oplus X \in \mathcal{U}(2n)$, for every $H \in M_n(\mathbb{C})^g$ of sufficiently small norm the tuple with j-th entry

$$\begin{pmatrix} X_j & H_j \\ 0 & X_j \end{pmatrix}$$

is in $\mathcal{U}(2n)$. Hence, for $z \in \mathbb{C}$ of small modulus, the tuple $Z(z)$ with j-th entry

$$Z_j(z) = \begin{pmatrix} X_j + zH_j & H_j \\ 0 & X_j \end{pmatrix}$$

is in $\mathcal{U}(2n)$. Note that the choice (when $z \neq 0$) of $\Gamma(z) = \frac{1}{z}$, $A = X + zH$ and $B = X$ in Lemma 2.6 gives this $Z(z)$. Hence, by Lemma 2.6,

$$f(Z(z)) = \begin{pmatrix} f(X + zH) & \frac{f(X+zH)-f(X)}{z} \\ 0 & f(X) \end{pmatrix}.$$

Since $Z(z)$ converges as z tends to 0 and $f[2n]$ is assumed continuous, the limit

$$\lim_{z \to 0} \frac{f(X + zH) - f(X)}{z}$$

exists. This proves that f is analytic at X. It also establishes the moreover portion of the proposition. $\blacksquare$

Remark 2.7. Kalyuzhnyi-Verbovetskiĭ and Vinnikov [KVV] are developing general results based on very weak hypotheses with the conclusion that f is (in our language) an analytic free map. Here we will assume continuity whenever expedient.

For perspective we mention power series. It is shown in [Vo2, Section 13] that an analytic free map f has a formal power series expansion in the non-commuting variables, which indeed is a powerful way to think of analytic free maps. Voiculescu also gives elegant formulas for the coefficients of the power series expansion of f in terms of clever evaluations of f. Convergence properties for bounded analytic free maps are studied in [Vo2, Sections 14-16]; see also [Vo2, Section 17] for a bad unbounded example. We do not dwell on this since power series are not essential to this paper.

3. A Proper Free Map is Bianalytic Free

Given non-commutative domains $\mathcal{U}$ and $\mathcal{V}$ in $M(\mathbb{C})^g$ and $M(\mathbb{C})^{\tilde{g}}$ respectively, a free map $f : \mathcal{U} \to \mathcal{V}$ is **proper** if each $f[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is proper in the sense that if $K \subseteq \mathcal{V}(n)$ is compact, then $f^{-1}(K)$ is compact. In other words: for all n, if (z_j) is a sequence in $\mathcal{U}(n)$ and $z_j \to \partial\mathcal{U}(n)$, then $f(z_j) \to \partial\mathcal{V}(n)$. In the case $g = \tilde{g}$ and both f and f^{-1} are (proper) analytic free maps we say f is a **bianalytic** free map. The following theorem is a central result of this paper.

Theorem 3.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be bounded non-commutative domains containing 0 in $M(\mathbb{C})^g$ and $M(\mathbb{C})^{\tilde{g}}$, respectively and suppose $f : \mathcal{U} \to \mathcal{V}$ is a free map.*

(1) *If f is proper, then it is one-to-one, and $f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is a free map.*
(2) *If, for each n and $Z \in M_n(\mathbb{C})^{\tilde{g}}$, the set $f[n]^{-1}(\{Z\})$ has compact closure in $\mathcal{U}$, then f is one-to-one and moreover, $f^{-1} : f(\mathcal{U}) \to \mathcal{U}$ is a free map.*
(3) *If $g = \tilde{g}$ and $f : \mathcal{U} \to \mathcal{V}$ is proper and continuous, then f is bianalytic.*

Corollary 3.2. *Suppose $\mathcal{U}$ and $\mathcal{V}$ are non-commutative domains in $M(\mathbb{C})^g$. If $f : \mathcal{U} \to \mathcal{V}$ is a free map and if each $f[n]$ is bianalytic, then f is a bianalytic free map.*

Proof. Since each $f[n]$ is bianalytic, each $f[n]$ is proper. Thus f is proper. Since also f is a free map, by Theorem 3.1(3) f is a bianalytic free map. ∎

Before proving Theorem 3.1 we establish the following preliminary result which is of independent interest and whose proof uses the full strength of Lemma 2.6.

