FEJÉR–RIESZ FACTORIZATION FOR FREE PRODUCTS

ABHAY JINDAL[1,Q], IGOR KLEP[2,Q,R], JACOB LEVENSON, AND SCOTT MCCULLOUGH

ABSTRACT. We establish an operator-valued Fejér–Riesz factorization theorem for the algebraic free product $\mathfrak{A} = \mathbb{C}[F_r] * \mathcal{A}_1 * \cdots * \mathcal{A}_s$, where F_r is a free group and each $\mathcal{A}_j$ is a finite-dimensional unital C^*-algebra. With respect to a natural reduced-word length, called extent, we show that every element $A \in B(\mathcal{H}) \otimes \mathfrak{A}$ of extent at most $2d$ that is positive under all finite-dimensional unital $*$-representations of $\mathfrak{A}$ admits a factorization $A = B^*B$, with B of extent at most d. This sharp half-extent bound holds without strict positivity. The proof combines a new finite-dimensional realization theorem for positive truncated kernels with operator-system methods. We also establish factorization results for tensor products with group algebras of arbitrary discrete groups under strict positivity in the maximal C^*-tensor product; for amenable groups, strict positivity in the minimal tensor product suffices.

CONTENTS

1. Introduction 2
1.1. The algebraic free product 3
1.2. Reduced words 3
1.3. Extent 4
1.4. Involution 4
1.5. The main result 4
1.6. Tensoring with a group algebra 5
1.7. Comparison with earlier approaches 8
1.8. Reader's guide 8
1.9. Use of AI 8
2. Preliminaries 9
3. Positive truncated kernels and representations 10

2020 *Mathematics Subject Classification.* Primary: 47A68, 46L07, 43A35; Secondary: 13J30, 47A56, 47B35.

Key words and phrases. Fejér–Riesz theorem; sums of squares; free products; extent filtration; positive truncated kernels; completely positive maps.

[1]Supported by the Slovenian Research Agency program P1-0222 and grant J1-50002.

[2]Supported by the Slovenian Research Agency program P1-0222 and grants J1-50002, N1-0217, J1-60011, J1-50001, J1-3004 and J1-60025.

[Q]This work was performed within the project COMPUTE, funded within the QuantERA II Programme that has received funding from the EU's H2020 research and innovation programme under the GA No. 101017733 .

[R]Part of this work was performed within the project SDPcode, funded within the QuantERA III Programme that has received funding from the EU's Horizon research and innovation programme under the GA No. 101212998 .

4. Proofs of the main results 18
4.1. Proof of Theorem 1.1 18
4.2. Proof of Corollary 1.2 20
4.3. Proof of Corollary 1.7 21
5. Proofs for tensor products 22
5.1. Proof of Theorem 1.4 22
5.2. Proof of Corollary 1.5 25
5.3. Limitations of spatial positivity 27
References 28
Appendix A. An elementary Følner-set proof of Corollary 1.5 29

1. Introduction

The classical Fejér–Riesz theorem asserts that a nonnegative trigonometric polynomial on the unit circle admits a square factorization. In operator-valued form, if

$$p(e^{i\theta}) = \sum_{n=-k}^{k} A_n e^{in\theta}, \qquad A_n \in B(\mathcal{H}), \tag{1.1}$$

is positive semidefinite for every θ, then there is an analytic polynomial $q(e^{i\theta}) = \sum_{n=0}^{k} B_n e^{in\theta}$ with coefficients in $B(\mathcal{H})$ such that $p = q^*q$; see [Rou68, DR10]. In real algebraic geometry [BCR98, Mar08, Sce24], the result is understood as a sums-of-squares representation or a Positivstellensatz. Beyond positivity certificates, Fejér–Riesz factorization is closely tied to the operator-algebraic theory of Toeplitz systems [CvS21] and has concrete applications in prediction theory [Roa58], digital filter design [WBV99, Dum17], and quantum signal processing [BS25].

Generalizations include multivariate Fejér–Riesz theorems [GW04, Dri04, DW05] and factorization and extension results for noncommutative polynomials and free groups [McC01, Hel02, BT07, Oza13]. Recent group-algebra results concern free products of copies of $\mathbb{Z}_2$ [KLM26] and free products of finite abelian groups [JKM26b]. The latter work introduced the extent filtration as a degree-like measure for reduced words.

In this paper we consider the algebraic free product

$$\mathfrak{A} = \mathbb{C}[F_r] * \mathcal{A}_1 * \cdots * \mathcal{A}_s,$$

where F_r is a free group and each $\mathcal{A}_j$ is a finite-dimensional unital C^*-algebra. Our main result establishes an exact Fejér–Riesz factorization with operator coefficients and the sharp half-extent bound, without assuming strict positivity. In particular, it covers free products with matrix-algebra factors and with group algebras of non-abelian finite groups.

A central ingredient here is a new realization theorem for positive truncated kernels, extending local left-multiplication operators to representations of the free-product factors. Together with operator-system methods, this yields the factorization and the stated extent bound.

We also study tensor products with group algebras of arbitrary discrete groups. In that setting, strict positivity in the maximal C^*-tensor product yields a factorization with the same half-extent bound. The maximal positivity assumption cannot in general be replaced by positivity in spatial representations; for amenable groups, however, strict positivity in the minimal tensor product suffices.

We next recall the free-product notation and the extent filtration needed to state the results precisely.

1.1. **The algebraic free product.** Let F_r denote the free group on the r generators $x_1, \dots, x_r$ and let $\mathbb{C}[F_r]$ denote the resulting group algebra:

$$\mathbb{C}[F_r] \;\cong\; \mathbb{C}[x_1, x_1^{-1}] * \cdots * \mathbb{C}[x_r, x_r^{-1}].$$

For $1 \le j \le s$, let $\mathcal{A}_j$ be a finite-dimensional unital C^*-algebra, and let

$$\mathfrak{A} \;=\; \mathbb{C}[F_r] * \mathcal{A}_1 * \cdots * \mathcal{A}_s \tag{1.2}$$

be the algebraic unital free product. For each infinite cyclic factor $\mathbb{C}[x_i, x_i^{-1}]$, let τ_{x_i} denote the *canonical trace*,

$$\tau_{x_i}(x_i^n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

For each finite-dimensional factor $\mathcal{A}_j$, fix a faithful tracial state τ_j and write

$$\mathcal{A}_j = \mathbb{C}\mathtt{1} \oplus \mathcal{A}_j^\circ, \qquad \mathcal{A}_j^\circ = \{a \in \mathcal{A}_j : \tau_j(a) = 0\}.$$

Choose an orthonormal basis $\mathcal{B}_j$ of $\mathcal{A}_j^\circ$ with respect to the inner product

$$\langle a, b \rangle_j = \tau_j(b^* a).$$

1.2. **Reduced words.** A *reduced word* in $\mathfrak{A}$ is either the identity $\mathtt{1}$ or a product

$$w = w_1 w_2 \cdots w_m$$

such that each syllable w_ℓ is either a nontrivial power x_i^n, with $n \in \mathbb{Z} \setminus \{0\}$, or an element of one of the bases $\mathcal{B}_j$, and consecutive syllables come from different free product factors. Thus two consecutive powers of the same generator x_i are not allowed, and two consecutive letters from the same finite-dimensional factor are not allowed. Let $\mathcal{W}$ denote the set of reduced words in $\mathfrak{A}$.

The *free-product trace*

$$\tau \;=\; \tau_{x_1} * \cdots * \tau_{x_r} * \tau_1 * \cdots * \tau_s$$

is the linear functional on $\mathfrak{A}$ that extracts the coefficient of the empty word. Equivalently, $\tau(\mathtt{1}) = 1$ and $\tau(w) = 0$ for every nonempty reduced word w. We refer the reader to [VDN92] for a detailed discussion of the free product of C^*-algebras.

1.3. **Extent.** Define the *weight* of a syllable by

$$\mathrm{wt}(w_\ell) = \begin{cases} |n|, & w_\ell = x_i^n,\ n \neq 0, \\ 1, & w_\ell \in \mathcal{B}_j. \end{cases}$$

The *extent* of a reduced word $w = w_1 \cdots w_m$ is

$$\mathrm{ext}(w) \ = \ \sum_{\ell=1}^{m} \mathrm{wt}(w_\ell), \qquad \mathrm{ext}(1) = 0.$$

Let $\mathcal{W}_{\mathtt{d}}$ be the finite set of reduced words of extent at most $\mathtt{d}$ and set

$$\mathfrak{A}_{\mathtt{d}} = \operatorname{span} \mathcal{W}_{\mathtt{d}} \subseteq \mathfrak{A}.$$

1.4. **Involution.** The involution on $\mathfrak{A}$ is induced by $(x_i^n)^* = x_i^{-n}$ and the involutions on the factors $\mathcal{A}_j$, with the order of syllables reversed. The adjoint of a basis element of $\mathcal{A}_j^\circ$ need not belong to the chosen basis $\mathcal{B}_j$, so the adjoint of a reduced word may be a linear combination of reduced words. Nevertheless, the spaces $\mathfrak{A}_{\mathtt{d}}$ are invariant under involution; see Lemma 2.2. For operator-valued polynomials we use $(T \otimes a)^* = T^* \otimes a^*$ and extend conjugate-linearly.

