ON REAL ONE-SIDED IDEALS IN A FREE ALGEBRA

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ABSTRACT. In real algebraic geometry there are several notions of the radical of an ideal I. There is the vanishing radical $\sqrt{I}$ defined as the set of all real polynomials vanishing on the real zero set of I, and the real radical $\sqrt[re]{I}$ defined as the smallest real ideal containing I. (Neither of them is to be confused with the usual radical from commutative algebra.) By the real Nullstellensatz, $\sqrt{I} = \sqrt[re]{I}$. This paper focuses on extensions of these to the free algebra $\mathbb{R}\langle x, x^* \rangle$ of noncommutative real polynomials in $x = (x_1, \ldots, x_g)$ and $x^* = (x_1^*, \ldots, x_g^*)$.

We work with a natural notion of the (noncommutative real) zero set $V(I)$ of a left ideal I in $\mathbb{R}\langle x, x^* \rangle$. The vanishing radical $\sqrt{I}$ of I is the set of all $p \in \mathbb{R}\langle x, x^* \rangle$ which vanish on $V(I)$. The earlier paper [CHMN+] gives an appropriate notion of $\sqrt[re]{I}$ and proves $\sqrt{I} = \sqrt[re]{I}$ when I is a finitely generated left ideal, a free $*-$Nullstellensatz. However, this does not tell us for a particular ideal I whether or not $I = \sqrt[re]{I}$, and that is the topic of this paper. We give a complete solution for monomial ideals and homogeneous principal ideals. We also present the case of principal univariate ideals with a degree two generator and find that it is very messy. We discuss an algorithm to determine if $I = \sqrt[re]{I}$ (implemented under `NCAlgebra`) with finite run times and provable effectiveness.

1. INTRODUCTION

The introduction begins with definitions and a little motivation for them. Then it sketches the main results of this paper together with links to where they are found.

1.1. **Zero Sets in Free Algebras.** Let $\langle x, x^* \rangle$ be the monoid freely generated by $x = (x_1, \ldots, x_g)$ and $x^* = (x_1^*, \ldots, x_g^*)$, i.e., $\langle x, x^* \rangle$ consists of **words** in the $2g$ noncommuting letters $x_1, \ldots, x_g, x_1^*, \ldots, x_g^*$ (including the empty word $\varnothing$ which plays the role of the identity

1). Let $\mathbb{R}\langle x, x^* \rangle$ denote the $\mathbb{R}$-algebra freely generated by x, x^*, i.e., the elements of $\mathbb{R}\langle x, x^* \rangle$ are **polynomials** in the noncommuting variables x, x^* with coefficients in $\mathbb{R}$. Equivalently, $\mathbb{R}\langle x, x^* \rangle$ is the **free $*$-algebra** on x. The length of the longest word in a noncommutative polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is the **degree** of f and is denoted by $\deg(f)$. The set of all words of degree at most k is $\langle x, x^* \rangle_k$, and $\mathbb{R}\langle x, x^* \rangle_k$ is the vector space of all noncommutative polynomials of degree at most k.

Given a g-tuple $X = (X_1, \ldots, X_g)$ of same size square matrices over $\mathbb{R}$, write $p(X)$ for the natural evaluation of p at X. For $S \subseteq \mathbb{R}\langle x, x^* \rangle$ we introduce

$$V(S)^{(n)} = \left\{ (X, v) \in (\mathbb{R}^{n \times n})^g \times \mathbb{R}^n \mid p(X)v = 0 \text{ for every } p \in S \right\},$$

and define the **zero set** of S to be

$$V(S) = \bigcup_{n \in \mathbb{N}} V(S)^{(n)} = \{ (X, v) \mid p(X)v = 0 \text{ for every } p \in S \}.$$

To each subset T of $\bigcup_{n \in \mathbb{N}} \left((\mathbb{R}^{n \times n})^g \times \mathbb{R}^n \right)$ we associate the **left ideal**

$$\mathcal{I}(T) = \{ p \in \mathbb{R}\langle x, x^* \rangle \mid p(X)v = 0 \text{ for every } (X, v) \in T \}.$$

For a left ideal I of $\mathbb{R}\langle x, x^* \rangle$, we call

$$\sqrt{I} := \mathcal{I}\big(V(I)\big)$$

the **vanishing radical** of I.[1] Evidently $\sqrt{I}$ is a left ideal. We say that I is **has the left Nullstellensatz property** if $\sqrt{I} = I$. Now we describe a class of ideals which has this property.

A polynomial p is **analytic** if it has no transpose variables, that is, no x_i^*. For example, $p(x) = 1 + x_1 x_2 + x_1^3 + x_2^5$ is analytic while $p(x) = 1 + x_1^* x_2 + (x_1^*)^3 + x_2^5$ is not analytic. There is a strong Nullstellensatz for left ideals generated by analytic polynomials in [HMP07]. This strengthens an earlier result proved by Bergman [HM04]; see also [BK11] for a survey on noncommutative Nullstellensätze.

Theorem 1.1. *Let $p_1, \ldots, p_m$ be analytic polynomials, and let I be the left ideal generated by those p_i. Then $\sqrt{I} = I$, i.e.,*

$$\big(\forall j \ p_j(X)v = 0\big) \implies q(X)v = 0 \qquad \textit{iff} \qquad q = f_1 p_1 + \cdots + f_m p_m.$$

We pause to make two remarks related to Theorem 1.1. First, no powers q^k are needed, contrary to the case in the classical commutative Hilbert Nullstellensatz or the real Nullstellensatz. This absence of powers seems to be the pattern for free algebra situations.

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[1] In [CHMN+] this radical was denoted $\sqrt[\Pi]{\ }$. Since in this article only radicals with respect to finite dimensional representations are considered, the Π has been dropped.

Secondly, while there are other notions of zero of a free polynomial, the one used here is particularly suited for studying left ideals and has proved fruitful in a variety of other contexts; e.g. [HMP07, HM12]. One alternative notion is to say that X is a zero of p if $p(X) = 0$. However, in this case, for $R \subseteq \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{n \times n})^g$, the set $\{p \mid p(X) = 0 \text{ for all } X \in R\}$ is a two-sided ideal. Another choice is to declare X a zero of p if $p(X)$ fails to be invertible, but then $\{p \mid \det(p(X)) = 0 \text{ for all } X \in R\}$ is not closed under sums.

What about ideals generated by p_j which are not analytic? To shed light on the basic question of which ideals have the left Nullstellensatz property, we seek an algebraic description of the vanishing radical $\sqrt{I}$ similar to the notion of real radical in the classical real algebraic geometry, cf. [BCR98, Chapter 4], [Mar08, Chapter 2], [PD01, Chapter 4] or [Sce09]. For this we introduced real ideals in [CHMN+]. Now we recall these definitions.

A left ideal I of $\mathbb{R}\langle x, x^* \rangle$ is said to be **real** if for every $a_1, \ldots, a_r \in \mathbb{R}\langle x, x^* \rangle$ such that

$$\sum_{i=1}^{r} a_i^* a_i \in I + I^*,$$

we have that $a_1, \ldots, a_r \in I$. Here I^* is the right ideal $I^* = \{a^* \mid a \in I\}$. An intersection of a family of real ideals is a real ideal. For a left ideal J of $\mathbb{R}\langle x, x^* \rangle$ we call the ideal

$$\sqrt[\text{re}]{J} = \bigcap_{\substack{I \supseteq J \\ I \text{ real}}} I = \text{the smallest real ideal containing } J$$

the real radical of J. It is not hard to show for any left ideal I that

$$I \subseteq \sqrt[\text{re}]{I} \subseteq \sqrt{I},$$

see [CHMN+]. The main result of [CHMN+] is a real Nullstellensatz which states:

Theorem 1.2 ([CHMN+, Theorem 1.6]). *A finitely generated left ideal I in $\mathbb{R}\langle x, x^* \rangle$ satisfies $\sqrt[\text{re}]{I} = \sqrt{I}$. Thus I has the left Nullstellensatz property if and only if it is real.*

This result is not true for infinitely generated ideals, as is shown in Example 2.2.

A quantitative version of this theorem gives bounds (which we shall need) on the degrees of the polynomials involved.

Theorem 1.3 ([CHMN+, Theorem 2.5]). *Let I be a left ideal in $\mathbb{R}\langle x, x^* \rangle$ generated by polynomials of degree bounded by d. Then I is real if and only if whenever $q_1, \ldots, q_k$ are polynomials with $\deg(q_j) < d$ for each j, and $\sum_{i=1}^{\ell} q_i^* q_i \in I + I^*$, then $q_j \in I$ for each j.*

These results give a clean equivalence but do not tell us whether or not a particular ideal has the left Nullstellensatz property. This paper focuses on examples of ideals I for which we can determine if $I = \sqrt{I}$.

1.2. **Main Results.** Our goal is to determine which left ideals I have the property $I = \sqrt{I}$. We give a complete solution for principle ideals generated by a degree 1 or degree 2 polynomial. Other results treat more general situations but are less complete; we list them below.

Monomial ideals. A **left monomial ideal** is a left ideal generated by monomials. A word $w \in \langle x, x^* \rangle$ is **left unshrinkable** [Lan97, Tap99] if it cannot be written as $w = uu^*v$ for some $u, v \in \langle x, x^* \rangle$ with $u \neq 1$. We show:

Theorem 1.4. *A left monomial ideal is real if and only if it is generated by left unshrinkable words. Hence, by Theorem 1.2, a finitely generated left monomial ideal is real if and only if it has the left Nullstellensatz property.*

We emphasize that first claim is proved in Section 2 and does not require finite generation. Absent finitely many generators we need something more to prove a version of the second claim, see Example 2.2.

Principal ideals.

Theorem 1.5. *Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is the left ideal generated by a nonzero polynomial p.*

(1) *Suppose p is homogeneous. Then I is real if and only if p is not of the form*

$$(1.1) \qquad\qquad\qquad p = (s + q)f,$$

   where s is a nonconstant sum of squares, $q^* = -q$, and $f \in \mathbb{R}\langle x, x^* \rangle$.
(2) *Even for nonhomogeneous p the condition in (1.1) on p rules out I being real.*
(3) *Suppose $p = a + b^*$, where a and b are analytic polynomials. Then I is not real if and only if $p = a - a^* + c$ (or equivalently, $p = -b + b^* + c$) for some nonzero constant c.*

Proving this theorem is the subject of Section 3.