Proposition 3.3. *Let $\mathcal{U} \subseteq M(\mathbb{C})^g$ be a non-commutative domain and suppose $f : \mathcal{U} \to M(\mathbb{C})^{\tilde{g}}$ is a free map. Suppose further that $X \in \mathcal{U}(n)$, $Y \in \mathcal{U}(m)$, Γ is an $n \times m$ matrix, and*

$$f(X)\Gamma = \Gamma f(Y).$$

If $f^{-1}\big(\{f(X) \oplus f(Y)\}\big)$ has compact closure in $\mathcal{U}$, then $X\Gamma = \Gamma Y$.

Proof. As in Lemma 2.6, let $C_j = X_j\Gamma - \Gamma Y_j$. For $0 < t$ sufficiently small, $Z(t) \in \mathcal{U}(n+m)$, where

$$(3.1) \qquad Z_j(t) = \begin{pmatrix} X_j & tC_j \\ 0 & Y_j \end{pmatrix}.$$

If $f(X)\Gamma = \Gamma f(Y)$, then, by Lemma 2.6,

$$f_j(Z(t)) = \begin{pmatrix} f_j(X) & t\big(f_j(X)\Gamma - \Gamma f_j(Y)\big) \\ 0 & f_j(Y) \end{pmatrix} = \begin{pmatrix} f_j(X) & 0 \\ 0 & f_j(Y) \end{pmatrix}.$$

Thus, $f_j(Z(t)) = f_j(Z(0))$. In particular,

$$f^{-1}\big(\{f(Z(0))\}\big) \supseteq \{Z(t) : t \in \mathbb{C}\} \cap \mathcal{U}.$$

Since this set has, by assumption, compact closure in $\mathcal{U}$, it follows that $C = 0$; i.e., $X\Gamma = \Gamma Y$. ∎

We are now ready to prove that a proper free map is one-to-one and even a bianalytic free map if continuous and mapping between domains of the same dimension.

Proof of Theorem 3.1. If f is proper, then $f^{-1}(\{Z\})$ has compact closure in $\mathcal{U}$ for every $Z \in M(\mathbb{C})^{\tilde{g}}$. Hence (1) is a consequence of (2).

For (2), invoke Proposition 3.3 with $\Gamma = \gamma I$ to conclude that f is injective. Thus $f : \mathcal{U} \to f(\mathcal{U})$ is a bijection from one non-commutative set to another. Given $W, Z \in f(\mathcal{U})$ there exists $X, Y \in \mathcal{U}$ such that $f(X) = W$ and $f(Y) = Z$. If moreover, $W\Gamma = \Gamma Z$, then $f(X)\Gamma = \Gamma f(Y)$ and Proposition 3.3 implies $X\Gamma = \Gamma Y$; i.e., $f^{-1}(W)\Gamma = \Gamma f^{-1}(Z)$. Hence f^{-1} is itself a free map.

Let us now consider (3). Using the continuity hypothesis and Proposition 2.5, for each n, the map $f[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is analytic. By hypothesis each $f[n]$ is also proper and hence its range is $\mathcal{V}(n)$ by [Rud, Theorem 15.1.5].

Now $f[n] : \mathcal{U}(n) \to \mathcal{V}(n)$ is one-to-one, onto and analytic, so its inverse is analytic. Further, by the already proved part of the theorem, f^{-1} is an analytic free map. ∎

For both completeness and later use we record the following companion to Lemma 2.6.

Proposition 3.4. *Let $\mathcal{U} \subseteq M(\mathbb{C})^g$ and $\mathcal{V} \subseteq M(\mathbb{C})^{\tilde{g}}$ be non-commutative domains. If $f : \mathcal{U} \to \mathcal{V}$ is a proper analytic free map and if $X \in \mathcal{U}(n)$, then $f'(X) : M_n(\mathbb{C})^g \to M_n(\mathbb{C})^{\tilde{g}}$ is one-to-one. In particular, if $g = \tilde{g}$, then $f'(X)$ is a vector space isomorphism.*