1.5. **The main result.** Let $\mathcal{H}$ be a Hilbert space and let $A \in B(\mathcal{H}) \otimes \mathfrak{A}_{2\mathtt{d}}$. Writing

$$A \ = \ \sum_{\ell=1}^{N(2\mathtt{d})} A_\ell \otimes r_\ell, \qquad A_\ell \in B(\mathcal{H}), \quad r_\ell \in \mathcal{W}_{2\mathtt{d}},$$

we say that A is *positive on finite-dimensional representations*, and write $A \succeq 0$, if $A = A^*$ and for every finite-dimensional Hilbert space $\mathcal{F}$ and every unital $*$-representation $\sigma : \mathfrak{A} \to B(\mathcal{F})$,

$$(\mathrm{id}_{B(\mathcal{H})} \otimes \sigma)(A) \ := \ \sum_{\ell=1}^{N(2\mathtt{d})} A_\ell \otimes \sigma(r_\ell) \succeq 0 \quad \text{on } \mathcal{H} \otimes \mathcal{F}.$$

Our first main result is the following.

Theorem 1.1. *Let $\mathtt{d}$ denote a positive integer and let $\mathcal{H}$ be a Hilbert space. If $A \in B(\mathcal{H}) \otimes \mathfrak{A}_{2\mathtt{d}}$ is positive on every finite-dimensional unital $*$-representation of $\mathfrak{A}$, then there exists a Hilbert space $\mathcal{E}$ and an element $B \in B(\mathcal{H}, \mathcal{E}) \otimes \mathfrak{A}_{\mathtt{d}}$ such that*

$$A = B^* B.$$

We can take $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^{N(\mathtt{d})}$, where $N(\mathtt{d}) = \dim(\mathfrak{A}_{\mathtt{d}})$.

The converse of Theorem 1.1 is immediate: a factorization $A = B^*B$ certifies positivity in every representation. The theorem requires no strict positivity and gives the uniform ampliation bound $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^{N(\mathtt{d})}$.

The free-group case ($s = 0$) has antecedents in [McC01, BT07, Oza13]; see, e.g., [Oza13, Theorem 26] and [JKM26a, Theorem 1.3] for operator-valued factorizations with prescribed support. For free products of finite abelian groups, [JKM26b, Theorem B] gives factors of extent at most $\mathtt{d}+1$ for an input of extent at most $2\mathtt{d}$. Theorem 1.1 improves this general bound to $\mathtt{d}$ and allows arbitrary finite-dimensional C^*-algebra factors.

As a consequence of Theorem 1.1, we obtain the corresponding Fejér–Riesz factorization on free products of a free group and finitely many finite groups.

Corollary 1.2. *Let*

$$\mathbb{G} = F_r * \mathbb{G}_1 * \cdots * \mathbb{G}_s, \tag{1.3}$$

where each $\mathbb{G}_j$ is a finite group. Equip $\mathbb{G}$ with the notion of extent defined above, using $\mathbb{G}_j \setminus \{1\}$ in place of $\mathcal{B}_j$. If

$$A = \sum_{\mathrm{ext}(g) \le 2\mathtt{d}} A_g \otimes g \in B(\mathcal{H}) \otimes \mathbb{C}[\mathbb{G}]$$

is a self-adjoint operator-valued group polynomial of extent at most $2\mathtt{d}$ such that for every finite-dimensional Hilbert space $\mathcal{F}$ and every unitary representation $\pi : \mathbb{G} \to B(\mathcal{F})$

$$(\mathrm{id}_{B(\mathcal{H})} \otimes \pi)(A) \succeq 0,$$

then there exists $B = \sum_{\mathrm{ext}(g) \le \mathtt{d}} B_g \otimes g \in B(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^{N(\mathtt{d})}) \otimes \mathbb{C}[\mathbb{G}]$ such that

$$A = B^* B.$$

The non-quantitative scalar sums-of-squares conclusion for the groups in Corollary 1.2 is already known. Indeed, a group of the form (1.3) is virtually free. Netzer–Thom [NT13, Theorem 6.1] establish the scalar free-group result and remark its extension to virtually free groups, with the proof of the latter later given in [Coh23, Theorem 3.2.1(b)]. These results cover the qualitative scalar conclusion, but do not yield either the prescribed extent bound or the operator-valued factorization. Thus Corollary 1.2 strengthens these conclusions by allowing operator coefficients, and by guaranteeing that an element of extent at most $2\mathtt{d}$ admits a factorization with factors of extent at most $\mathtt{d}$.

Remark 1.3. In Theorem 1.1 and Corollary 1.2, in the case that $\mathcal{H}$ has finite dimension K, it suffices to check that A is positive against unital $*$-representations of $\mathfrak{A}$ on a Hilbert space of dimension at most

$$\mathrm{lcm}\Big\{1, 2, \ldots, K\, N(\mathtt{d}), \dim \mathcal{A}_1, \ldots, \dim \mathcal{A}_s\Big\}.$$

1.6. **Tensoring with a group algebra.** Let $\mathfrak{H}$ be a discrete group. Given a subset $\mathbf{y} \subseteq \mathfrak{H}$, let $\mathcal{Y}$ denote the monoid generated by $\mathbf{y}$. For $M \ge 0$, let $\mathcal{Y}_{\le M}$ consist of the elements of $\mathcal{Y}$ representable by words in $\mathbf{y}$ of length at most M. For a subset E of a group G, let

$$\ell\text{-frac}\, E = \{u^{-1}v : u, v \in E\}.$$

Our next theorem gives an operator-valued factorization for arbitrary additional groups under maximal C^*-tensor-product positivity. Here $C^*_u(\mathfrak{A})$ denotes the universal C^*-completion of $\mathfrak{A}$, and $C^*(\mathfrak{H})$ is the full group C^*-algebra of $\mathfrak{H}$.

Theorem 1.4. *Let $\mathfrak{H}$ be an arbitrary discrete group, let $\mathtt{d}$ be a positive integer, and let $\mathcal{H}$ be a Hilbert space. Suppose that*

$$A = A^* \in B(\mathcal{H}) \otimes \mathfrak{A}_{2\mathtt{d}} \otimes \mathbb{C}[\mathfrak{H}]$$

and

$$A \succ 0 \quad \textit{in} \quad B(\mathcal{H}) \otimes_{\max} C_u^*(\mathfrak{A}) \otimes_{\max} C^*(\mathfrak{H}). \tag{1.4}$$

(Here $A \succ 0$ means that A is positive and invertible in the indicated C^*-algebra.)

Then there exist a positive integer n, a finite subset $F \subseteq \mathfrak{H}$, and operators

$$B_{q,g} \in B(\mathcal{H}, \mathcal{H}^{\oplus n}), \qquad q \in \mathcal{W}_{\mathtt{d}}, \quad g \in F,$$

such that

$$B = \sum_{\substack{q \in \mathcal{W}_{\mathtt{d}} \\ g \in F}} B_{q,g} \otimes (q \otimes g) \qquad \textit{satisfies} \qquad A = B^*B.$$

In particular, the target space $\mathcal{E} = \mathcal{H}^{\oplus n}$ is separable whenever $\mathcal{H}$ is separable.

Suppose in addition that $\mathfrak{H} = \ell\text{-frac}\,\mathcal{Y}$, where $\mathcal{Y}$ is the monoid generated by a finite set $\mathbf{y} \subseteq \mathfrak{H}$, and that A has the form

$$A = \sum_{\substack{a \in \mathcal{W}_{2\mathtt{d}} \\ g \in \ell\text{-frac}\,\mathcal{Y}_{\le M}}} A_{a,g} \otimes (a \otimes g), \qquad A_{a,g} \in B(\mathcal{H}),$$

for some positive integer M. Then F can be chosen to be contained in $\mathcal{Y}_{\le M'}$ for some $M' \ge M$.