Analytic homogeneous ideals. Turning to the question of identifying not necessarily finitely generated left ideals with the left Nullstellensatz property, it turns out that an ideal generated by analytic homogeneous polynomials has the left Nullstellensatz property. The formal result is stated as Proposition 2.3.

Algorithms. We give a computer algorithm to determine if a given I is a real ideal, hence if I has the left Nullstellensatz property, see Subsection 3.5. This is a more practical offshoot of an algorithm given in [CHMN+].

   Our classification of univariate real ideals generated by p of degree 2 is done in Section 4 and it illustrates techniques derived in Section 3.

2. Monomial Ideals

In this section we describe what is known about monomial ideals both finitely and infinitely generated. We start with basic facts about monomial ideals.

Lemma 2.1. *For a left ideal $I \subseteq \mathbb{R}\langle x, x^* \rangle$ the following are equivalent:*

(i) *I is a left monomial ideal;*

(ii) *A polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is in I iff all of the monomials appearing in f are in I.*

(iii) *A polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is in $I + I^*$ iff all of the monomials appearing in f are in $I + I^*$.*

(iv) *A polynomial $f \in \mathbb{R}\langle x, x^* \rangle$ is in $I + I^*$ iff all of the monomials appearing in f are in I or I^*.*

Proof. (i) $\Rightarrow$ (ii) is obvious. If (ii) holds, then I is generated by the set of all monomials in I so (i) holds. The equivalence of (i) and (iii) follows similarly. It is clear that (iii) and (iv) are equivalent. ∎

2.1. **Proof of Theorem 1.4.** Suppose I is generated by left unshrinkable words. Our goal is to show that I is a real ideal. (We emphasize that this part of the proof does not require finite generation.)

Assume I is not real and consider a finite sum of hermitian squares

$$s = \sum_j g_j^* g_j$$

such that $s \in I + I^*$ and $g_j \notin I$ for some j. Decompose each g_j as

$$g_j = \sum_{w \in \langle x, x^* \rangle} a_{w,j} w = \sum_{\substack{w \in \langle x, x^* \rangle \\ w \in I}} a_{w,j} w + \sum_{\substack{u \in \langle x, x^* \rangle \\ u \notin I}} a_{u,j} u = p_j + q_j$$

for some $a_{v,k} \in \mathbb{R}$. Here $p_j = \sum_{w \in I} a_{w,j} w \in I$ and $q_j = \sum_{u \notin I} a_{u,j} u$ is either zero (if all of its coefficients are zero) or does not belong to I (it it has a nonzero coefficient). Since

$$s = \sum_j g_j^* g_j = \sum_j (p_j^* + q_j^*)(p_j + q_j) = \sum_j \left(p_j^* p_j + p_j^* q_j + q_j^* p_j + q_j^* q_j \right)$$

belongs to $I + I^*$, it follows that

(2.1) $$\sum_j q_j^* q_j \in I + I^*.$$

Moreover, since $g_j \notin I$ for some j, also $q_j \neq 0$ for some j. We may assume that all q_j in (2.1) are nonzero and hence each nonzero term of each q_j in (2.1) lies outside I. Since the highest degree terms in a sum of squares cannot cancel, the highest degree terms in $\sum_j q_j^* q_j$ are multiples of words of the form $w^* w$ for words $w \notin I$. Lemma 2.1 implies that $w^* w \in I + I^*$

for each such w. Since I is a left monomial ideal, it follows that w^*w is of the form v_1v_2, where v_2 is one of the unshrinkable words which generate I.

If $\deg(w) \geq \deg(v_2)$, then this implies that $w = v'v_2$, where v' is the right-hand piece of v_1, which is a contradiction since $w \notin I$.

If $\deg(w) < \deg(v_2)$, then $v_2 = w'w$, where w' is the left-hand piece of v_2, and $\deg(w') > 0$. Thus $v_1v_2 = v_1w'w = w^*w$, which implies that $w^* = v_1w'$, which implies that $w = (w')^*v_1^*$. Therefore $v_2 = w'(w')^*v_1^*$, which is a contradiction since v_2 is unshrinkable.

Conversely, suppose I is a real left monomial ideal. Let $\mathcal{B}$ be a minimal set of words which generate I. Without loss of generality, we may assume that for each $w \in \mathcal{B}$, there exist no $u, v \in \mathbb{R}\langle x, x^* \rangle$, with $v \in I$, such that $w = uv$. Assume $w \in \mathcal{B}$ is shrinkable. Let $w = u^*uv$, with u some nonconstant word. Thus $v^*u^*uv \in I + I^*$, which implies that $uv \in I$ by I being real. This however contradicts the minimality of $\mathcal{B}$.

The second part of the theorem now follows from Theorem 1.2, and this does require finite generation; cf. Example 2.2. ∎

2.2. Real monomial ideals without the left Nullstellensatz property.

When does a left monomial ideal have the left Nullstellensatz property? Theorem 1.4 gives us a necessary condition, namely that it is generated by left unshrinkable words. However, this condition is not sufficient, see Example 2.2. In the positive direction, some infinitely unshrinkably generated left ideals have the left Nullstellensatz property, see Section 2.3.

We begin with the following basic fact: if I is the intersection of some family of finitely generated real left ideals I_α, then

$$(2.2) \qquad \sqrt{I} \subseteq \bigcap_\alpha \sqrt{I_\alpha} = \bigcap_\alpha I_\alpha = I \subseteq \sqrt{I}.$$

Therefore, to show that a left ideal I satisfies $I = \sqrt{I}$, it suffices to show it is an intersection of finitely generated real left ideals.

Now we give an example of a real monomial ideal on which the real Nullstellensatz fails.

Example 2.2. *If $x = (x_1, x_2)$ and $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is the left ideal generated by the set*

$$\left\{ x_1(x_2^*x_2)^d x_1 \mid d \geq 2 \right\}$$

then I is real but $\sqrt{I} \neq I$.

Proof. If

$$u^*uv = x_1(x_2^*x_2)^d x_1$$

for some words u, v and for some d, then either $\deg(u) = 0$ or u^* starts with x_1. The latter case cannot happen since this would imply that u ends with x_1^* and there is no x_1^* in

$x_1(x_2^* x_2)^d x_1$. Therefore for each d, the monomial $x_1(x_2^* x_2)^d x_1$ is left unshrinkable, which by Theorem 1.4 implies that the left ideal I is real.

Next, by definition

$$\sqrt{I} = \bigcap_{(X,v)\in V(I)} \mathcal{I}\big(\{(X,v)\}\big),$$

so we will show that there is a polynomial $p \notin I$ contained in each left ideal $\mathcal{I}\big(\{(X,v)\}\big)$ with $(X,v) \in V(I)$. Suppose $(X,v) \in V(I)$, where $X = (X_1, X_2)$ is a pair of $n \times n$ matrices and $v \in \mathbb{R}^n$. The matrix $X_2^* X_2$ is positive semidefinite, so we can decompose it as

$$X_2^* X_2 = U^* \Lambda U,$$

where U is unitary and Λ is a diagonal matrix with nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $w = UX_1v$ be a vector with i^{th} entry denoted w_i, and let $Y = X_1 U^*$ be a matrix with (i,j) entry y_{ij}. Then for each $d \geq 2$, we see

$$X_1(X_2^* X_2)^d X_1 v = Y\Lambda^d w = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1^d & 0 & \cdots & 0 \\ 0 & \lambda_2^d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^n y_{1j}\lambda_j^d w_j \\ \sum_{j=1}^n y_{2j}\lambda_j^d w_j \\ \vdots \\ \sum_{j=1}^n y_{nj}\lambda_j^d w_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If $\Lambda = 0$, then $X_1 X_2^* X_2 X_1 v = 0$. Otherwise, let $\xi_1, \ldots \xi_k$ be the distinct nonzero values of $\lambda_1, \ldots, \lambda_n$. For each $d \geq 2$ and each i,

$$\sum_{j=1}^k \left(\sum_{\lambda_\ell = \xi_j} y_{i\ell} w_\ell \right) \xi_j^d = 0.$$

Since the matrix

$$\begin{bmatrix} \xi_1^2 & \xi_2^2 & \cdots & \xi_k^2 \\ \xi_1^3 & \xi_2^3 & \cdots & \xi_k^3 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{k+1} & \xi_2^{k+1} & \cdots & \xi_k^{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_k \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{k-1} & \xi_2^{k-1} & \cdots & \xi_k^{k-1} \end{bmatrix} \begin{bmatrix} \xi_1^2 & 0 & \cdots & 0 \\ 0 & \xi_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_k^2 \end{bmatrix}$$

has nonzero determinant, each $\sum_{\lambda_i = \xi_j} y_{ij} w_j = 0$. Therefore

$$X_1^* X_2^* X_2 X_1 v = Y \Lambda w = \begin{bmatrix} \sum_{j=1}^n y_{1j} \lambda_j w_j \\ \sum_{j=1}^n y_{2j} \lambda_j w_j \\ \vdots \\ \sum_{j=1}^n y_{nj} \lambda_j w_j \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^k \left(\sum_{\lambda_\ell = \xi_j} y_{1\ell} w_\ell \right) \xi_j \\ \sum_{j=1}^k \left(\sum_{\lambda_\ell = \xi_j} y_{2\ell} w_\ell \right) \xi_j \\ \vdots \\ \sum_{j=1}^k \left(\sum_{\lambda_\ell = \xi_j} y_{k\ell} w_\ell \right) \xi_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $x_1 x_2^* x_2 x_1 \in \sqrt{I}$. However, $x_1 x_2^* x_2 x_1 \notin I$ since it is not a multiple of any of the monomial generators of I, because it has degree 4 and the homogeneous generators of I each have degree of at least 6. ∎

Note that what the above implies is that if $I \subseteq J$, where J is a finitely generated real left ideal, then $x_1 x_2^* x_2 x_1 \in J \setminus I$. Therefore, the intersection of all real finitely generated ideals containing I necessarily contains $x_1 x_2^* x_2 x_1$.