Proof. Suppose $f'(X)[H] = 0$. We scale H so that $\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} \in \mathcal{U}$. From Proposition 2.5

$$f\begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X) & f'(X)[H] \\ 0 & f(X) \end{pmatrix} = \begin{pmatrix} f(X) & 0 \\ 0 & f(X) \end{pmatrix} = f\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}.$$

By the injectivity of f established in Theorem 3.1, $H = 0$. ∎

3.1. **The Main Result is Sharp.** Key to the proof of Theorem 3.1 is testing f on the special class of matrices of the form (3.1). One naturally asks if the hypotheses of the theorem in fact yield stronger conclusions, say by plugging in richer classes of test matrices. The answer to this question is no: suppose f is any analytic free map from g to g variables defined on a neighborhood $\mathcal{N}_\epsilon$ of 0 with $f(0) = 0$ and $f[1]'(0)$ invertible. Under mild additional assumptions (e.g. the lowest eigenvalue of $f'(X)$ or the norm $\|f'(X)\|$ is bounded away from 0 for $X \in \mathcal{N}_\epsilon(n)$ independently of the size n) then there are non-commutative domains $\mathcal{U}$ and $\mathcal{V}$ with $f : \mathcal{U} \to \mathcal{V}$ meeting the hypotheses of the theorem.

Indeed, consider (for fixed n) the analytic function $f[n]$ on $\mathcal{N}_\epsilon(n)$. Its derivative at 0 is invertible; in fact, $f[n]'(0)$ is unitarily equivalent to $I_n \otimes f[1]'(0)$, cf. Lemma 4.2 below. By the implicit function theorem, there is a small δ-neighborhood of 0 on which $f[n]^{-1}$ is defined and analytic. By our assumptions and the bounds on the size of this neighborhood given in [Wan], $\delta > 0$ may be chosen to be independent of n. This gives rise to a non-commutative domain $\mathcal{V}$ and the analytic free map $f^{-1} : \mathcal{V} \to \mathcal{U}$, where $\mathcal{U} = f^{-1}(\mathcal{V})$. Note $\mathcal{U}$ is open (and hence a non-commutative domain) since $f^{-1}(n)$ is analytic and one-to-one. It is now clear that $f : \mathcal{U} \to \mathcal{V}$ satisfies the hypotheses of Theorem 3.1.

We just saw that absent more conditions on the non-commutative domains $\mathcal{D}$ and $\tilde{\mathcal{D}}$, nothing beyond bianalytic free can be concluded about f. The authors, for reasons not gone into here, are particularly interested in convex domains, the paradigm being those given by what are called LMIs. These will be discussed in Section 5. Whether or not convexity of the domain or range of an analytic free f has a highly restrictive impact on f is a serious open question.

4. Several Analogs to Classical Theorems

The conclusion of Theorem 3.1 is sufficiently strong that most would say that it does not have a classical analog. In this section analytic free map analogs of classical several complex variable theorems are obtained by combining the corresponding classical theorem and Theorem 3.1. Indeed, hypotheses for these analytic free map results are weaker than their classical analogs would suggest.

4.1. **A Free Caratheodory-Cartan-Kaup-Wu (CCKW) Theorem.** The commutative Caratheodory-Cartan-Kaup-Wu (CCKW) Theorem [Kr, Theorem 11.3.1] says that if f is an analytic self-map of a bounded domain in $\mathbb{C}^g$ which fixes a point P, then the eigenvalues of $f'(P)$ have modulus at most one. Conversely, if the eigenvalues

all have modulus one, then f is in fact an automorphism; and further if $f'(P) = I$, then f is the identity. The CCKW Theorem together with Corollary 3.2 yields Corollary 4.1 below. We note that Theorem 3.1 can also be thought of as a non-commutative CCKW theorem in that it concludes, like the CCKW Theorem does, that a map f is bianalytic, but under the (rather different) assumption that f is proper.

Corollary 4.1. *Let $\mathcal{D}$ be a given bounded non-commutative domain which contains 0. Suppose $f : \mathcal{D} \to \mathcal{D}$ is an analytic free map. Let ϕ denote the mapping $f[1] : \mathcal{D}(1) \to \mathcal{D}(1)$ and assume $\phi(0) = 0$.*

(1) If all the eigenvalues of $\phi'(0)$ have modulus one, then f is a bianalytic free map; and
(2) if $\phi'(0) = I$, then f is the identity.