Since the group support of A is finite, Lemma 2.4 shows that, in the first assertion of Theorem 1.4, we may assume without loss of generality that $\mathfrak{H}$ is finitely generated.

For amenable supporting subgroups, the maximal positivity hypothesis can be expressed in terms of spatial evaluations. This gives the following corollary of Theorem 1.4.

Corollary 1.5. *Let $\mathfrak{H}$ be an amenable group, and let $\mathcal{Y}$ be the monoid generated by a finite set $\mathbf{y} \subseteq \mathfrak{H}$. Suppose that*

$$\mathfrak{H} = \ell\text{-frac}\,\mathcal{Y}.$$

Let $\mathtt{d}, M$ be positive integers and let $\mathcal{H}$ be a Hilbert space. Suppose that $A = A^*$ has the form

$$A = \sum_{\substack{a \in \mathcal{W}_{2\mathtt{d}} \\ g \in \ell\text{-frac}\,\mathcal{Y}_{\le M}}} A_{a,g} \otimes (a \otimes g), \qquad A_{a,g} \in B(\mathcal{H}).$$

If

$$A \succ 0 \quad \textit{in} \quad B(\mathcal{H}) \otimes_{\min} C_u^*(\mathfrak{A}) \otimes_{\min} C^*(\mathfrak{H}), \tag{1.5}$$

then there exist an integer $M' \ge M$, a Hilbert space $\mathcal{E}$, and operators

$$B_{q,u} \in B(\mathcal{H}, \mathcal{E}), \qquad q \in \mathcal{W}_{\mathtt{d}}, \quad u \in \mathcal{Y}_{\le M'},$$

such that

$$B = \sum_{\substack{q \in \mathcal{W}_{\mathtt{d}} \\ u \in \mathcal{Y}_{\le M'}}} B_{q,u} \otimes (q \otimes u) \qquad \textit{satisfies} \qquad A = B^*B.$$

The space $\mathcal{E}$ can be chosen to be a finite ampliation of $\mathcal{H}$. Thus $\mathcal{E}$ can be chosen separable whenever $\mathcal{H}$ is separable. If $\mathcal{H}$ is infinite-dimensional, the coefficients of B may be taken in $B(\mathcal{H})$.

The strict positivity condition in equation (1.5) is equivalent to the existence of an $\varepsilon > 0$ such that

$$(\mathrm{id}_{B(\mathcal{H})} \otimes \pi)(A) \succeq \varepsilon I$$

for every unital $*$-representation π of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$ on a Hilbert space.

For matrix coefficients, the amenability assumption can be omitted.

Remark 1.6. For $\mathcal{H} = \mathbb{C}^K$, the conclusion of Corollary 1.5 holds without assuming $\mathfrak{H}$ amenable if (1.5) is replaced by $(\mathrm{id}_{M_K} \otimes \pi)(A) \succ 0$ for every unital $*$-representation π of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$. Indeed, put $D = C_u^*(\mathfrak{A}) \otimes_{\max} C^*(\mathfrak{H})$. If A were not strictly positive in $M_K(D)$, a state attaining the minimum of its spectrum would yield a GNS representation of $M_K(D)$ in which A is not strictly positive. Since every unital representation of $M_K(D)$ is of the form $\mathrm{id}_{M_K} \otimes \pi$, up to unitary equivalence, this contradicts the hypothesis. Thus $A \succ 0$ in $M_K(D) = M_K(\mathbb{C}) \otimes_{\max} D$, and Theorem 1.4 applies. As $\mathfrak{H}$ is finitely generated, separable representations suffice.

Strict positivity in Remark 1.6 is essential in general, as the conclusion can fail even for scalar-valued nonnegative trigonometric polynomials in three commuting variables (cf. [Sce00]). Example 5.3 shows that, even for an amenable additional group, strict positivity of the quadratic form in every spatial evaluation does not suffice for $B(\mathcal{H})$-valued coefficients. Even a uniform lower bound in all spatial evaluations need not suffice for general additional groups, as Example 5.2 in Section 5 shows. This explains the role of maximal tensor-product positivity in Theorem 1.4. Corollary 1.7 below illustrates how the positivity hypothesis can be relaxed when the additional group $\mathfrak{H}$ is finite.

Remark 1.6 is related to [KLM26, Theorem 1.1], which already establishes a strict matrix-valued mixed-group factorization with preservation of support in a free coordinate. [KLM26] also treats non-strict operator-valued factorization for free products of copies of $\mathbb{Z}_2$ [KLM26, Theorem 1.6]; see also [KLM26, Remark 1.7]. The support conventions, however, are different: [KLM26] employs shortlex and analytic/hereditary support restrictions, whereas the present results use extent and allow inverses in the free-group coordinate. (Compare Examples 8.1 and 8.2 of [KLM26]).

Unlike the compactness argument of [KLM26, Sections 6–7], the matrix-coefficient assertion of Remark 1.6 follows directly from Theorem 1.4, including the extent bound $\mathtt{d}$ for the present class of free products.

When $\mathfrak{H}$ is finite, Theorem 1.1 yields the following stronger conclusion.

Corollary 1.7. *Let $\mathfrak{H}$ be a finite group, let $\mathtt{d}$ be a positive integer, and let $\mathcal{H}$ be a Hilbert space. If*

$$A \in B(\mathcal{H}) \otimes \mathfrak{A}_{2\mathtt{d}} \otimes \mathbb{C}[\mathfrak{H}]$$

is nonnegative against every unital $*$-representation of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$ on a finite-dimensional Hilbert space, then there exists

$$B \in B(\mathcal{H}, \mathcal{E}) \otimes \mathfrak{A}_{\mathtt{d}} \otimes \mathbb{C}[\mathfrak{H}], \qquad \mathcal{E} = \mathcal{H} \otimes \mathbb{C}^{N(\mathtt{d})},$$

such that $A = B^*B$.

Further, if $\mathcal{H}$ is finite-dimensional, then it suffices to verify that A is positive against unital $*$-representations of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$ on Hilbert spaces of dimension at most

$$\operatorname{lcm}\Big\{1, 2, \dots, \dim \mathcal{H}\,|\mathfrak{H}|\,N(\mathtt{d}), \dim \mathcal{A}_1, \dots, \dim \mathcal{A}_s\Big\}.$$

1.7. **Comparison with earlier approaches.** The approach of [JKM26b] to free products of finite abelian groups uses a Positivstellensatz for positive operator-valued measures and Boca's theorem on free products of completely positive maps. It exploits the commutative spectral structure of finite abelian group algebras, which is absent for general finite-dimensional C^*-algebras. The methods of [KLM26] rely on positive semidefinite matrix completions adapted to the shortlex order on free semigroups and free products of copies of $\mathbb{Z}_2$. These constructions do not directly yield representations respecting the multiplication relations within arbitrary finite-dimensional C^*-algebra factors.

The principal novelty here underlying Theorems 1.1 and 1.4 and their corollaries is the realization of positive truncated kernels by representations respecting the relations of all free-product factors, while retaining control of the extent. See Theorem 3.1, which extends local left-multiplication operators to representations of the free-product factors. Combined with operator-system methods, this yields Theorem 1.1, including the sharp half-extent factorization without strict positivity.

1.8. **Reader's guide.** In outline, the proof of Theorem 1.1 proceeds as follows. The positivity of A is interpreted as complete positivity of a linear map Φ_A defined on an operator system $\mathcal{O}_{\mathtt{d}}$. When $\mathfrak{A} = \mathbb{C}[x, x^{-1}]$, this operator system is, after ordering the basis appropriately, the space of $(2\mathtt{d}+1)\times(2\mathtt{d}+1)$ Toeplitz matrices. Positive elements of $\mathcal{O}_{\mathtt{d}} \otimes M_K(\mathbb{C})$, where $M_K(\mathbb{C})$ is the space of complex $K \times K$ matrices, are seen, via a finite-dimensional dilation result for positive truncated kernels (Theorem 3.1), to arise from representations of $\mathfrak{A}$. Applying the Arveson Extension Theorem to Φ_A and factoring the resulting positive semidefinite Choi matrix associated to the extended version of Φ_A completes the proof.

For Theorem 1.4, we use an algebraic separation argument, in the spirit of [HM04], and a group-equivariant form of the truncated-kernel realization. If $A - \delta I$ is not a sum of squares of elements of $\mathfrak{A}$-extent at most $\mathtt{d}$, an algebraic order-unit separation argument produces a separating linear functional Λ. Applying Theorem 3.1 with $G = \mathcal{U}(B(\mathcal{H})) \times \mathfrak{H}$ realizes Λ as a vector functional on a representation of the maximal C^*-tensor product, contradicting the strict positivity of A. Combining Theorem 1.4 with Pisier's free-product theorem, Lemma 5.1, and the nuclearity of $C^*(\mathfrak{H})$ for amenable $\mathfrak{H}$ yields Corollary 1.5.