2.3. **Several non-finitely generated ideals with the left Nullstellensatz property.** In this section we study non-finitely generated left ideals that satisfy the left Nullstellensatz property.

A bright spot in our understanding are left ideals with analytic monomial generators. Analytic monomials are certainly unshrinkable and as we shall see always generate real ideals. In fact, this works more generally than for just monomials:

Proposition 2.3. *Suppose I is generated by homogeneous analytic polynomials $p_1, \ldots, p_k, \ldots$. Then $\sqrt{I} = I$.*

Proof. For each $d \in \mathbb{N}$, let $I^{(d)}$ be the ideal generated by all polynomials p_i of degree $\leq d$ as well as all analytic words of degree $d + 1$. In this case, each $p_j \in I^{(d)}$, and each $I^{(d)}$ is finitely generated. Further, since $I^{(d)}$ is generated by analytic polynomials, it is real. Therefore

$$\sqrt{I} \subseteq \bigcap_{d=1}^\infty I^{(d)}.$$

At the same time

$$I = \bigcap_{d=1}^\infty I^{(d)},$$

since the elements of $I^{(e)}$ of degree bounded by d, where $e \geq d$, are precisely the elements of I of degree bounded by d, cf. Lemma 2.1. ∎

This compares to [HMP07] which required finite-generation by analytic polynomials (which may not be homogeneous).

In distinction to the analytic case the next monomial ideal I we treat has generators with adjoints. The generators are very simple left unshrinkable monomials but still it is a bit tricky to prove the ideal is real. Indeed we shall write I as the intersection of finitely-generated ideals which are not monomial ideals and which are not even generated by homogeneous polynomials.

Example 2.4. *Let $x = (x_1, x_2)$ and let $I \subseteq \mathbb{R}\langle x, x^* \rangle$ be the left ideal generated by the set*

$$\{x_1(x_2^* x_2)^d \mid d \geq 0\}.$$

Then $\sqrt{I} = I$.

Proof. Let I_λ be the left ideal generated by x_1 and $x_2^* x_2 - \lambda$, where $\lambda > 0$. By the discussion at the beginning of section 2.2, the claim that $\sqrt{I} = I$ follows from Steps 1 and 2 below.

STEP 1: I_λ is real.
Since I_λ is generated by polynomials of degree bounded by 2, we only have to check that no sum of squares of polynomials outside of I of degree 1 or less is in $I_\lambda + I_\lambda^*$. Since $x_1 \in I_\lambda$, we need only consider squares of polynomials in the span of all monomials of degree 1 or less which are not equal to I. Suppose that

$$p = \begin{bmatrix} x_1^* \\ x_2 \\ x_2^* \\ 1 \end{bmatrix}^* A \begin{bmatrix} x_1^* \\ x_2 \\ x_2^* \\ 1 \end{bmatrix} \in (I_\lambda + I_\lambda^*)$$

is a symmetric polynomial, i.e., A is symmetric. It is straightforward to show that p must be in the span of $x_2^* x_2 - \lambda$ and $x_1 + x_1^*$. Therefore,

$$A = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & -\lambda\alpha \end{bmatrix}.$$

If p is a nonzero sum of squares, then $(0,0) \neq (\alpha, \beta) \in \mathbb{R}^2$, which implies that

$$\beta = 0, \; \alpha \neq 0, \quad \text{and} \quad \alpha \begin{bmatrix} 1 & 0 \\ 0 & -\lambda \end{bmatrix} \succeq 0.$$

But this contradicts $\lambda > 0$.

STEP 2: $\bigcap_{\lambda > 0} I_\lambda = I$.
First, consider

$$x_1(x_2^* x_2)^d.$$

If $d = 0$, then $x_1 \in I_\lambda$. Further, we see that

$$x_1(x_2^* x_2)^d = \lambda x_1 (x_2^* x_2)^{d-1} + x_1(x_2^* x_2)^{d-1}(x_2^* x_2 - \lambda),$$

hence by induction $x_1(x_2^* x_2)^d \in I_\lambda$ for all d. Since I_λ contains the generators of I,

$$I \subseteq \bigcap_{\lambda > 0} I_\lambda.$$

Next, suppose $p \in \bigcap_{\lambda > 0} I_\lambda$. First, $p \in I_1$, so p is of the form

$$p = q x_1 + \sum_{i=0}^{d} r_i (x_2^* x_2 - 1)^i,$$

for some d, where $r_i, q \in \mathbb{R}\langle x, x^* \rangle$. Without loss of generality, assume that the terms of the r_i are not divisible on the right by $x_2^* x_2$; for example, if $r_i = \tilde{r}_i x_2^* x_2$ for some $\tilde{r}_i$, then

$$r_i(x_2^* x_2 - 1)^i = \tilde{r}_i(x_2^* x_2)(x_2^* x_2 - 1)^i = \tilde{r}_i(x_2^* x_2 - 1)^{i+1} + \tilde{r}_i(x_2^* x_2 - 1)^i.$$

Given another $\lambda > 0$, $p \in I_\lambda$. We see that

$$x_2^* x_2 - 1 + I_\lambda = \lambda - 1 + I_\lambda,$$

and inductively that

$$(x_2^* x_2 - 1)^i + I_\lambda = (\lambda - 1)^i + I_\lambda.$$

Therefore

$$\sum_{i=0}^{d} r_i (\lambda - 1)^i \in I_\lambda.$$

The leading term of this resulting polynomial must be divisible on the right by x_1 or $x_2^* x_2$ since

$$\{x_1, x_2^* x_2 - \lambda\}$$

is a left Gröbner basis. Since the terms of each r_i are not divisible on the right by $x_2^* x_2$ by construction, the leading term of $\sum_{i=0}^{d} r_i (\lambda - 1)^i$ is divisible on the right by x_1. Since this is true for arbitrary $\lambda > 0$, the leading term of some r_i is divisible on the right by x_1. Let this term be denoted r_i'. Then

$$p - r_i'(x_2^* x_2)^i \in \bigcap_{\lambda > 0} I_\lambda$$

since $r_i'(x_2^* x_2)^i \in I \subseteq \bigcap_{\lambda > 0} I_\lambda$ and is in I if and only if p is. We can reduce p inductively to deduce that $p \in I$. ∎

3. Principal Left Ideals

We now turn our attention to principal (i.e., singly generated) left ideals. Our focus here is on homogeneous generators. The main results in this section are Theorem 3.11 that precisely describes when a principal homogeneous left ideal is real, and the accompanying Algorithm 3.6, producing a practical test for determining whether a given homogeneous polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ generates a real left ideal.

3.1. **Factorization of Homogeneous Polynomials.**

Checking whether a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ factors as a product $p = p_1 p_2$ of polynomials of smaller degree amount to solving a large system of linear equations. For homogeneous noncommutative polynomials factorization can be done more efficiently, and this is what we describe in this subsection.

Lemma 3.1. *Let $V(x) = (v_1, \ldots, v_r)^* \in \mathbb{R}\langle x, x^* \rangle^r$ be a vector of linearly independent homogeneous degree d_r polynomials, and let $W(x) = (w_1, \ldots, w_s)^* \in \mathbb{R}\langle x, x^* \rangle^s$ be a vector of linearly independent homogeneous degree d_s polynomials. For $A \in \mathbb{R}^{r \times s}$, we have*

$$V(x)^* A W(x) = 0 \quad \Longleftrightarrow \quad A = 0.$$

Proof. First consider the case where $V = M_{d_r}$ is a vector whose entries are all the words of degree d_r, and where $W = M_{d_s}$ is a vector whose entries are all the words of degree d_s. Let $A \in \mathbb{R}^{r \times s}$ satisfy $M_r(x)^* A M_s(x) = 0$. Each word w of degree $d_r + d_s$ can be expressed uniquely as a product

$$w = w_r w_s,$$

where w_r is the word consisting of the first d_r letters of w and w_s is the word consisting of the last d_s letters of w. Let $A_{w_r^*, w_s}$ be the entry of A corresponding to w_r^* on the left and w_s on the right. Then,

$$M_r(x)^* A M_s(x) = \sum_{\deg(w_r) = d_r} \sum_{\deg(w_s) = d_s} A_{w_r^*, w_s} w_r w_s = 0.$$

This implies that each $A_{w_r^*, w_s} = 0$, or in other words, that $A = 0$.

Next consider the case where the entries of V span the set of all homogeneous degree d_r polynomials, and where the entries of W span the set of all homogeneous degree d_s polynomials. The elements of V may be expressed uniquely as a linear combination of the elements of M_r. Therefore there exists an invertible matrix R such that

$$RV(x) = M_r(x).$$

Similarly, there exists an invertible matrix S such that

$$SW(x) = M_s(x).$$

Therefore
$$V(x)^* A W(x) = M_r^* (R^{-1})^* A S^{-1} M_s = 0$$
which implies that $(R^{-1})^* A S^{-1} = 0$. Since R and S are invertible, it follows that $A = 0$.

Finally, consider the general case. Let V' be a vector whose entries, together with the entries of V, form a basis for the set of homogeneous degree d_r polynomials, and let W' be a vector whose entries, together with the entries of W, form a basis for the set of homogeneous degree d_s polynomials. Suppose $V(x)^T A W = 0$. Then

$$V^T A W = \begin{bmatrix} V \\ V' \end{bmatrix}^T \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W \\ W' \end{bmatrix} = 0,$$

which implies that $A = 0$ by previously established case. ∎

Remark 3.2. Let $M_{d_1} = (w_1^{(1)}, \ldots, w_k^{(1)})$ be a vector whose entries are all words of length d_1 and $M_{d_2} = (w_1^{(2)}, \ldots, w_\ell^{(2)})$ be a vector whose entries are all words of length d_2. Let p be homogeneous of degree $d_1 + d_2$. Then the unique matrix A such that

$$p = M_{d_1}^* A M_{d_2}$$

is the matrix whose (i,j)-entry is the coefficient of $(w_i^{(1)})^* w_j^{(2)}$ in p. We call it the (d_1, d_2)-**Gram matrix** for p. (Observe that the uniqueness of the Gram matrix fails for nonhomogeneous polynomials.)