The proof uses the following lemma, whose proof is trivial if it is assumed that f is continuous (and hence analytic) and then one works with the formal power series representation for a free analytic function.

Lemma 4.2. *Keep the notation and hypothesis of Corollary 4.1. If n is a positive integer and Φ denotes the mapping $f[n] : \mathcal{D}(n) \to \mathcal{D}(n)$, then $\Phi'(0)$ is unitarily equivalent to $I_n \otimes \phi'(0)$.*

Proof. Let $E_{i,j}$ denote the matrix units for $M_n(\mathbb{C})$. Fix $h \in \mathbb{C}^g$. Arguing as in the proof of Proposition 3.4 gives, for $k \neq \ell$ and $z \in \mathbb{C}$ of small modulus,

$$\Phi((E_{k,k} + E_{k,\ell}) \otimes zh) = (E_{k,k} + E_{k,\ell}) \otimes \phi(zh).$$

It follows that

$$\Phi'(0)[(E_{k,k} + E_{k,\ell}) \otimes h] = (E_{k,k} + E_{k,\ell})\phi'(0)[h].$$

On the other hand,

$$\Phi'(0)[E_{k,k} \otimes h] = E_{k,k} \otimes \phi'(0)[h].$$

By linearity of $\Phi'(0)$, it follows that

$$\Phi'(0)[E_{k,\ell} \otimes h] = E_{k,\ell} \otimes \phi'(0)[h].$$

Thus, $\Phi'(0)$ is unitarily equivalent to $I_n \otimes \phi'(0)$.   ∎

Proof of Corollary 4.1. The hypothesis that $\phi'(0)$ has eigenvalues of modulus one, implies, by Lemma 4.2, that, for each n, the eigenvalues of $f[n]'(0)$ all have modulus one. Thus, by the CCKW Theorem, each $f[n]$ is an automorphism. Now Corollary 3.2 implies f is a bianalytic free map.

Similarly, if $\phi'(0) = I_g$, then $f[n]'(0) = I_{ng}$ for each n. Hence, by the CCKW Theorem, $f[n]$ is the identity for every n and therefore f is itself the identity.   ∎

Note a classical bianalytic function f is completely determined by its value and differential at a point (cf. a remark after Theorem 11.3.1 in [Kr]). Much the same is true for analytic free maps and for the same reason.

Proposition 4.3. *Suppose* $\mathcal{U}, \mathcal{V} \subseteq M(\mathbb{C})^g$ *are non-commutative domains,* $\mathcal{U}$ *is bounded, both contain* 0, *and* $f, g : \mathcal{U} \to \mathcal{V}$ *are proper analytic free maps. If* $f(0) = g(0)$ *and* $f'(0) = g'(0)$, *then* $f = g$.

Proof. By Theorem 3.1 both f and g are bianalytic free maps. Thus $h = f \circ g^{-1} : \mathcal{U} \to \mathcal{U}$ is a bianalytic free map fixing 0 with $h[1]'(0) = I$. Thus, by Corollary 4.1, h is the identity. Consequently $f = g$. ∎

4.2. **Circular Domains.** A subset S of a complex vector space is **circular** if $\exp(it)s \in S$ whenever $s \in S$ and $t \in \mathbb{R}$. A non-commutative domain $\mathcal{U}$ is circular if each $\mathcal{U}(n)$ is circular.

Compare the following theorem to its commutative counterpart [Kr, Theorem 11.1.2] where the domains $\mathcal{U}$ and $\mathcal{V}$ are the same.

Theorem 4.4. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be bounded non-commutative domains in* $M(\mathbb{C})^g$ *and* $M(\mathbb{C})^{\tilde{g}}$, *respectively, both of which contain* 0. *Suppose* $f : \mathcal{U} \to \mathcal{V}$ *is a proper analytic free map with* $f(0) = 0$. *If* $\mathcal{U}$ *and the range* $\mathcal{R} := f(\mathcal{U})$ *of* f *are circular, then* f *is linear.*

The domain $\mathcal{U} = (\mathcal{U}(n))$ is **convex** if each $\mathcal{U}(n)$ is a convex set.