Section 2 collects some needed facts about $\mathfrak{A}$. Section 3 develops the representation theory of positive truncated kernels and proves Theorem 3.1. Section 4 establishes Theorem 1.1 and the associated free-product and finite-group Corollaries 1.2 and 1.7. Section 5 proves the results on tensor products with group algebras (Theorem 1.4 and Corollary 1.5), and gives an obstruction for general groups in Example 5.2.

1.9. **Use of AI.** The authors acknowledge the use of LLMs during the late stages of work on this paper. It responded to our prompts to unify two instances of the matrix-coefficient factorization now stated in Remark 1.6 - one without the free group factor in $\mathfrak{A}$ and general

$\mathfrak{H}$ and the other with general $\mathfrak{A}$ and $\mathfrak{H}$ an amenable group - using Lemma 3.3. It also produced Example 5.2 and was used to assist in final editing. All mathematical details were independently checked and verified by the authors, who assume full responsibility for their mathematical correctness.

2. Preliminaries

Lemma 2.1. *The reduced words in $\mathfrak{A}$ form an orthonormal basis of the pre-Hilbert space obtained from $\mathfrak{A}$ with the inner product*

$$\langle u, v\rangle = \tau(v^* u).$$

Proof. For each cyclic factor put

$$\mathbb{C}[x_i, x_i^{-1}]^\circ = \operatorname{span}\{x_i^n : n \neq 0\}.$$

As a vector space, the algebraic free product decomposes as

$$\mathfrak{A} \;=\; \mathbb{C}\mathbf{1} \oplus \bigoplus_{m\geq 1} \bigoplus_{\ell_j \neq \ell_{j+1}\ (1\leq j<m)} E^\circ_{\ell_1} \otimes E^\circ_{\ell_2} \otimes \cdots \otimes E^\circ_{\ell_m},$$

where either $E^\circ_\ell = \mathbb{C}[x_i, x_i^{-1}]^\circ$ or $E^\circ_\ell = \mathcal{A}^\circ_j$. The free-product inner product is the tensor-product inner product on each summand, and distinct summands are orthogonal to one another. Since

$$\{x_i^n : n \neq 0\}$$

is an orthonormal basis for $\mathbb{C}[x_i, x_i^{-1}]^\circ$ and $\mathcal{B}_j$ is an orthonormal basis for $\mathcal{A}^\circ_j$, the corresponding reduced words form an orthonormal basis. □

Lemma 2.2. *For every nonnegative integer $\mathtt{d}$, the subspace $\mathfrak{A}_\mathtt{d}$ is closed under the involution.*

Proof. It suffices to consider a reduced word $w = w_1 \cdots w_m \in \mathfrak{A}_\mathtt{d}$. If $w_\ell = x_i^n$, then $w_\ell^* = x_i^{-n}$ and the weight is unchanged. If $w_\ell \in \mathcal{B}_j$, then $w_\ell^* \in \mathcal{A}^\circ_j$ and hence w_ℓ^* is a linear combination of elements of $\mathcal{B}_j$. Reversing the order of the syllables preserves the condition that neighboring syllables come from different free product factors. Therefore w^* is a linear combination of reduced words with the same extent as w. □

Lemma 2.3. *For every nonnegative integer $\mathtt{d}$,*

$$\mathfrak{A}_{2\mathtt{d}} \;=\; \operatorname{span}\{v^* u \;:\; u, v \in \mathfrak{A}_\mathtt{d}\}.$$

Proof. The inclusion from right to left follows from Lemma 2.2 and the subadditivity of extent under multiplication.

For the reverse inclusion, let $w = w_1 \cdots w_m$ be a reduced word with $\operatorname{ext}(w) \leq 2\mathtt{d}$. Split w as $w = pq$ with $\operatorname{ext}(p) \leq \mathtt{d}$ and $\operatorname{ext}(q) \leq \mathtt{d}$. If the splitting point falls inside a syllable x_i^n, write $x_i^n = x_i^t x_i^{n-t}$, choosing t with the same sign as n so that the two new powers have the desired extents. No symbol from $\mathcal{B}_j$ needs to be split, since all weights are integral and such syllables have weight one. Consequently, $p, q \in \mathfrak{A}_\mathtt{d}$ and

$$w = pq = (p^*)^* q.$$

By Lemma 2.2, $p^* \in \mathfrak{A}_\mathtt{d}$, so w belongs to the stated span. □

The following subgroup reduction is a standard consequence of induction of representations.

Lemma 2.4. *Let $\mathfrak{H}$ be a subgroup of $\mathfrak{Y}$ and let D be a unital C^*-algebra. For $\alpha \in \{\min, \max\}$, the canonical map*

$$D \otimes_\alpha C^*(\mathfrak{H}) \longrightarrow D \otimes_\alpha C^*(\mathfrak{Y})$$

is injective. Consequently, for every $A = A^* \in D \otimes \mathbb{C}[\mathfrak{H}]$ and $\varepsilon \geq 0$,

$$A \succeq \varepsilon I \quad \text{in } D \otimes_\alpha C^*(\mathfrak{H}) \quad \iff \quad A \succeq \varepsilon I \quad \text{in } D \otimes_\alpha C^*(\mathfrak{Y}).$$

The same equivalence holds for strict positivity in these C^*-algebras.

In particular, taking $D = M_K(C_u^*(\mathfrak{A}))$ and $\alpha = \max$ recovers the corresponding equivalence for pointwise strict positivity under all unital $*$-representations of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$ and $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{Y}]$.

Proof. The inclusion $C^*(\mathfrak{H}) \subseteq C^*(\mathfrak{Y})$ is standard; see [BO08, Proposition 2.5.8]. Injectivity of the minimal tensor product gives the assertion for $\alpha = \min$. For $\alpha = \max$, induction of representations from $\mathfrak{H}$ to $\mathfrak{Y}$, with D acting fiberwise, preserves the maximal tensor norm. The positivity equivalences follow by spectral permanence. The final assertion follows from the characterization of strict positivity by representations; see [Dix77, 2.12.12]. □

3. Positive truncated kernels and representations

Let k, d be positive integers, let $\mathfrak{H}$ be a group and let $S \subseteq \mathfrak{H}$. A linear map

$$\Theta : \mathfrak{A}_{2d} \otimes \operatorname{span}\{g : g \in \ell\text{-frac } S\} \longrightarrow M_k(\mathbb{C})$$

is a *positive truncated kernel on S* if, for every finite $G \subseteq S$,

$$\Big[\Theta(p^*q \otimes u^{-1}v)\Big]_{(p,u),(q,v) \in \mathcal{W}_d \times G} \succeq 0. \tag{3.1}$$

We also write $\Theta(a, g)$ for $\Theta(a \otimes g)$, and suppress the group factor when the group is trivial. If $\mathcal{Y} \subseteq \mathfrak{H}$ is a monoid such that $\mathfrak{H} = \{u^{-1}v : u, v \in \mathcal{Y}\}$, then $\operatorname{span} \ell\text{-frac } \mathcal{Y} = \mathbb{C}[\mathfrak{H}]$.

Theorem 3.1. *Suppose $\mathfrak{H}$ is an arbitrary discrete group and $\mathcal{Y} \subseteq \mathfrak{H}$ is a monoid such that*

$$\mathfrak{H} = \{u^{-1}v : u, v \in \mathcal{Y}\}.$$

If the linear map

$$\Theta : \mathfrak{A}_{2d} \otimes \mathbb{C}[\mathfrak{H}] \longrightarrow M_k(\mathbb{C})$$

is a positive truncated kernel on $\mathcal{Y}$, then there exist a Hilbert space $\mathcal{E}$, a unital $*$-representation

$$\pi : \mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}] \longrightarrow B(\mathcal{E}),$$

and a bounded linear map $V : \mathbb{C}^k \to \mathcal{E}$ such that

$$\Theta(p^*q \otimes u^{-1}v) = V^*\pi\Big((p^*q) \otimes (u^{-1}v)\Big)V \tag{3.2}$$

for all $p, q \in \mathfrak{A}_d$ and $u, v \in \mathfrak{H}$. Moreover, $V^*V = \Theta(1, e)$. If $\mathfrak{H}$ is at most countable, $\mathcal{E}$ can be chosen to be separable.