The following is obvious and well-known.

Lemma 3.3 (cf. [Har12, KP10]). *Let $p \in \mathbb{R}\langle x, x^* \rangle$ be a homogeneous degree $2d$ polynomial. Then p is a sum of squares if and only if its (d,d)-Gram matrix is positive semidefinite.*

We call a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ **irreducible** if it cannot be written as a product of two polynomials of smaller degree, i.e., if $p = qr$ with $q, r \in \mathbb{R}\langle x, x^* \rangle$, then q is constant or r is constant.

Proposition 3.4 (cf. [Co63]). *Let $p \in \mathbb{R}\langle x, x^* \rangle$ be a noncommutative homogeneous polynomial. Then p can be factored as*

$$p = p_1 \cdots p_k$$

for some irreducible homogeneous polynomials $p_1, \ldots, p_k$. Further, such a representation is unique in that if $p = q_1 \cdots q_\ell$ is another decomposition, where each q_j is irreducible, then $k = \ell$ and there exist nonzero scalars $\lambda_1, \ldots, \lambda_k$ such that $q_i = \lambda_i p_i$ for $i = 1, \ldots, k$.

We refer the reader to [Co06] for a detailed study of factorization in free algebras, and to [GGRW05] for another take on noncommutative factorization.

It is straightforward to compute a factorization of a homogeneous noncommutative polynomial. There is an algorithm described in [Ca+]; see also [KS93]. An alternate way of looking at the algorithm is presented in Algorithm 3.6 below.

Lemma 3.5. *A homogeneous polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ can be factored as $p = p_1 p_2$, for $\deg(p_1) = d_1$ and $\deg(p_2) = d_2$ if and only if the (d_1, d_2)-Gram matrix A for p has rank 1. Indeed, a decomposition $p = p_1 p_2$ is then given by decomposing $A = \alpha_1 \alpha_2^*$ and setting $p_1 = M_{d_1}^* \alpha_1$ and $p_2 = \alpha_2^* M_{d_2}$.*

Proof. Straightforward. ∎

Algorithm 3.6. *Suppose $p \in \mathbb{R}\langle x, x^* \rangle$ is a homogeneous degree d polynomial. If its (d_1, d_2)-Gram matrix is of rank > 1 for all $d_1, d_2 \in \mathbb{N}$ with $d_1 + d_2 = d$, then p is irreducible. Otherwise choose the smallest possible d_1 producing a factorization $p = p_1 \tilde{p}_1$ as in Lemma 3.5. Then p_1 is irreducible. Repeating the procedure on $\tilde{p}_1$ yields a factorization $\tilde{p}_1 = p_2 \cdots p_k$, where each p_j is irreducible. Then $p = p_1 p_2 \cdots p_k$.*

3.2. **Homogeneous Principal Left Ideals.** There is a clean test for determining whether or not a principal left ideal is real. This test does not require homogeneity.

Proposition 3.7. *Let I be the left ideal generated by a* (*not necessarily homogeneous*) *polynomial $p \in \mathbb{R}\langle x, x^* \rangle$. Then I is real if and only if there exists no nonzero sum of squares equal to $qp + p^* q^*$ for a polynomial $q \in \mathbb{R}\langle x, x^* \rangle$ with $\deg(q) < \deg(p)$.*

Proof. The left ideal I is equal to

$$I = \{ qp \mid q \in \mathbb{R}\langle x, x^* \rangle \}.$$

A polynomial qp has $\deg(qp) = \deg(q) + \deg(p)$, which implies that I contains no nonzero polynomials of degree less than $\deg(p)$.

By Theorem 1.3, I is real if and only if there exists no sum of squares of the form

$$(3.1) \qquad\qquad q_1^* q_1 + \cdots + q_k^* q_k \in I + I^*,$$

with $\deg(q_j) < \deg(p)$ for each j, and $q_j \notin I$. Since I is principal, as shown above there are no nonzero $q_j \in I$ with $\deg(q_j) < \deg(p)$. Further, a sum of squares of the form (3.1) has degree equal to $2 \max\{\deg(q_j)\}$. Therefore I is real if and only if there exists no nonzero sum of squares in $I + I^*$ with degree less than $2 \deg(p)$.

[CHMN+, Proposition 2.18] implies that $(I + I^*)_{2d-1} = I_{2d-1} + I_{2d-1}^*$. The set I_{2d-1} is equal to

$$I_{2d-1} = \{ qp \mid \deg(q) < \deg(p) \}$$

since $\deg(qp) = \deg(q) + \deg(p)$. Therefore an element of $(I+I^*)_{2d-1}$ is of the form $q_1p + p^*q_2^*$, with $\deg(q_1), \deg(q_2) < \deg(p)$. Further, if $q_1p + p^*q_2^*$ is symmetric, then

$$q_1p + p^*q_2^* = \frac{1}{2}(q_1p + p^*q_2) + \frac{1}{2}(q_1p + p^*q_2)^* = \left(\frac{1}{2}(q_1 + q_2)\right)p + p^*\left(\frac{1}{2}(q_1 + q_2)\right)^*.$$

Therefore the set of symmetric elements of $(I + I^*)_{2d-1}$ is $\{qp + p^*q^* \mid \deg(q) < \deg(p)\}$. Hence I is real if and only if there exists no nonzero sum of squares of the form $qp + p^*q^*$ with $\deg(q) < \deg(p)$. ∎

Example 3.8. *Suppose $I \subseteq \mathbb{R}\langle x, x^*\rangle$ is the left ideal generated by a homogeneous polynomial p, and p has no terms containing $x_ix_i^*$ or $x_i^*x_i$ for any i. Then I is real.*

Proof. By Proposition 3.7, it suffices to assume that there exists a polynomial q such that $qp + p^*q^*$ is a nonzero sum of squares, with $\deg(q) < \deg(p)$. Then $\deg(qp + p^*q^*) \le \deg(q) + \deg(p) < 2\deg(p)$. A nonzero sum of squares s necessarily contains some terms of the form w^*w, where $2\deg(w) = \deg(s)$. Such a term contains a $x_ix_i^*$ or a $x_i^*x_i$ at the $\deg(s)/2$ through $\deg(s)/2 + 1$ position. However, no term of $qp + p^*q^*$ contains such a hermitian square since p contains no such terms and $2\deg(p) > \deg(qp + p^*q^*)$. ∎

Lemma 3.9. *Let p, q be homogeneous polynomials with $\deg(q) \le \deg(p)$. Then $qp + p^*q^*$ is a sum of squares if and only if $p = rq^*$ for some polynomial r such that $r + r^*$ is a sum of squares.*

Proof. Suppose $\deg(qp + p^*q^*) = 2d$ so that $d \le \deg(p)$. Let $w_1q^*, \ldots, w_kq^*, r_1, \ldots, r_\ell$ be a basis for homogeneous polynomials of degree d so that $w_1, \ldots, w_k$ are all words of length $d - \deg(q)$. Then there exists a unique decomposition of qp as

$$qp = \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}^* \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}$$

for some block matrices A and B. Further,

$$qp + p^*q^* = \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}^* \begin{bmatrix} A + A^* & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} w_1q^* \\ \vdots \\ w_kq^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}.$$

Since $qp + p^*q^*$ is a sum of squares, and by uniqueness of the Gram matrix representation, we have $B = 0$. Therefore

$$
qp = \begin{bmatrix} w_1 q^* \\ \vdots \\ w_k q^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 q^* \\ \vdots \\ w_k q^* \\ r_1 \\ \vdots \\ r_\ell \end{bmatrix} = q \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}^* A \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix} q^*
$$

Hence

$$
r = \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}^* A \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix}
$$

gives $p = rq^*$ and $r + r^*$ is a sum of squares. The converse implication is trivial. ∎

Corollary 3.10. *Let $I \subseteq \mathbb{R}\langle x, x^* \rangle$ be the left ideal generated by a single homogeneous polynomial p. Let $p = p_1 \cdots p_k$ be a factorization of p into irreducible factors. Then I is real if and only if none of the $2k$ polynomials*

$$
\pm(p_1 + p_1^*), \ \pm(p_1 p_2 + p_2^* p_1^*), \ \ldots, \ \pm(p_1 \cdots p_k + p_k^* \cdots p_1^*)
$$

is a nonzero sum of squares.

For the sake of convenience, we call the highest degree homogeneous part of a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ its **leading polynomial**.

Proof. By Proposition 3.7, I is real if and only if there is no nonzero sum of squares of the form $qp + p^*q^*$, where $\deg(q) < \deg(p)$. We claim that we can restrict without loss of generality to q being homogeneous, i.e., $q = q'$. Indeed, if q' is the leading polynomial of q, then either $q'p + p^*(q')^* = 0$ or $q'p + p^*(q')^*$ is the leading polynomial of $qp + p^*q^*$. In the former case,

$$
qp + p^*q^* = (q - q')p + p^*(q - q')^*,
$$

so we can look at the leading polynomial of $q - q'$ and repeat. In the latter case, $q'p + p^*(q')^*$ is a sum of squares since $qp + p^*q^*$ is.

If $s = (-1)^t(p_1 p_2 \cdots p_j + p_j^* \cdots p_2^* p_1^*)$ is a nonzero sum of squares for some $j < k$ and some t, then let $q = (-1)^t p_k^* p_{k-1}^* \cdots p_{j+1}^*$. We have $\deg(q) < \deg(p)$ and

$$
qp + p^*q^* = p_k^* p_{k-1}^* \cdots p_{j+1}^* s p_{j+1} \cdots p_{k-1} p_k
$$

is a nonzero sum of squares in $I + I^*$. If $s = (-1)^t(p + p^*)$ is a nonzero sum of squares for some t, then $q = (-1)^t$ works.