Corollary 4.5. *Let* $\mathcal{U}$ *and* $\mathcal{V}$ *be bounded non-commutative domains in* $M(\mathbb{C})^g$ *both of which contain* 0. *Suppose* $f : \mathcal{U} \to \mathcal{V}$ *is a proper analytic free map with* $f(0) = 0$. *If both* $\mathcal{U}$ *and* $\mathcal{V}$ *are circular and if one is convex, then so is the other.*

This corollary is an immediate consequence of Theorem 4.4 and the fact (see Theorem 3.1(3)) that f is onto $\mathcal{V}$.

We admit the hypothesis that the range $\mathcal{R} = f(\mathcal{U})$ of f in Theorem 4.4 is circular seems pretty contrived when the domains $\mathcal{U}$ and $\mathcal{V}$ have a different number of variables. On the other hand if they have the same number of variables it is the same as $\mathcal{V}$ being circular since by Theorem 3.1, f is onto.

Proof of Theorem 4.4. Because f is a proper free map it is injective and its inverse (defined on $\mathcal{R}$) is a free map by Theorem 3.1. Moreover, using the analyticity of f, its derivative is pointwise injective by Proposition 3.4. It follows that each $f[n] : \mathcal{U}(n) \to$

$M_n(\mathbb{C})^{\tilde{g}}$ is an embedding [GP, p. 17]. Thus, each $f[n]$ is a homeomorphism onto its range and its inverse $f[n]^{-1} = f^{-1}[n]$ is continuous.

Define $F : \mathcal{U} \to \mathcal{U}$ by

(4.1)
$$F(x) := f^{-1}\big(e^{-i\theta} f(e^{i\theta} x)\big)$$

This function respects direct sums and similarities, since it is the composition of maps which do. Moreover, it is continuous by the discussion above. Thus F is an analytic free map.

Using the relation $e^{i\theta} f(F(x)) = f(e^{i\theta})$ we find $e^{i\theta} f'(F(0))F'(0) = f'(0)$. Since $f'(0)$ is injective, $e^{i\theta} F'(0) = I$. It follows from Corollary 4.1(2) that $F(x) = e^{i\theta} x$ and thus, by (4.1), $f(e^{i\theta} x) = e^{i\theta} f(x)$. Since this holds for every θ, it follows that f is linear. ∎

If f is not assumed to map 0 to 0 (but instead fixes some other point), then a proper self-map need not be linear. This follows from the example we discuss in Section 5.2.

Remark 4.6. A consequence of the Kaup-Upmeier series of papers [BKU, KU] shows that given two bianalytically equivalent bounded circular domains in $\mathbb{C}^g$, there is a *linear* bianalytic map between them. We believe this result extends to the present non-commutative setting.

5. Maps in One Variable, Examples

This section contains two examples. The first shows that the circled hypothesis is needed in Theorem 4.4. Our second example concerns $\mathcal{D}$, a non-commutative domain in one variable containing the origin, and $b : \mathcal{D} \to \mathcal{D}$ a proper analytic free map with $b(0) = 0$. It follows that b is bianalytic and hence $b[1]'(0)$ has modulus one. Our second example shows that this setting can force further restrictions on $b[1]'(0)$. The non-commutative domains of both examples are LMI domains; i.e., they are the non-commutative solution set of a linear matrix inequality (LMI). Such domains are convex, and play a major role in the important area of semidefinite programming; see [WSV] or the excellent survey [Nem].

5.1. **LMI Domains.** A special case of the non-commutative domains are those described by a linear matrix inequality. Given a positive integer d and $A_1, \ldots, A_g \in M_d(\mathbb{C})$, the linear matrix-valued polynomial

$$L(x) = \sum A_j x_j \in M_d(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \ldots, x_g \rangle$$

is a **truly linear pencil**. Its adjoint is, by definition, $L(x)^* = \sum A_j^* x_j^*$. Let

$$\mathcal{L}(x) = I_d + L(x) + L(x)^*.$$

If $X \in M_n(\mathbb{C})^g$, then $\mathcal{L}(X)$ is defined by the canonical substitution,

$$\mathcal{L}(X) = I_d \otimes I_n + \sum A_j \otimes X_j + \sum A_j^* \otimes X_j^*,$$

and yields a symmetric $dn \times dn$ matrix. The inequality $\mathcal{L}(X) \succ 0$ for tuples $X \in M(\mathbb{C})^g$ is a **linear matrix inequality (LMI)**. The sequence of solution sets $\mathcal{D}_\mathcal{L}$ defined by

$$\mathcal{D}_\mathcal{L}(n) = \{X \in M_n(\mathbb{C})^g : \mathcal{L}(X) \succ 0\}$$

is a non-commutative domain which contains a neighborhood of 0. It is called a **non-commutative (NC) LMI domain**.