More precisely, let $\mathcal{H}_\Theta$ be the Hilbert space obtained by quotienting $\mathfrak{A}_d \otimes \mathbb{C}[\mathfrak{H}] \otimes \mathbb{C}^k$ by the null space of the positive semidefinite form

$$\langle p \otimes u \otimes \xi, q \otimes v \otimes \eta \rangle_\Theta = \langle \Theta(q^*p, v^{-1}u)\xi, \eta \rangle_{\mathbb{C}^k}$$

and completing. Write $[p\otimes u\otimes\xi]$ *for the resulting classes, and let* τ_Θ *be the canonical unitary representation of* $\mathfrak{H}$ *on* $\mathcal{H}_\Theta$*,*

$$\tau_\Theta(g)[p\otimes u\otimes\xi]=[p\otimes gu\otimes\xi].$$

One can choose the realization in (3.2) *so that*

$$\begin{aligned}\mathcal{E}=\mathcal{H}_\Theta\otimes\ell^2(\mathbb{N}),\qquad V\xi=[1\otimes e\otimes\xi]\otimes e_0,\\ \pi(1\otimes g)=\tau_\Theta(g)\otimes I_{\ell^2(\mathbb{N})},\qquad g\in\mathfrak{H},\end{aligned}\tag{3.3}$$

where e_0 *is the first standard basis vector of* $\ell^2(\mathbb{N})$*. In this realization,*

$$\pi(p\otimes g)V\xi=[p\otimes g\otimes\xi]\otimes e_0,\qquad p\in\mathfrak{A}_d,\quad g\in\mathfrak{H},\quad \xi\in\mathbb{C}^k.\tag{3.4}$$

In the case that $\mathfrak{H}=\{e\}$ *is trivial (so that* $\mathfrak{A}\otimes\mathbb{C}[\mathfrak{H}]\cong\mathfrak{A}$*), the realization in* (3.2) *can be chosen with* $\mathcal{E}$ *of dimension at most*

$$\operatorname{lcm}\Big\{1,2,\dots,kN(d),\dim\mathcal{A}_1,\dots,\dim\mathcal{A}_s\Big\}.$$

The proof of Theorem 3.1 uses the following two lemmas. The first lemma extends the positivity condition from finite subsets of $\mathcal{Y}$ to finite subsets of $\mathfrak{H}$. For $\mathfrak{H}=\mathbb{Z}$ and $\mathcal{Y}=\mathbb{N}_0$, it gives the passage from unilateral to bilateral Toeplitz positivity. It follows from a standard fact from noncommutative ring theory. See [KLM26, Lemma 3.6] for instance. A proof is given here for the reader's convenience.

Lemma 3.2. *Suppose that* $\mathcal{Y}\subseteq\mathfrak{H}$ *is a monoid and* $\mathfrak{H}=\ell\text{-frac}\,\mathcal{Y}$*. If*

$$\Theta:\mathfrak{A}_{2d}\otimes\mathbb{C}[\mathfrak{H}]\longrightarrow M_k(\mathbb{C})$$

is a positive truncated kernel on $\mathcal{Y}$*, then, for every finite* $G\subseteq\mathfrak{H}$*,*

$$\Big[\Theta(p^*q\otimes u^{-1}v)\Big]_{(p,u),(q,v)\in\mathcal{W}_d\times G}\succeq 0.\tag{3.5}$$

Proof. We first find $c\in\mathcal{Y}$ such that $cG\subseteq\mathcal{Y}$. Write $G=\{g_1,\dots,g_m\}$ and set $c_0=\mathrm{e}$. Inductively, write

$$c_{j-1}g_j=a_j^{-1}b_j,\qquad a_j,b_j\in\mathcal{Y},$$

and put $c_j=a_jc_{j-1}$. Then $c_jg_i\in\mathcal{Y}$ for $1\le i\le j$. Thus $c=c_m$ has the required property.

For $u,v\in G$,

$$(cu)^{-1}(cv)=u^{-1}v.$$

Therefore the matrix in (3.5) is the positive matrix associated with the finite subset $cG\subseteq\mathcal{Y}$, after reindexing u as cu. □

Lemma 3.3. *Let* $\tau:G\to B(\mathcal{H})$ *be a unitary representation of a group* G*, and let* $\mathcal{M},\mathcal{N}$ *be closed reducing subspaces for* τ*. Set*

$$\widetilde{\mathcal{H}}=\mathcal{H}\otimes\ell^2(\mathbb{N}),\qquad\widetilde{\tau}(g)=\tau(g)\otimes I_{\ell^2(\mathbb{N})},$$

and identify $\mathcal{H}$ *with the first summand of* $\widetilde{\mathcal{H}}$*. Write* $\widetilde{\tau}_{\mathcal{M}}$ *and* $\widetilde{\tau}_{\mathcal{N}}$ *for the restrictions to* $\widetilde{\mathcal{H}}\ominus\mathcal{M}$ *and* $\widetilde{\mathcal{H}}\ominus\mathcal{N}$*, respectively.*

(i) There is a unitary $V : \widetilde{\mathcal{H}} \ominus \mathcal{M} \to \widetilde{\mathcal{H}}$ *such that*

$$V\tilde{\tau}_{\mathcal{M}}(g) = \tilde{\tau}(g)V, \qquad g \in G.$$

(ii) There is a unitary $R : \widetilde{\mathcal{H}} \ominus \mathcal{M} \to \widetilde{\mathcal{H}} \ominus \mathcal{N}$ *such that*

$$R\tilde{\tau}_{\mathcal{M}}(g) = \tilde{\tau}_{\mathcal{N}}(g)R, \qquad g \in G.$$

(iii) If $L : \mathcal{M} \to \mathcal{N}$ *is unitary and intertwines* $\tau|_{\mathcal{M}}$ *and* $\tau|_{\mathcal{N}}$*, then* L *extends to a unitary on* $\widetilde{\mathcal{H}}$ *commuting with* $\tilde{\tau}$*.*

(iv) If $\mathcal{A}$ *is a finite-dimensional unital* C^**-algebra and* $L : \mathcal{A} \to B(\mathcal{M})$ *is a unital* $*$*-representation commuting with* $\tau|_{\mathcal{M}}$*, then there is a unital* $*$*-representation*

$$\rho : \mathcal{A} \to B(\widetilde{\mathcal{H}} \ominus \mathcal{M})$$

commuting with $\tilde{\tau}_{\mathcal{M}}$*. Consequently,* $\sigma = L \oplus \rho$ *is a unital* $*$*-representation on* $\widetilde{\mathcal{H}}$ *commuting with* $\tilde{\tau}$*.*

Proof. Note that $\mathcal{H} = \{h \otimes e_0 : h \in \mathcal{H}\}$, where (e_n) is the standard orthonormal basis for $\ell^2(\mathbb{N})$. Since $\mathcal{M}$ reduces τ, we have the direct sum decomposition $\tau = \tau|_{\mathcal{M}} \oplus \tau|_{\mathcal{M}^\perp}$. Consequently,

$$\begin{aligned}\tilde{\tau}|_{\widetilde{\mathcal{H}}\ominus\mathcal{M}} &\sim_{\mathrm{u}} \tau|_{\mathcal{M}^\perp} \oplus (\tau|_{\mathcal{M}} \oplus \tau|_{\mathcal{M}^\perp}) \oplus (\tau|_{\mathcal{M}} \oplus \tau|_{\mathcal{M}^\perp}) \oplus \cdots \\ &\sim_{\mathrm{u}} (\tau|_{\mathcal{M}^\perp} \oplus \tau|_{\mathcal{M}}) \oplus (\tau|_{\mathcal{M}^\perp} \oplus \tau|_{\mathcal{M}}) \oplus \cdots \sim_{\mathrm{u}} \tilde{\tau},\end{aligned}$$

where $\sim_{\mathrm{u}}$ indicates unitary equivalence. Likewise, $\tilde{\tau}|_{\widetilde{\mathcal{H}}\ominus\mathcal{N}} \sim_{\mathrm{u}} \tilde{\tau}$. Hence there are unitary operators

$$V : \widetilde{\mathcal{H}} \ominus \mathcal{M} \longrightarrow \widetilde{\mathcal{H}}, \qquad R : \widetilde{\mathcal{H}} \ominus \mathcal{M} \longrightarrow \widetilde{\mathcal{H}} \ominus \mathcal{N}$$

such that

$$V\tilde{\tau}_{\mathcal{M}}(g) = \tilde{\tau}(g)V, \qquad R\tilde{\tau}_{\mathcal{M}}(g) = \tilde{\tau}_{\mathcal{N}}(g)R, \qquad g \in G. \tag{3.6}$$