Conversely, suppose $qp + p^*q^*$ is homogeneous of degree $2d$, for some $d < \deg(p)$, and assume it is a nonzero sum of squares. By Lemma 3.9, $p = rq^*$ for some r such that $r + r^*$ is a sum of squares. By the uniqueness of the factorization,

$$r = \lambda p_1 \cdots p_j \quad \text{and} \quad q^* = \frac{1}{\lambda} p_{j+1} \cdots p_k,$$

for some scalar λ and some j, which implies, since $r + r^*$ is a nonzero sum of squares, that

$$(-1)^t (p_1 \cdots p_j + p_j^* \cdots p_1^*)$$

is a nonzero sum of squares for some t.   ∎

Irreducibility of the factors p_j allow a rephrasing of Corollary 3.10 which lends itself to computation.

Theorem 3.11. *Let p be a nonconstant homogeneous polynomial, and let I be the left ideal generated by p. Decompose p as $p = p_1 \cdots p_k$, where the p_i are irreducible and nonconstant. Then I is real if and only if neither of the following two hold:*

(1) *There exists j with $2j \leq k$ such that*

$$p_1 = \lambda_1 p_{2j}^*, \ p_2 = \lambda_2 p_{2j-1}^*, \ \ldots, p_j = \lambda_j p_{j+1}^*$$

   for some scalars λ_i.

(2) *There exists j with $2j + 1 \leq k$ such that*

$$p_1 = \lambda_1 p_{2j+1}^*, \ p_2 = \lambda_2 p_{2j}^*, \ \ldots, \ p_j = \lambda_j p_{j+2}^*$$

   for some scalars λ_i, and $(-1)^t (p_{j+1} + p_{j+1}^*)$ is a nonzero sum of squares for some t.

Remark 3.12. *This theorem implies that to test the conditions of Corollary 3.10 for $p = p_1 \cdots p_k$, one only needs to do the following. For each j with $2j \leq k$, examine whether for each i the polynomials p_i and p_{2j-i}^* are scalar multiples of each other. For each j with $2j + 1 \leq k$, examine whether for each i the polynomials p_i and p_{2j+1-i}^* are scalar multiples of each other, and if they are, check to see if $p_{j+1} + p_{j+1}^*$ is plus or minus a sum of squares. Checking if a homogeneous degree $2d$ noncommutative polynomial p is a sum of squares is straightforward: by Lemma 3.3 we only need to find its (d, d)-Gram matrix (by solving a linear system) and test whether it is positive semidefinite.*

Proof of Theorem 3.11. By Corollary 3.10, I is not real if and only if for some ℓ and t,

$$(-1)^t (p_1 \cdots p_\ell + p_\ell^* \cdots p_1^*)$$

is a nonzero sum of squares. We claim that the latter is true if and only if either item (1) or item (2) of Theorem 3.11 holds. We will prove the nontrivial direction of this claim by induction on ℓ.

If $\ell = 1$, then $(-1)^t(p_1 + p_1^*)$ is a sum of squares for some t and so item (2) holds.

If $\ell > 1$, suppose that $\deg(p_\ell) \leq \frac{1}{2}\deg(p_1 \cdots p_\ell)$. If this were not the case, a similar argument could be made using p_1 noting that $\deg(p_\ell) > \frac{1}{2}\deg(p_1 \ldots p_\ell)$ implies that $\deg(p_1) \leq \frac{1}{2}\deg(p_1 \cdots p_\ell)$. Note that $(-1)^t(p_1 \cdots p_\ell + p_\ell^* \cdots p_1^*)$ is a sum of homogeneous degree $\deg(p_1 \cdots p_\ell)$ polynomials and is a nonzero sum of squares. Therefore

$$\deg(p_1 \cdots p_\ell + p_\ell^* \cdots p_1^*) = \deg(p_1 \cdots p_\ell)$$

and this degree is even. By Lemma 3.9, this implies that p_1^* factors $p_2 \cdots p_\ell$ on the right. Since p_1 is irreducible, by uniqueness of factorization there exists some nonzero λ_1 such that $p_1 = \lambda_1 p_\ell^*$. Now either $\ell = 2$ and so item (1) holds, or $p_2 \cdots p_{\ell-1}$ satisfies

$$(-1)^t(p_2 \cdots p_{\ell-1} + p_{\ell-1}^* \cdots p_2^*)$$

is a sum of squares. The result follows by induction.   ∎

Now we prove Theorem 1.5 (1).

Corollary 3.13. *Let $0 \neq p \in \mathbb{R}\langle x, x^* \rangle$ be homogeneous and let I be the left ideal generated by p. Then I is real if and only if p is not of the form*

(3.2) $$p = (s + q)f,$$

where s is a nonconstant sum of squares, $q^* = -q$, and $f \in \mathbb{R}\langle x, x^* \rangle$.

Proof. Theorem 3.11 implies that I is not real if and only if one of two cases hold. In case (1), p is equal to $p = u^*uv$ for some nonconstant polynomial u. Here $s = u^*u$ and $q = 0$ gives the result. In case (2), p is equal to $p = u^*tuv$, with $t + t^*$ plus or minus a sum of squares. By multiplying v by ± 1, we can assume without loss of generality that $t + t^*$ is a sum of squares. Setting $s = \frac{1}{2}u^*(t + t^*)u$, $q = \frac{1}{2}u^*(t - t^*)u$, and $f = v$ gives the result. The converse implication (if p is of the form (3.2) then I is non-real) is trivial; see Proposition 3.15 below.   ∎

Example 3.14. *Let $w \in \mathbb{R}\langle x, x^* \rangle$. If $w = (s + q)r$, for some nonconstant sum of squares s, some antisymmetric q, and some r, then w is a monomial if and only if $s + q$ and r are monomials, that is, $s = u^*u$ for some monomial $u \in \mathbb{R}\langle x, x^* \rangle$, $q = 0$, and r is a monomial. Therefore, Corollary 3.13 in the monomial case states that the left ideal I generated by a monomial w is real if and only if w is left unshrinkable.*

3.3. General Principal Left Ideals.
Apart from Proposition 3.7 we have to this point considered only homogeneous ideals. Now we consider principal left ideals with a nonhomogeneous generator. Here the results are less definitive than before.

Now we prove Theorem 1.5 (2). If $p \in \mathbb{R}\langle x, x^* \rangle$ is not homogeneous but has a similar structure to that of Corollary 3.13, then the following holds.

Proposition 3.15. *If p is of the form*

$$(3.3) \qquad\qquad p = (s + q)f,$$

where s is a nonzero sum of squares, $q^* = -q$, $f \neq 0$, and $\deg(s + q) > 0$, then the left ideal I generated by p is not real.

Proof. Since $\deg(s + q) > 0$, and $\deg(p) = \deg(f) + \deg(s + q)$, we have $\deg(f) < \deg(p)$. We see that

$$f^*p + p^*f = 2f^*sf \in I + I^*$$

is a nonzero sum of squares. By Proposition 3.7, this implies that I is not real. ■

Example 3.16. *Not every non-real principal left ideal is generated by a polynomial of the form* (3.3). *Consider*

$$p = (x_1 - x_1^* + 1)(x_2 - x_2^*) + 1.$$

One can show p is irreducible. Further,

$$p + p^* = (x_1 - x_1^*)(x_2 - x_2^*) + (x_2 - x_2^*)(x_1 - x_1^*) + 2$$

which is not a $(\pm)$ sum of squares. However,

$$(x_2 - x_2^*)p + p^*(x_2 - x_2^*)^* = 2(x_2 - x_2^*)^*(x_2 - x_2^*)$$

is a sum of squares of elements not in the left ideal generated by p.

Proposition 3.17. *If p is of the form*

$$(3.4) \qquad\qquad p = (s + q_1)q_2 + c,$$

where each q_i is antisymmetric, $q_2 \neq 0$, s is a nonzero sum of squares, c is a constant, and $\deg(s + q_1) > 0$ then the left ideal I generated by p is not real.

Proof. Since $\deg(s + q_1) > 0$, we see $\deg(q_2) < \deg(p)$. We see

$$q_2^*p + p^*q_2 = 2q_2^*sq_2 \in I + I^*$$

is a nonzero sum of squares, which implies that I is not real. ■

However, Propositions 3.15 and 3.17 do not describe all polynomials generating non-real principal left ideals.

Example 3.18. *Consider the following univariate polynomial*

$$p = xx^* - (x^*)^2 + 2x + 4.$$

As is easily seen, p is not of the form (3.3) *or* (3.4). *On the other hand, the left ideal I generated by p is non-real. Indeed,*

$$(x + 2)p + p^*(x + 2)^* = (4 + 2x^*)^*(4 + 2x^*) \in I + I^*,$$

but $4 + 2x^* \notin I$. We shall investigate reality of ideals generated by quadratics in more detail in Section 4 below.

Proposition 3.19. *Let I be the left ideal generated by some polynomial p. Let p' be the leading polynomial of p, and let $p' = p_1 \cdots p_k$ be a factorization of p' into irreducible parts. If I is not real, then at least one of*

$$\pm(p_1 + p_1^*), \pm(p_1 p_2 + p_2^* p_1^*), \ldots, \pm(p_1 \cdots p_k + p_k^* \cdots p_1^*)$$

is a sum of squares.

Proof. By Proposition 3.7, if I is not real, then there exists a q with $\deg(q) < \deg(p)$ and $qp + p^* q^*$ being a nonzero sum of squares. Let q' be the leading polynomial of q. Then

$$q'p' + (p')^*(q')^*$$

is either 0 or the leading polynomial of $qp + p^* q^*$, in which case it is a sum of squares. Hence, in either case, $q'p' + (p')^*(q')^*$ is a sum of squares. By Lemma 3.9, this necessarily implies that $p' = p_1 \cdots p_j(\lambda q')$ for some j and some nonzero λ, and that $\pm(p_1 \cdots p_j + p_j^* \cdots p_1^*)$ is a sum of squares. ∎

3.4. **Analytic plus antianalytic generators.** Recall that a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ is analytic if it contains no x_j^*, i.e., if $p \in \mathbb{R}\langle x \rangle$. Likewise, a polynomial $p \in \mathbb{R}\langle x^* \rangle$ is called **antianalytic**.