5.2. A Concrete Example of a Nonlinear Bianalytic Self-map on an NC LMI Domain.
It is surprisingly difficulty to find proper self-maps on LMI domains which are not linear. This section contains the only such example, up to trivial modification, of which we are aware. Of course, by Theorem 4.4 the underlying domain can not be circular.

In this example the domain is a one-variable LMI domain. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $\mathcal{L}$ denote the univariate 2×2 linear pencil,

$$\mathcal{L}(x) := I + Ax + A^*x^* = \begin{pmatrix} 1 + x + x^* & x \\ x^* & 1 \end{pmatrix}.$$

Then

$$\mathcal{D}_\mathcal{L} = \{X \mid \|X - 1\| < \sqrt{2}\}.$$

To see this note $\mathcal{L}(X) \succ 0$ if and only if $1 + X + X^* - XX^* \succ 0$, which is in turn equivalent to $(1 - X)(1 - X)^* \prec 2$.

Proposition 5.1. *For real θ, consider*

$$f_\theta(x) := \frac{e^{i\theta} x}{1 + x - e^{i\theta} x}.$$

(1) *$f_\theta : \mathcal{D}_\mathcal{L} \to \mathcal{D}_\mathcal{L}$ is a proper analytic free map, $f_\theta(0) = 0$, and $f'_\theta(0) = \exp(i\theta)$.*
(2) *Every proper analytic free map $f : \mathcal{D}_\mathcal{L} \to \mathcal{D}_\mathcal{L}$ fixing the origin equals one of the f_θ.*

Proof. Item (1) follows from a straightforward computation:

$$(1 - f_\theta(X))(1 - f_\theta(X))^* \prec 2 \iff \left(1 - \frac{e^{i\theta}X}{1 + X - e^{i\theta}X}\right)\left(1 - \frac{e^{i\theta}X}{1 + X - e^{i\theta}X}\right)^* \prec 2$$

$$\iff \left(\frac{1 + X - 2e^{i\theta}X}{1 + X - e^{i\theta}X}\right)\left(\frac{1 + X - 2e^{i\theta}X}{1 + X - e^{i\theta}X}\right)^* \prec 2$$

$$\iff \left(1 + X - 2e^{i\theta}X\right)\left(1 + X - 2e^{i\theta}X\right)^* \prec 2\left(1 + X - e^{i\theta}X\right)\left(1 + X - e^{i\theta}X\right)^*$$

$$\iff 1 + X + X^* - XX^* \succ 0 \iff (1 - X)(1 - X)^* \prec 2.$$

Statement (2) follows from the uniqueness of a bianalytic map carrying 0 to 0 with a prescribed derivative. ■

5.3. Example of Nonexistence of a Bianalytic Self-map on an NC LMI Domain.

Recall that a bianalytic f with $f(0) = 0$ is completely determined by its differential at a point. Clearly, when $f'(0) = 1$, then $f(x) = x$. Does a proper analytic free self-map exist for each $f'(0)$ of modulus one? In the previous example this was the case. For the domain in the example in this subsection, again in one variable, there is no proper analytic free self-map whose derivative at the origin is i.

The domain will be a "non-commutative ellipse" described as $\mathcal{D}_\mathcal{L}$ with $\mathcal{L}(x) := I + Ax + A^*x^*$ for A of the form

$$A := \begin{pmatrix} C_1 & C_2 \\ 0 & -C_1 \end{pmatrix},$$

where $C_1, C_2 \in \mathbb{R}$. There is a choice of parameters in $\mathcal{L}$ such that there is no proper analytic free self-map b on $\mathcal{D}_\mathcal{L}$ with $b(0) = 0$, and $b'(0) = i$.