This proves (i) and (ii). For (iii), let $U = L \oplus R$ and observe that, for $g \in G$,

$$\begin{aligned}U\tilde{\tau}(g) &= (L \oplus R)(\tau(g)|_{\mathcal{M}} \oplus \tilde{\tau}_{\mathcal{M}}(g)) \\ &= L\tau(g)|_{\mathcal{M}} \oplus R\tilde{\tau}_{\mathcal{M}}(g) \\ &= \tau(g)|_{\mathcal{N}}L \oplus \tilde{\tau}_{\mathcal{N}}(g)R \\ &= (\tau(g)|_{\mathcal{N}} \oplus \tilde{\tau}_{\mathcal{N}}(g))(L \oplus R) = \tilde{\tau}(g)U.\end{aligned}$$

To prove (iv), choose a unital $*$-representation $\nu : \mathcal{A} \to B(\ell^2(\mathbb{N}))$ and define $\rho : \mathcal{A} \to B(\widetilde{\mathcal{H}} \ominus \mathcal{M})$ by

$$\rho(a) = V^*(I_{\mathcal{H}} \otimes \nu(a))V, \qquad a \in \mathcal{A},$$

set $\tilde{\nu} = I_{\mathcal{H}} \otimes \nu$ and observe, by equation (3.6) and the fact that V is unitary,

$$\begin{aligned}\rho(a)\tilde{\tau}_{\mathcal{M}}(g) &= V^*\tilde{\nu}(a)V\tilde{\tau}_{\mathcal{M}}(g) = V^*\tilde{\nu}(a)\tilde{\tau}(g)V \\ &= V^*\tilde{\tau}(g)\tilde{\nu}(a)V = \tilde{\tau}_{\mathcal{M}}(g)(V^*\tilde{\nu}(a)V) = \tilde{\tau}_{\mathcal{M}}(g)\rho(a).\end{aligned}$$

Define

$$\sigma : \mathcal{A} \longrightarrow B\Big(\mathcal{M} \oplus (\widetilde{\mathcal{H}} \ominus \mathcal{M})\Big) = B(\widetilde{\mathcal{H}})$$

by $\sigma(a) = L(a) \oplus \rho(a)$. Thus σ is a unital $*$-representation of $\mathcal{A}$ on $\widetilde{\mathcal{H}}$ and, for $a \in \mathcal{A}$ and $g \in G$,

$$\begin{aligned}\sigma(a)\widetilde{\tau}(g) &= [L(a) \oplus \rho(a)][\tau(g)|_{\mathcal{M}} \oplus \widetilde{\tau}_{\mathcal{M}}(g)] \\ &= L(a)\tau(g)|_{\mathcal{M}} \oplus \rho(a)\widetilde{\tau}_{\mathcal{M}}(g) = \tau(g)|_{\mathcal{M}}L(a) \oplus \widetilde{\tau}_{\mathcal{M}}(g)\rho(a) = \widetilde{\tau}(g)\sigma(a).\end{aligned}$$

□

Proof of Theorem 3.1. By Lemma 3.2, the kernel $\Theta : \mathfrak{A}_{2d} \otimes \mathbb{C}[\mathfrak{H}] \to M_k(\mathbb{C})$ is positive on every finite subset of $\mathfrak{H}$. On $\mathfrak{A}_d \otimes \mathbb{C}[\mathfrak{H}] \otimes \mathbb{C}^k$, define

$$\langle p \otimes u \otimes \xi, q \otimes v \otimes \eta\rangle_\Theta = \langle \Theta(q^*p, v^{-1}u)\xi, \eta\rangle_{\mathbb{C}^k}.$$

By equation (3.1), $\langle \cdot, \cdot \rangle_\Theta$ is a positive semidefinite sesquilinear form. Let

$$\mathcal{N}_\Theta = \{\omega : \langle \omega, \omega\rangle_\Theta = 0\},$$

and let $\mathcal{H}_\Theta$ be the Hilbert-space completion of

$$(\mathfrak{A}_d \otimes \mathbb{C}[\mathfrak{H}] \otimes \mathbb{C}^k)/\mathcal{N}_\Theta$$

under the inner product induced by $\langle \cdot, \cdot \rangle_\Theta$ and denote the resulting equivalence classes by $[p \otimes u \otimes \xi]$. If $\mathfrak{H}$ is at most countable, then the finite-dimensionality of $\mathfrak{A}_d$ and $\mathbb{C}^k$ implies that $\mathcal{H}_\Theta$ is separable. The construction also applies when $\mathfrak{H}$ is uncountable; in that case no separability of $\mathcal{H}_\Theta$ is asserted.

Define the canonical representation $\tau = \tau_\Theta : \mathfrak{H} \to B(\mathcal{H}_\Theta)$ as follows. Since $(gv)^{-1}(gu) = v^{-1}u$ for $g, u, v \in \mathfrak{H}$, for $g \in \mathfrak{H}$ setting

$$\tau(g)[p \otimes u \otimes \xi] = [p \otimes gu \otimes \xi]$$

and extending by linearity and continuity produces a unitary operator $\tau(g) \in B(\mathcal{H}_\Theta)$. Since $\tau(gh) = \tau(g)\tau(h)$ as is readily checked, τ is a unitary representation of $\mathfrak{H}$ and extends to a unital $*$-representation of $\mathbb{C}[\mathfrak{H}]$.

Fix $i \in \{1, \dots, r\}$, and let $z = x_i$ or $z = x_i^{-1}$. Before passing to the quotient, put

$$\mathcal{M}_z^0 = \operatorname{span}\{p \otimes u \otimes \xi : p \in \mathfrak{A}_{d-1} + z\mathfrak{A}_{d-1},\ u \in \mathfrak{H},\ \xi \in \mathbb{C}^k\},$$

and let $\mathcal{M}_z$ be the closure of its image in $\mathcal{H}_\Theta$. Left multiplication by z defines a bijection

$$L_z^0 : \mathcal{M}_{z^{-1}}^0 \to \mathcal{M}_z^0, \qquad L_z^0(p \otimes u \otimes \xi) = zp \otimes u \otimes \xi,$$

with inverse $L_{z^{-1}}^0$. For $p, q \in \mathfrak{A}_{d-1} + z^{-1}\mathfrak{A}_{d-1}$,

$$\begin{aligned}&\langle L_z^0(p \otimes u \otimes \xi), L_z^0(q \otimes v \otimes \eta)\rangle_\Theta \\ &\quad= \langle \Theta(q^*z^*zp \otimes v^{-1}u)\xi, \eta\rangle \\ &\quad= \langle \Theta(q^*p \otimes v^{-1}u)\xi, \eta\rangle \\ &\quad= \langle p \otimes u \otimes \xi, q \otimes v \otimes \eta\rangle_\Theta.\end{aligned}$$

By sesquilinearity, this identity holds throughout $\mathcal{M}^0_{z^{-1}}$. Thus L^0_z descends to the quotient and extends to a unitary

$$L_z : \mathcal{M}_{z^{-1}} \to \mathcal{M}_z, \qquad L_z^{-1} = L_{z^{-1}}.$$

Fix $j \in \{1, \dots, s\}$. Before passing to the quotient, let

$$\mathcal{M}^0_j = \operatorname{span}\{ap \otimes u \otimes \xi : a \in \mathcal{A}_j,\ p \in \mathfrak{A}_{d-1},\ u \in \mathfrak{H},\ \xi \in \mathbb{C}^k\}.$$

Define $L^0_j : \mathcal{A}_j \to \mathcal{L}(\mathcal{M}^0_j)$, where $\mathcal{L}(\mathcal{M}^0_j)$ denotes the linear maps from $\mathcal{M}^0_j$ to itself, by

$$L^0_j(a)\left(\sum_\ell b_\ell p_\ell \otimes u_\ell \otimes \xi_\ell\right) = \sum_\ell a b_\ell p_\ell \otimes u_\ell \otimes \xi_\ell.$$

One has

$$\langle L^0_j(a)\zeta, \omega\rangle_\Theta = \langle \zeta, L^0_j(a^*)\omega\rangle_\Theta.$$

Put $C = \|a\|$. Since $C^2 - a^*a \succeq 0$, there is $b \in \mathcal{A}_j$ such that $C^2 - a^*a = b^*b$. It follows that

$$\|L^0_j(a)\zeta\|_\Theta \le \|a\|\,\|\zeta\|_\Theta, \qquad \zeta \in \mathcal{M}^0_j.$$

Thus $L^0_j(a)$ descends to the quotient and extends to a bounded operator $L_j(a)$ on

$$\mathcal{M}_j = \overline{\mathcal{M}^0_j/(\mathcal{M}^0_j \cap \mathcal{N}_\Theta)}.$$

The map $L_j : \mathcal{A}_j \to B(\mathcal{M}_j)$ is a unital $*$-representation.