Lemma 3.20. *Let $a, b \in \mathbb{R}\langle x, x^* \rangle$ be homogeneous analytic polynomials such that $\deg(a) = \deg(b) > 0$. Then $a + b^*$ is irreducible.*

Proof. Assume that $a + b^*$ is reducible. Proposition 3.4 then implies that $a + b^*$ is a product of two nonconstant homogeneous polynomials, i.e.

$$a + b^* = \Big(\sum_{\substack{u \in \langle x, x^* \rangle \\ \deg(u) = d}} A_u u \Big) \Big(\sum_{\substack{w \in \langle x, x^* \rangle \\ \deg(w) = e}} B_w w \Big),$$

for some $d, e \in \mathbb{N}$. The coefficient of uw in $a + b^*$ is $A_u B_w$. We see that $a + b^*$ has only terms which are analytic or antianalytic. Suppose u_0, w_0 are such that $u_0 w_0$ is analytic. Then u_0, w_0 are analytic. This implies that for any other u, either $A_u = 0$ or uw_0 is analytic or antianalytic, which necessarily implies that u is analytic. Similarly, either $B_w = 0$ or w is analytic. This implies that $a + b^*$ is analytic, which is a contradiction. ∎

Now we prove Theorem 1.5 (3). First we recall its statement.

Theorem 1.5 (3). *Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is the left ideal generated by a nonconstant polynomial $p = a + b^*$, where a and b are analytic polynomials. Then I is not real if and only if $p = a - a^* + c$ for some nonzero constant c.*

Proof. First, if $p = a - a^* + c$, then $p + p^* = 2c \in I + I^*$, which implies that $1 \in \sqrt{I}$. Therefore I is not real.

Conversely, suppose I is not real. Let $d = \deg(p)$. We have $\deg(p) = \max\{\deg(a), \deg(b)\}$ since the leading polynomials of a and b^* are analytic and antianalytic respectively, and hence cannot cancel each other out. Let a', b' be the degree d terms of a and b respectively, so that $p' = a' + (b')^*$ is the leading polynomial of p.

If $b' = 0$, then $p' = a'$. By Proposition 3.19, $p' = p_1 \cdots p_j f$ for some f and some nonconstant irreducible factors p_i, and

$$\pm(p_1 \cdots p_j + p_j^* \cdots p_1^*)$$

is a sum of squares. However, since p' is analytic, this implies that $p_1 \cdots p_j$ is analytic. An analytic polynomial has no terms of the form m^*m, whereas a nonconstant sum of squares does. This is a contradiction, so $b' \neq 0$. Similar reasoning shows $a' \neq 0$.

Next, by Lemma 3.20, $a' + (b')^*$ is irreducible. By Proposition 3.19,

$$a' + b' + (a')^* + (b')^*$$

is a sum of squares. A sum of an analytic and an antianalytic polynomial has no terms of the form m^*m, whereas a nonconstant sum of squares does. Therefore $a' + b' + (a')^* + (b')^* = 0$, which implies that $a' = -b'$.

Next, since I is not real, there exists a q such that $\deg(q) < \deg(p)$ and $qp + p^*q^*$ is a nonzero sum of squares. Let q' be the leading polynomial of q. Then

$$q'\big(a' - (a')^*\big) + \big(a' - (a')^*\big)^*(q')^*$$

is either 0 or is the leading polynomial of $qp + p^*q^*$. In either case, it is a sum of squares. Since $\deg(q) < \deg(p) = \deg\big(a' - (a')^*\big)$, by Lemma 3.9, this implies that $(q')^*$ is a factor of $a' - (a')^*$ on the right. However, by Lemma 3.20, $a - a^*$ is irreducible. Therefore q' is constant, which implies that q is constant. Therefore $\pm(p + p^*)$ is a nonzero sum of squares.

Let $e = \deg(p + p^*)$. Let a'_e, b'_e be the degree e elements of a and b respectively so that $a'_e + (b'_e)^*$ is the leading polynomial of $p + p^*$. As before, a nonconstant sum of squares cannot be expressed in this form. Hence $e = 0$, and so p is of the form $a - a^* + c$, where $c = b_e$ is a nonzero constant. ∎

Example 3.21. *Theorem 1.5 (3) only applies to principal left ideals. Indeed, if I is generated by x_1^2 and $x_1 + x_1^*$, then I is not real. We have $-x_1^2 + x_1(x_1 + x_1^*) = x_1 x_1^* \in I + I^*$ but $x_1 \notin I$.*

Corollary 3.22. *Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is the left ideal generated by a polynomial p with $\deg(p) = 1$. Then I is real if and only if p is not of the form $a - a^* + c$, where a is analytic and homogeneous of degree 1 and $c \neq 0$ is a constant.*

Proof. Every polynomial p of degree 1 is of the form $p = a + b^*$, where a, b are analytic, and a has no constant term. The result follows from Theorem 1.5 item 3. ∎

3.5. **Algorithm for checking whether a principal ideal is real.** The paper [CHMN+] gives an algorithm for computing the real radical of any finitely generated left ideal I. Here we discuss an improvement of a more limited algorithm which determines whether or not a given left ideal I is real. A test version has been implemented, and we give an example.

Suppose $p \in \mathbb{R}\langle x, x^* \rangle_d$ is given and let I be the left ideal generated by p. We employ Proposition 3.7 to test reality of I. Consider the feasibility problem

(3.5)
$$s \in \mathbb{R}\langle x, x^* \rangle \quad \text{is a nonzero sum of squares}$$
$$\text{s.t. } s = qp + p^*q^* \quad \text{for some } q \text{ with } \deg(q) < \deg(p).$$

This is an instance of an LMI. Namely, let $M_{<d}$ be a vector whose entries are all monomials of degree $< d$. Then s is a sum of squares if and only if there is a $G \succeq 0$ with $M_{<d}^* G M_{<d} = s$; cf. [MP05, KP10, Hel02]. The equation $s = qp + p^*q^*$ translates into a system of linear equations involving the coefficients of q and entries of G. Thus (3.5) is a feasibility semidefinite program (SDP)

(3.6)
$$\text{Find } 0 \neq G \succeq 0$$
$$\text{s.t. } M_{<d}^* G M_{<d} = qp + p^*q^*, \quad \deg(q) < d$$

and can thus be solved using a standard SDP solver. To search for a nonzero G we normalize (3.6) by requiring $\text{tr}(G) = 1$. The left ideal I is non-real if and only if (3.6) is feasible.

Remark 3.23. *We remark that checking if a noncommutative polynomial is a sum of squares can be done exactly using quantifier elimination (cf. [PW98]), but this is only viable for problems of small size (since the complexity for quantifier elimination is doubly exponential [BPR06, Section 11]). Hence in practice we employ SDPs (which typically run in polynomial time, cf. [WSV00]) for numerical verification; cf.* NCSOStools *[CKP11] for a computer algebra package which does this.*

We demonstrate the above with a simple example.

Example 3.24. *Let* $p = xx^* - x^*x - 1$. *The corresponding left ideal* I *is real. Indeed, write* $q = q_0 + q_1 x + q_2 x^*$, *and* $G = [g_{ij}]_{i,j=1}^3$. *Then* (3.6) *becomes*

$$G \succeq 0,$$

$$g_{11} + g_{22} + g_{33} - 1 = 0 \qquad g_{11} + 2q_0 = 0 \qquad g_{11} + g_{22} + 4q_0 + 2q_2 = 0$$

$$g_{11} - 2g_{12} - 2g_{13} + g_{22} + 2g_{23} + 2g_{33} + 2q_1 = 0$$

$$g_{11} + g_{12} + g_{13} - g_{22} - 3g_{23} - 2g_{33} + 4q_0 - q_1 - 3q_2 = 0$$

$$g_{11} - 4g_{12} - 4g_{13} + 5g_{22} + 8g_{23} + 4g_{33} + 4q_0 - 8q_1 - 6q_2 = 0$$

$$g_{11} + 2g_{12} + 2g_{13} + 2g_{22} + 2g_{33} + 2q_0 + 2q_1 + 2q_2 = 0$$

$$g_{11} - 2g_{12} - 2g_{13} + g_{22} + 2g_{23} + g_{33} + 2q_0 - 2q_1 - 2q_2 = 0.$$

Solving the above linear system yields

$$g_{11} = 1, \quad g_{13} = -g_{12}, \quad g_{22} = 1, \quad g_{23} = 0, \quad g_{33} = -1, \quad q_0 = -\frac{1}{2}, \quad q_1 = 0, \quad q_2 = 0.$$

Putting this solution back into G *leads to the LMI*

$$G = \begin{bmatrix} 1 & g_{12} & -g_{12} \\ g_{12} & 1 & 0 \\ -g_{12} & 0 & -1 \end{bmatrix} \succeq 0,$$

which is clearly infeasible.

The algorithm based on (3.6) extends to non-principal left ideals I in a straightforward way. To check for membership in $I + I^*$ (needed to encode the linear constraints in (3.6)) any generators p_j for I will do, the fewer the better. Finding a smallest generating set for an ideal is hard, hence a reduced Gröbner basis is a reasonable choice. We refer the reader to [Gr00, Mor94, Rei95, Kel97, Hey01, Lev05] and the references therein for the theory of noncommutative Gröbner bases.

We have implemented the algorithm for test purposes under `NCAlgebra` [HOSM12]. A major limitation is construction of the LMI required. This arises from manipulation of the large number of monomials in $M_{<d}^* G M_{<d}$. One could improve performance by storing these in advance. Also, the benefits when the generators p_j of I have few terms could be explored further.

4. PRINCIPAL IDEALS GENERATED BY UNIVARIATE QUADRATIC NC POLYNOMIALS

In Subsection 3.4 we characterized linear polynomials giving rise to real principal left ideals. In this section we discuss a complete classification whether or not the left ideal generated by a univariate quadratic polynomial is real. Of course we hope that an elegant general result will emerge, and perhaps the issues and structures exposed in this particular nontrivial case will provide some guidance in that direction.

4.1. **A sum of squares tool.** We start by characterizing univariate quadratics which are sums of squares, since we need this for our classification. Its straightforward proof is left as an exercise for the reader.