Suppose $b : \mathcal{D}_\mathcal{L} \to \mathcal{D}_\mathcal{L}$ is a proper analytic free self-map with $b(0) = 0$, and $b'(0) = i$. By Theorem 3.1, b is bianalytic. In particular, $b[1] : \mathcal{D}_\mathcal{L}(1) \to \mathcal{D}_\mathcal{L}(1)$ is bianalytic. By the Riemann mapping theorem there is a conformal map f of the unit disk onto $\mathcal{D}_\mathcal{L}(1)$ satisfying $f(0) = 0$. Then

$$(5.1) \qquad\qquad b[1](z) = f\big(if^{-1}(z)\big).$$

(Note that $b[1] \circ b[1] \circ b[1] \circ b[1]$ is the identity.)

To give an explicit example, we recall some special functions involving elliptic integrals. Let $K(z, t)$ and $K(t)$ be the normal and complete elliptic integrals of the first kind, respectively, that is,

$$K(z, t) = \int_0^z \frac{dx}{\sqrt{(1 - x^2)(1 - t^2 x^2)}}, \qquad K(t) = K(1, t).$$

Furthermore, let

$$\mu(t) = \frac{\pi}{2} \frac{K(\sqrt{1 - t^2})}{K(t)}.$$

Choose the axis for the non-commutative ellipse as follows:

$$a = \cosh\left(\frac{1}{2}\mu(\tfrac{2}{3})\right), \quad b = \sinh\left(\frac{1}{2}\mu(\tfrac{2}{3})\right).$$

Then

$$C_1 = \frac{1}{2}\sqrt{\frac{1}{a^2} - \frac{1}{b^2}}, \quad C_2 = \frac{1}{b}.$$

The desired conformal mapping is [Scw, Sze]

$$f(z) = \sin\left(\frac{\pi}{2K(\frac{2}{3})}K\left(\frac{z}{\sqrt{\frac{2}{3}}}, \frac{2}{3}\right)\right).$$

Hence $b[1]$ in (5.1) can be explicitly computed (for details see the Mathematica notebook `Example53.nb` available under *Preprints* on `http://srag.fmf.uni-lj.si`). It has a power series expansion

$$(5.2) \qquad b[1](z) = iz - \frac{1}{27}i\left(9 - \frac{52K\left(\frac{4}{9}\right)^2}{\pi^2}\right)z^3 + i\frac{\left(9\pi^2 - 52K\left(\frac{4}{9}\right)^2\right)^2}{486\pi^4}z^5 + O(z^7)$$

$$\approx i\left(1 + 0.30572z^3 + 0.140197z^5\right).$$

This power series expansion has a radius of convergence $\geq \epsilon > 0$ and thus induces an analytic free mapping $\mathcal{N}_\epsilon \to M(\mathbb{C})$. By analytic continuation, this function coincides with b. This enables us to evaluate $b(zN)$ for a nilpotent N.

Let N be an order 3 nilpotent,

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $r \in \mathbb{R}$ satisfies $rN \in \mathcal{D}_\mathcal{L}$ if and only if $-1.00033 \leq r \leq 1.00033 =: r_0$. (This has been computed symbolically in the exact arithmetic using Mathematica, and the bounds given here are just approximations.) However, $b(r_0N) \in \mathcal{D}_\mathcal{L} \setminus \partial\mathcal{D}_\mathcal{L}$ contradicting the properness. (This was established by computing the 8×8 matrix $\mathcal{L}\left(b(r_0N)\right)$ symbolically thus ensuring it is exact. Then we apply a numerical eigenvalue solver to see that it is positive definite with smallest eigenvalue $0.0114903\ldots$.) We conclude that the proper analytic free self-map b does not exist.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO

E-mail address: helton@math.ucsd.edu

UNIVERZA V LJUBLJANI, FAKULTETA ZA MATEMATIKO IN FIZIKO, AND UNIVERZA V MARIBORU, FAKULTETA ZA NARAVOSLOVJE IN MATEMATIKO

E-mail address: igor.klep@fmf.uni-lj.si

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE

E-mail address: sam@math.ufl.edu

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