We pause here to prove the finite-dimensional alternative when $\mathfrak{H} = \{\mathrm{e}\}$. In this case, $\mathcal{H}_\Theta$ is a quotient of

$$\mathfrak{A}_d \otimes \mathbb{C}^k.$$

Since $\dim \mathfrak{A}_d = N(d)$, we have

$$\dim \mathcal{H}_\Theta \le kN(d).$$

Set

$$H = kN(d) \qquad \text{and} \qquad D = \operatorname{lcm}\Big\{1, 2, \dots, H, \dim \mathcal{A}_1, \dots, \dim \mathcal{A}_s\Big\}. \tag{3.7}$$

In particular,

$$D \ge H \ge \dim \mathcal{H}_\Theta.$$

Choose a Hilbert space $\mathcal{K}_\Theta$ such that

$$\mathcal{E}_\Theta = \mathcal{H}_\Theta \oplus \mathcal{K}_\Theta \qquad \text{and} \qquad \dim \mathcal{E}_\Theta = D.$$

For each $i \in \{1, \dots, r\}$, the local left-multiplication operator

$$L_{x_i} : \mathcal{M}_{x_i^{-1}} \longrightarrow \mathcal{M}_{x_i}$$

is unitary. Consequently,

$$\dim \mathcal{M}_{x_i^{-1}} = \dim \mathcal{M}_{x_i},$$

and hence

$$\dim(\mathcal{E}_\Theta \ominus \mathcal{M}_{x_i^{-1}}) = \dim(\mathcal{E}_\Theta \ominus \mathcal{M}_{x_i}).$$

Thus L_{x_i} extends to a unitary

$$U_i \in B(\mathcal{E}_\Theta).$$

Define

$$\pi_{x_i} : \mathbb{C}[x_i, x_i^{-1}] \longrightarrow B(\mathcal{E}_\Theta), \qquad \pi_{x_i}(x_i) = U_i.$$

We next extend the local representations associated with the finite-dimensional factors. Fix $j \in \{1, \dots, s\}$ and put

$$m_j = \dim \mathcal{M}_j.$$

Then

$$0 \leq m_j \leq \dim \mathcal{H}_\Theta \leq H.$$

Suppose first that $m_j > 0$. Since $m_j \leq H$, the definition of D implies that m_j divides D. Hence

$$D - m_j = \left(\frac{D}{m_j} - 1 \right) m_j.$$

Recall that

$$L_j : \mathcal{A}_j \longrightarrow B(\mathcal{M}_j)$$

is a unital $*$-representation. Let

$$\widehat{\rho}_j = L_j^{\oplus (D/m_j - 1)}$$

be the direct sum of $D/m_j - 1$ copies of L_j. This is a unital $*$-representation on a Hilbert space of dimension $D - m_j$. After identifying that Hilbert space unitarily with $\mathcal{E}_\Theta \ominus \mathcal{M}_j$, we obtain a unital $*$-representation

$$\rho_j : \mathcal{A}_j \longrightarrow B(\mathcal{E}_\Theta \ominus \mathcal{M}_j).$$

Define

$$\pi_j(a) = L_j(a) \oplus \rho_j(a), \qquad a \in \mathcal{A}_j,$$

with respect to the orthogonal decomposition

$$\mathcal{E}_\Theta = \mathcal{M}_j \oplus (\mathcal{E}_\Theta \ominus \mathcal{M}_j).$$

Suppose instead that $m_j = 0$. Equip $\mathcal{A}_j$ with the Hilbert-space inner product

$$\langle a, b \rangle_j = \tau_j(b^* a)$$

and let

$$\lambda_j : \mathcal{A}_j \longrightarrow B(\mathcal{A}_j)$$

denote the left regular representation,

$$\lambda_j(a) b = ab, \qquad a, b \in \mathcal{A}_j.$$

The faithfulness of τ_j implies that this inner product is positive definite, and the tracial property gives

$$\langle \lambda_j(a) b, c \rangle_j = \langle b, \lambda_j(a^*) c \rangle_j.$$

Thus λ_j is a unital $*$-representation on a Hilbert space of dimension $\dim \mathcal{A}_j$. Since $\dim \mathcal{A}_j$ divides D, the direct sum

$$\lambda_j^{\oplus (D/\dim \mathcal{A}_j)}$$

is a unital $*$-representation on a Hilbert space of dimension D. After identifying this Hilbert space unitarily with $\mathcal{E}_\Theta$, let

$$\pi_j : \mathcal{A}_j \longrightarrow B(\mathcal{E}_\Theta)$$

denote the resulting representation.

By the universal property of the algebraic unital free product, the representations

$$\pi_{x_1}, \dots, \pi_{x_r}, \pi_1, \dots, \pi_s$$

determine a unital $*$-representation

$$\pi_\Theta : \mathfrak{A} \longrightarrow B(\mathcal{E}_\Theta). \tag{3.8}$$

By construction, π_Θ extends each of the local left-multiplication operators used above. Define

$$V_\Theta : \mathbb{C}^k \to \mathcal{E}_\Theta, \qquad V_\Theta \xi = [\mathtt{1} \otimes \xi].$$

Induction on extent, using the local left-multiplication operators, gives

$$\pi_\Theta(q) V_\Theta \xi = [q \otimes \xi], \qquad q \in \mathfrak{A}_d.$$

For inverse generators, use the fact that U_i^* extends $L_{x_i^{-1}}$, since U_i extends the unitary L_{x_i} between its initial and final subspaces. Consequently, for $p, q \in \mathfrak{A}_d$,

$$\begin{aligned} \langle V_\Theta^* \pi_\Theta(p^* q) V_\Theta \xi, \eta \rangle &= \langle [q \otimes \xi], [p \otimes \eta] \rangle_\Theta \\ &= \langle \Theta(p^* q) \xi, \eta \rangle. \end{aligned}$$

Lemma 2.3 now gives

$$\Theta(a) = V_\Theta^* \pi_\Theta(a) V_\Theta, \qquad a \in \mathfrak{A}_{2d}.$$

Also $V_\Theta^* V_\Theta = \Theta(\mathtt{1})$. This proves the finite-dimensional alternative, including the stated dimension bound.

We now return to the general case and construct the canonical amplified realization for an arbitrary group $\mathfrak{H}$, including the trivial group. Set

$$\mathcal{E} = \widetilde{\mathcal{H}} = \mathcal{H}_\Theta \otimes \ell^2(\mathbb{N}), \qquad \widetilde{\tau}(g) = \tau(g) \otimes I_{\ell^2(\mathbb{N})},$$

and identify $\mathcal{H}_\Theta$ with its first summand, by $h \mapsto h \otimes e_0$. Since $\tau(g)$ changes only the group coordinate, it maps each of $\mathcal{M}_z$ and $\mathcal{M}_{z^{-1}}$ onto itself. These spaces therefore reduce $\tau(\mathfrak{H})$. For $p \in \mathfrak{A}_{d-1} + z^{-1}\mathfrak{A}_{d-1}$, $u, g \in \mathfrak{H}$ and $\xi \in \mathbb{C}^k$,

$$\begin{aligned} L_z \tau(g)[p \otimes u \otimes \xi] &= [zp \otimes gu \otimes \xi] \\ &= \tau(g) L_z [p \otimes u \otimes \xi]. \end{aligned}$$

By linearity and continuity, this proves the intertwining identity

$$L_z \tau(g)|_{\mathcal{M}_{z^{-1}}} = \tau(g)|_{\mathcal{M}_z} L_z, \qquad g \in \mathfrak{H}.$$

Hence, by Lemma 3.3(iii), for each $i \in \{1, \dots, r\}$ there is a unitary operator U_i on $\widetilde{\mathcal{H}}$ extending L_{x_i} and satisfying

$$U_i \widetilde{\tau}(g) = \widetilde{\tau}(g) U_i, \qquad g \in \mathfrak{H}. \tag{3.9}$$

Since U_i extends the unitary $L_{x_i} : \mathcal{M}_{x_i^{-1}} \to \mathcal{M}_{x_i}$, its adjoint U_i^* extends $L_{x_i^{-1}}$. As F_r is the free group on $x_1, \ldots, x_r$, the unitaries $U_1, \ldots, U_r$ determine a unitary representation of F_r that commutes with $\widetilde{\tau}$.