Lemma 4.1. *For a symmetric univariate quadratic polynomial*

$$p = a_0 + a_1(x + x^*) + a_2\big(x^2 + (x^*)^2\big) + a_3 xx^* + a_4 x^* x,$$

the following are equivalent:

(i) *p is a sum of two squares;*

(ii) *p is a sum of squares;*

(iii) *the following LMI is feasible*

(4.1) $G = \begin{bmatrix} a_0 & \lambda & a_1 - \lambda \\ \lambda & a_3 & a_2 \\ a_1 - \lambda & a_2 & a_4 \end{bmatrix} \succeq 0;$

(iv) *$p(X) \succeq 0$ for all $n \in \mathbb{N}$ and all $X \in \mathbb{R}^{n \times n}$;*

(v) *$p(X) \succeq 0$ for all $X \in \mathbb{R}^{2 \times 2}$;*

(vi) *$-a_1^2 + a_0(2a_2 + a_3 + a_4) \geq 0$, $a_0 \geq 0$, and*

(4.2) $\begin{bmatrix} a_3 & a_2 \\ a_2 & a_4 \end{bmatrix} \succeq 0.$

The quantitative strengthening of the sum of squares theorem tells us that a p as in Lemma 4.1 is a sum of squares iff it is positive semidefinite for all $X \in \mathbb{R}^{3 \times 3}$ (cf. [Hel02, MP05, McC01]). Item (v) above improves this size bound a bit.

4.2. **The p which generate real ideals.** Let p be an arbitrary univariate and quadratic polynomial, and let I be the principal left ideal it generates. We would like to determine when I is real.

Proposition 4.2. *Let*

(4.3) $p = a_0 + a_1 x + a_2 x^* + a_3 x^2 + a_4 xx^* + a_5 x^* x + a_6 (x^*)^2 \in \mathbb{R}\langle x, x^* \rangle$

be an arbitrary quadratic univariate polynomial (i.e., at least one of a_3, a_4, a_5, a_6 is nonzero). Then the left principal ideal generated by p is non-real if and only if either (1) or (2) holds.

(1) *Either $a_4 + a_6$ or $a_3 + a_5$ is nonzero and there exists an integer t such that*

$(-1)^t \begin{bmatrix} 2a_4 & a_3 + a_6 \\ a_3 + a_6 & 2a_5 \end{bmatrix} \succeq 0$, $(-1)^t a_0 \geq 0$ *and* $-(a_1 + a_2)^2 + 4a_0(a_3 + a_4 + a_5 + a_6) \geq 0.$

(2) *$a_3 + a_5 = 0 = a_4 + a_6$ and either*

$a_3 + a_4 = 0$ *and* $a_1 + a_2 = 0$ *and* $\big(a_2 a_4 \neq 0$ *or* $(a_0 a_4 \geq 0$ *and* $a_0^2 + a_4^2 \neq 0)\big)$

or

$a_4 + a_3 \neq 0$ *and* $a_1 + a_2 \neq 0$ *and* $a_0 = \dfrac{(a_1 + a_2)(a_1 a_4 - a_2 a_3)}{(a_3 + a_4)^2}.$

Proof. The proof illustrates the theory in Section 3 and requires calculations stemming from that. These we found tricky, so we include them. By Proposition 3.7, I is non-real if and only if there is a linear

$$q = q_0 + q_1 x + q_2 x^* \in \mathbb{R}\langle x, x^* \rangle \tag{4.4}$$

such that $qp + p^* q^*$ is a nonzero sum of squares. Compute $qp + p^* q^*$ and verify that it equals

$$\begin{aligned}
(4.5) \quad 2a_0 q_0 &+ x\big(a_0 q_1 + a_0 q_2 + a_1 q_0 + a_2 q_0\big) + x^*\big(a_0 q_1 + a_0 q_2 + a_1 q_0 + a_2 q_0\big) \\
&+ x^2\big(a_1 q_1 + a_2 q_2 + a_3 q_0 + a_6 q_0\big) + (x^*)^2\big(a_1 q_1 + a_2 q_2 + a_3 q_0 + a_6 q_0\big) \\
&+ 2x^* x\big(a_1 q_2 + a_5 q_0\big) + + 2xx^*\big(a_2 q_1 + a_4 q_0\big) \\
&+ x^3\big(a_3 q_1 + a_6 q_2\big) + (x^*)^3\big(a_3 q_1 + a_6 q_2\big) + x^2 x^*\big(a_6 q_1 + a_4 q_1\big) + x(x^*)^2\big(a_6 q_1 + a_4 q_1\big) \\
&+ xx^* x\big(a_4 q_2 + a_5 q_1\big) + x^* x^2\big(a_3 q_2 + a_5 q_2\big) + (x^*)^2 x\big(a_3 q_2 + a_5 q_2\big) + x^* xx^*\big(a_4 q_2 + a_5 q_1\big).
\end{aligned}$$

Here line 1 of (4.5) collects degree 0 and 1 terms of $qp + p^* q^*$, lines 2 and 3 contain degree 2 terms, while lines 4 and 5 contain degree 3 terms.

If (4.5) is a sum of squares, all degree 3 terms must vanish (and we are thrown into the case of Lemma 4.1). All degree 3 terms vanishing is equivalent to the following system of equations:

$$(4.6) \qquad \begin{aligned} a_3 q_1 + a_6 q_2 &= 0 & \qquad (a_4 + a_6) q_1 &= 0 \\ a_4 q_2 + a_5 q_1 &= 0 & \qquad (a_3 + a_5) q_2 &= 0. \end{aligned}$$

There are two cases to consider.

(a) If either $a_4 + a_6 \neq 0$ or $a_3 + a_5 \neq 0$, then the system (4.6) has only the trivial solution $q_1 = q_2 = 0$. In this case I is non-real iff

$$p + p^* = 2a_0 + (a_1 + a_2)x + (a_1 + a_2)x^* + (a_3 + a_6)x^2 + 2a_4 xx^* + 2a_5 x^* x + (a_3 + a_6)(x^*)^2$$

is a $\pm$ sum of squares. By Lemma 4.1(vi), this is true if either

$$\begin{bmatrix} 2a_4 & a_3 + a_6 \\ a_3 + a_6 & 2a_5 \end{bmatrix} \succeq 0 \text{ and } -(a_1 + a_2)^2 + 4a_0(a_3 + a_4 + a_5 + a_6) \geq 0 \text{ and } a_0 \geq 0$$

or

$$\begin{bmatrix} 2a_4 & a_3 + a_6 \\ a_3 + a_6 & 2a_5 \end{bmatrix} \preceq 0 \text{ and } -(a_1 + a_2)^2 + 4a_0(a_3 + a_4 + a_5 + a_6) \geq 0 \text{ and } a_0 \leq 0.$$

(b) If $a_3 + a_5 = 0 = a_4 + a_6$ then $a_5 = -a_3$ and $a_6 = -a_4$, and the system (4.6) reduces to the single equation

$$-a_3 q_1 + a_4 q_2 = 0.$$

If $a_3 = a_4 = 0$, then p is linear and then Corollary 3.22 tells us when I is real. If $a_3 \neq 0$ or $a_4 \neq 0$ then there exists $t \in \mathbb{R}$ such that $q_1 = ta_4$ and $q_2 = ta_3$. We reevaluate $qp + p^*q^*$:

$$(4.7) \quad 2a_0q_0 + x\Big(a_1q_0 + a_2q_0 + a_0a_4t + a_0a_3t\Big) + x^*\Big(a_1q_0 + a_2q_0 + a_0a_4t + a_0a_3t\Big)$$
$$+ x^2\Big(a_3q_0 - a_4q_0 + a_1a_4t + a_2a_3t\Big) + (x^*)^2\Big(a_3q_0 - a_4q_0 + a_1a_4t + a_2a_3t\Big)$$
$$+ 2xx^*\Big(a_4q_0 + a_2a_4t\Big) + 2x^*x\Big(-a_3q_0 + a_1a_3t\Big)$$

Then I is non-real if and only if there are $q_0, t \in \mathbb{R}$ making (4.7) a nonzero sum of squares. By assertion (vi) of Lemma 4.1, (4.7) is a sum of squares if and only if the system of the following polynomial inequalities is feasible:

$$(4.8) \quad \begin{aligned} a_4(a_2t + q_0) &\geq 0 \\ a_3(a_1t - q_0) &\geq 0 \\ (a_3 + a_4)q_0 &= (a_1a_4 - a_2a_3)t \\ (a_1 + a_2)q_0 &= a_0(a_3 + a_4)t \\ a_0q_0 &\geq 0 \end{aligned}$$

We also require that (4.7) has at least one nonzero coefficient.

If the linear system (for q_0, t) of the third and the fourth equation in (4.8) has a nonzero determinant, then $q_0 = t = 0$ which implies a contradiction that all coefficients of (4.7) are zero. Therefore

$$(4.9) \quad (a_1 + a_2)(a_1a_4 - a_2a_3) = a_0(a_3 + a_4)^2.$$

If $a_3 + a_4 = 0$, then by (4.9) also $a_1 + a_2 = 0$ and (4.8) simplifies considerably. The inequality in the second line of (4.8) is now equivalent to the first, so we are left with the system of the first and the fifth inequality. Moreover, (4.7) has a nonzero coefficient iff $a_4(a_2t + q_0) \neq 0$ or $a_0q_0 \neq 0$. This system has a solution iff either $a_2a_4 \neq 0$ or $a_0a_4 \geq 0$ and $a_0^2 + a_4^2 \neq 0$.

If $a_3 + a_4 \neq 0$ we can compute a_0 from (4.9) and q_0 from the third equation of (4.8). In this case the system (4.8) is equivalent to $\dfrac{(a_1 + a_2)t}{a_3 + a_4} \geq 0$. Moreover, (4.7) has a nonzero coefficient iff $(a_1 + a_2)t \neq 0$. This system has a solution iff $a_1 + a_2 \neq 0$.

Therefore, in case (b), the ideal I is non-real iff either

$$a_3 + a_4 = 0 \quad \text{and} \quad a_1 + a_2 = 0 \quad \text{and} \quad \Big(a_2a_4 \neq 0 \ \text{or} \ (a_0a_4 \geq 0 \ \text{and} \ a_0^2 + a_4^2 \neq 0)\Big)$$

or

$$a_4 + a_3 \neq 0 \quad \text{and} \quad a_1 + a_2 \neq 0 \quad \text{and} \quad a_0 = \frac{(a_1 + a_2)(a_1a_4 - a_2a_3)}{(a_3 + a_4)^2}. \quad \blacksquare$$

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APPENDIX A. NOT FOR PUBLICATION

In this section we collect some supplementary material that was not included for publication.

A.1. **Proof of Lemma** 4.1. It is clear that (i) implies (ii). By noting that G in (4.1) is the general form for the Gram matrix of p, we can deduce that items (ii) and (iii) are equivalent. Now suppose (iii) holds. By choosing the smallest λ making $G = G(\lambda) \succeq 0$, the rank of the corresponding G is ≤ 2. As in the proof of Lemma 3.3 this implies p is a sum of ≤ 2 squares, so (i) holds. By the sum of squares theorem [Hel02], (ii) and (iv) are equivalent. All this shows that (i), (ii), (iii) and (iv) are equivalent.

Clearly, (iv) implies (v). Let us now establish (v) $\Rightarrow$ (vi). First of all, $a_0 = p(0) \geq 0$. Since p is positive semidefinite on $\mathbb{R}^{2\times2}$, its homogeneous degree 2 part, $\hat{p}$, is also positive semidefinite on 2×2 matrices. It follows that

$$(A.1) \qquad \operatorname{tr}\left(\hat{p}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right) = a_3 + a_4 \geq 0,$$

and

$$(A.2) \qquad \det\left(\hat{p}\left(\begin{bmatrix} 0 & 1 \\ 0 & c \end{bmatrix}\right)\right) = a_3 a_4 + c^2(a_3 a_4 - a_2^2) \geq 0 \quad \text{for all } c \in \mathbb{R}.$$

From (A.2) we immediately obtain

$$(A.3) \qquad a_3 a_4 - a_2^2 \geq 0.$$

Together (A.1) and (A.3) show that the matrix (4.2) has nonnegative trace and nonnegative determinant, so is positive semidefinite. In particular,

$$2a_2 + a_3 + a_4 = \left\langle \begin{bmatrix} a_3 & a_2 \\ a_2 & a_4 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle \geq 0.$$

If $2a_2 + a_3 + a_4 = 0$, then $p(c) = a_0 + 2ca_1 \geq 0$ for all $c \in \mathbb{R}$, whence $a_1 = 0$. In this case $-a_1^2 + a_0(2a_2 + a_3 + a_4) = 0$. If $2a_2 + a_3 + a_4 > 0$, then

$$p\left(-\frac{a_1}{2a_2 + a_3 + a_4}\right) = \frac{-a_1^2 + a_0(2a_2 + a_3 + a_4)}{2a_2 + a_3 + a_4} \geq 0$$

shows $-a_1^2 + a_0(2a_2 + a_3 + a_4) \geq 0$, as desired.

Finally, assume (vi) holds. If $2a_2 + a_3 + a_4 = 0$, then $a_1 = 0$, and hence p has no linear terms. Since (4.2) is positive semidefinite, the homogeneous quadratic part of p is a sum

of squares. As also $a_0 \geq 0$, p is a sum of squares. We can thus assume $2a_2 + a_3 + a_4 > 0$. Perfoming an affine linear change of variables $x \mapsto x - \frac{a_1}{2a_2+a_3+a_4}$ leads to the polynomial

$$\tilde{p}(x) = p\left(x - \frac{a_1}{2a_2 + a_3 + a_4}\right) = \frac{-a_1^2 + a_0(2a_2 + a_3 + a_4)}{2a_2 + a_3 + a_4} + a_2(x^2 + (x^*)^2) + a_3 xx^* + a_4 x^* x$$

having no linear term. Obviously, p is positive (resp., sum of squares) iff $\tilde{p}$ is positive (resp., sum of squares), and the latter is the case iff its constant term is nonnegative and its homogeneous quadratic part is a sum of squares, i.e., iff $-a_1^2 + a_0(2a_2 + a_3 + a_4) \geq 0$ and (4.2) is positive semidefinite. ∎

A.2. **Left Gröbner Basis Algorithm on $\mathbb{R}\langle x, x^* \rangle$.** A useful tool for the general algorithm for testing reality of non-principal left ideals is computation of a left Gröbner basis for an ideal. This was implicit in [CHMN+] but here we give a cleaner description which lends itself to our computation. For details we refer the reader to the extensive literature on noncommutative Gröbner bases; see e.g. [Mor94, Rei95, Kel97, Hey01, Lev05] and the references therein.

Fix a monomial order $\succ$ on $\langle x, x^* \rangle$. For convenience, we choose an order such that $\deg(u) < \deg(v)$ implies $u \prec v$. By definition, $\succ$ satisfies the descending chain condition.

Let $p_1, \ldots, p_k$ generate a left ideal $I \subseteq \mathbb{R}\langle x, x^* \rangle$, and assume that they are monic (i.e. the coefficient of the leading monomial is 1). Let $u_1, \ldots, u_k$ be the leading monomials of $p_1, \ldots, p_k$ respectively. If $u_i \mid u_j$ for any $i \neq j$, let ω be a monomial such that $u_i = \omega u_j$. Replace p_j by $A(p_j - \omega p_i)$, where A is a normalization making the latter polynomial monic. In this case, the leading monomial of this new p_j is lower than the leading monomial of the old p_j. If any of the new p_j are 0, remove them from our set.

Repeat this until $u_i \nmid u_j$ for any $i \neq j$. Then $\{p_1, \ldots, p_k\}$ is a left Gröbner basis. This Algorithm is guaranteed to terminate since at each step where the algorithm does not stop, we replace a polynomial with another polynomial whose leading monomial is lower than that of the polynomial being replaced.

Proposition A.1. *Let $p_1, \ldots, p_k$ be a left Gröbner basis for a left ideal I, and suppose $\deg(p_i) \leq d$ for each i. Then for each $e \geq d$, a basis for I_e is*

(A.4) $$\{vp_i \mid 1 \leq i \leq k, \ v \in \langle x, x^* \rangle_{e - \deg(p_i)}\}.$$

Proof. Any polynomial $\iota \in I$ is equal to

$$\iota = q_1 p_1 + \ldots + q_k p_k$$

for some polynomials q_i since the p_i generate I. Let ω_i be the leading monomial of each q_i, and let u_i be the leading monomial of each p_i so that the leading monomial of $q_i p_i$ is $\omega_i u_i$.

If $i \neq j$, then $\omega_i u_i \neq \omega_j u_j$; indeed, suppose $\omega_i u_i = \omega_j u_j$, and suppose $\deg(u_i) \leq \deg(u_j)$. Then $\omega_i u_i = \omega_j u_j$ implies that $u_i \mid u_j$, which is a contradiction by the construction of the left Gröbner basis.

Therefore none of the leading monomials of the $q_i p_i$ cancel each other out. In particular, $\deg(\iota) = \max\{\deg(q_i p_i)\}$. This shows that the set (A.4) spans I_e for each $e \geq d$. Likewise, since the leading monomials of $v_i p_i$ cannot cancel each other out, the set (A.4) is linearly independent. ∎

A.3. **The Real Ideal Algorithm for finitely generated left ideals.** Suppose $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is a finitely generated left ideal with left Gröbner basis $\{p_1, \ldots, p_r\} \subseteq \mathbb{R}\langle x, x^* \rangle_d$. Consider the feasibility problem

$$s \in \mathbb{R}\langle x, x^* \rangle \quad \text{is a nonzero sum of squares}$$

(A.5)
$$\text{s.t. } s = \sum_{j=1}^{r} \left(q_j p_j + p_j^* q_j^* \right), \quad \deg(q_j p_j) < 2d.$$

As in Subsection 3.5, this is an instance of an LMI,

$$\text{Find } 0 \neq G \succeq 0$$

(A.6)
$$\text{s.t. } M_{<d}^* G M_{<d} = \sum_{j=1}^{r} \left(q_j p_j + p_j^* q_j^* \right), \quad \deg(q_j p_j) < 2d.$$

and can thus be solved using a standard SDP solver. To search for a nonzero G we normalize (A.6) by requiring $\operatorname{tr}(G) = 1$. The ideal I is real if and only if (A.6) is infeasible.

Example A.2. *Here is a simple univariate example. Let $\succ$ be the following monomial order on $\langle x, x^* \rangle$: for $u, w \in \langle x, x^* \rangle$, we define $u \prec w$ if $\deg(u) < \deg(w)$ or if $\deg(u) = \deg(w)$ and $u = rx^*s$ and $w = rxt$ for some $r, s, t \in \langle x, x^* \rangle$.*

Let the following set generate a left ideal

$$S = \{x^3 + 1, x^2 + (x^*)^2, xx^* - (x^*)^2, x^*x - 5\}.$$

Observe that these polynomials are presented in an $\succ$-decreasing manner. We see x^2 divides x^3, so we replace $x^3 + 1$ in S with

$$-\left(x^3 + 1 - x(x^2 + (x^*)^2) \right) = x(x^*)^2 - 1.$$

We thus obtain the generating set

(A.7)
$$S' = \{x(x^*)^2 - 1, x^2 + (x^*)^2, xx^* - (x^*)^2, x^*x - 5\}.$$

Clearly, S' is a left Gröbner basis. We used a Mathematica implementation of (A.6) based on NCAlgebra [HOSM12] *to verify that the left ideal generated by S' is real.*

To construct (A.5) or (A.6), any generators p_j for I will do, the fewer the better. Finding a smallest generating set for an ideal is hard, hence a reduced Gröbner basis is a reasonable choice. The left Gröber basis algorithm is useful in producing a fairly small basis for the ideal I.

We have implemented the algorithm for test purposes. A major limitation is construction of the LMI required. This arises from manipulation of the large number of monomials in $M_{<d}^* G M_{<d}$. One could improve performance by storing these in advance. Also, the benefits when the p_j have few terms could be explored further.

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