Similarly, $\tau(g)$ maps $\mathcal{M}_j$ onto itself, so $\mathcal{M}_j$ reduces τ. For $a, b \in \mathcal{A}_j$, $p \in \mathfrak{A}_{d-1}$, $u, g \in \mathfrak{H}$ and $\xi \in \mathbb{C}^k$,

$$\begin{aligned} L_j(a)\tau(g)[bp \otimes u \otimes \xi] &= [abp \otimes gu \otimes \xi] \\ &= \tau(g)L_j(a)[bp \otimes u \otimes \xi]. \end{aligned}$$

Again, linearity and continuity give

$$L_j(a)\tau(g)|_{\mathcal{M}_j} = \tau(g)|_{\mathcal{M}_j} L_j(a), \qquad a \in \mathcal{A}_j, \quad g \in \mathfrak{H}.$$

By Lemma 3.3(iv), there is a unital $*$-representation

$$\rho_j : \mathcal{A}_j \longrightarrow B(\widetilde{\mathcal{H}} \ominus \mathcal{M}_j)$$

such that the unital $*$-representation $\sigma_j = L_j \oplus \rho_j$ commutes with $\widetilde{\tau}$:

$$\sigma_j(a)\widetilde{\tau}(g) = \widetilde{\tau}(g)\sigma_j(a), \qquad a \in \mathcal{A}_j, \quad g \in \mathfrak{H}. \tag{3.10}$$

The universal property of the algebraic unital free product gives a unital $*$-representation

$$\sigma : \mathfrak{A} \longrightarrow B(\widetilde{\mathcal{H}})$$

extending the representations determined by the U_i and the σ_j. It commutes with $\widetilde{\tau}$ by (3.9) and (3.10). Consequently,

$$\pi(a \otimes g) = \sigma(a)\widetilde{\tau}(g) \tag{3.11}$$

defines a unital $*$-representation of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$, with $\pi(\mathtt{1} \otimes g) = \tau_\Theta(g) \otimes I_{\ell^2(\mathbb{N})}$. Define

$$V : \mathbb{C}^k \longrightarrow \widetilde{\mathcal{H}}, \qquad V\xi = [\mathtt{1} \otimes \mathrm{e} \otimes \xi] \otimes e_0.$$

This map is bounded, since

$$\|V\xi\|^2 = \langle \Theta(\mathtt{1}, \mathrm{e})\xi, \xi \rangle \leq \|\Theta(\mathtt{1}, \mathrm{e})\| \, \|\xi\|^2.$$

Thus the realization satisfies (3.3).

We verify the vector identity (3.4). In the following calculation the kernel classes are viewed in the first summand of $\widetilde{\mathcal{H}}$. For the identity word,

$$\widetilde{\tau}(u)V\xi = [\mathtt{1} \otimes u \otimes \xi].$$

Suppose that the identity (3.4) has been established for words of extent less than $m \leq d$. A nonempty reduced word p of extent m can be written as $p = zq$, where $z = x_i$ or x_i^{-1} and $q \in \mathcal{W}_{m-1}$, or as $p = aq$, where $a \in \mathcal{B}_j \subseteq \mathcal{A}_j$ and $q \in \mathcal{W}_{m-1}$. In the first case, $[q \otimes u \otimes \xi] \in \mathcal{M}_{z^{-1}}$, and $\sigma(z)$ extends L_z. In the second case, $[q \otimes u \otimes \xi] \in \mathcal{M}_j$, and $\sigma(a)$ extends $L_j(a)$. Thus, respectively,

$$\begin{aligned} \sigma(zq)\widetilde{\tau}(u)V\xi &= \sigma(z)[q \otimes u \otimes \xi] = [zq \otimes u \otimes \xi], \\ \sigma(aq)\widetilde{\tau}(u)V\xi &= \sigma(a)[q \otimes u \otimes \xi] = [aq \otimes u \otimes \xi], \end{aligned}$$

completing the induction. By linearity and equation (3.11),

$$\pi(p \otimes u)V\xi = \sigma(p)\widetilde{\tau}(u)V\xi = [p \otimes u \otimes \xi], \qquad p \in \mathfrak{A}_d, \quad u \in \mathfrak{H},$$

which is (3.4) under the first-summand identification.

Finally, for $p, q \in \mathfrak{A}_d$, $u, v \in \mathfrak{H}$ and $\xi, \eta \in \mathbb{C}^k$,

$$\begin{aligned}\langle V^*\pi\Big((p^*q)\otimes(u^{-1}v)\Big)V\xi,\eta\rangle &= \langle \sigma(q)\tilde{\tau}(v)V\xi, \sigma(p)\tilde{\tau}(u)V\eta\rangle \\ &= \langle [q\otimes v\otimes\xi], [p\otimes u\otimes\eta]\rangle_\Theta \\ &= \langle \Theta(p^*q, u^{-1}v)\xi,\eta\rangle,\end{aligned}$$

proving (3.2) for all $u, v \in \mathfrak{H}$. Taking $p = q = \mathtt{1}$ and $u = v = \mathrm{e}$ gives $V^*V = \Theta(\mathtt{1}, \mathrm{e})$. If $\mathfrak{H}$ is at most countable, then $\mathcal{H}_\Theta$ and its countable amplification $\mathcal{E}$ are separable. □

4. Proofs of the main results

In this section we prove Theorem 1.1 and Corollaries 1.2 and 1.7. The proofs of Theorem 1.4 and Corollary 1.5 are given in Section 5.

4.1. **Proof of Theorem 1.1.** Enumerate $\mathcal{W}_\mathtt{d}$,

$$\mathcal{W}_\mathtt{d} = \{q_1, \dots, q_{N(\mathtt{d})}\}.$$

Let $\mathfrak{A}^*_{2\mathtt{d}}$ be the set of linear functionals on $\mathfrak{A}_{2\mathtt{d}}$. Define

$$\mathcal{O}_\mathtt{d} = \left\{T^\lambda = \Big(\lambda(q_\alpha^* q_\beta)\Big)_{\alpha,\beta=1}^{N(\mathtt{d})} \ : \ \lambda \in \mathfrak{A}^*_{2\mathtt{d}}\right\} \subseteq M_{N(\mathtt{d})}(\mathbb{C}).$$

By Lemma 2.3, the map $\lambda \mapsto T^\lambda$ is injective. The space $\mathcal{O}_\mathtt{d}$ is an operator system. Indeed, it is self-adjoint because

$$(T^\lambda)^* = T^{\lambda^\sharp}, \qquad \lambda^\sharp(a) = \overline{\lambda(a^*)},$$

and it contains the identity because Lemma 2.1 gives

$$T^\tau = I_{N(\mathtt{d})}.$$

Let

$$A = \sum_{\ell=1}^{N(2\mathtt{d})} A_\ell \otimes r_\ell \in B(\mathcal{H}) \otimes \mathfrak{A}_{2\mathtt{d}}$$

be positive on finite-dimensional representations. Using the fact that the map $\mathfrak{A}^*_{2\mathtt{d}} \ni \lambda \mapsto T^\lambda$ is one-to-one, we obtain a well-defined map $\Phi_A : \mathcal{O}_\mathtt{d} \to B(\mathcal{H})$ defined by

$$\Phi_A(T^\lambda) = (\mathrm{id}_{B(\mathcal{H})} \otimes \lambda)(A) = \sum_{\ell=1}^{N(2\mathtt{d})} A_\ell \lambda(r_\ell).$$

Lemma 4.1. *The map $\Phi_A : \mathcal{O}_\mathtt{d} \to B(\mathcal{H})$ is completely positive.*

Proof. Let

$$X = [T^{\lambda_{\mu,\nu}}]_{\mu,\nu=1}^k \in M_k(\mathcal{O}_\mathtt{d})$$

be positive. Define

$$\Theta : \mathfrak{A}_{2\mathtt{d}} \to M_k(\mathbb{C}), \qquad \Theta(a) = \Big(\lambda_{\mu,\nu}(a)\Big)_{\mu,\nu=1}^k.$$

The positivity of X is precisely the assertion that Θ is a positive truncated kernel. Let $\mathcal{E}_\Theta$, π_Θ and V be as in the finite-dimensional alternative in Theorem 3.1. In particular,

$$\Theta(a) = V^*\pi_\Theta(a)V, \qquad a \in \mathfrak{A}_{2\mathtt{d}}.$$

Under the standard identification $M_k(B(\mathcal{H})) \cong B(\mathcal{H} \otimes \mathbb{C}^k)$,

$$\begin{aligned}\Phi_A^{(k)}(X) &= \sum_{\ell=1}^{N(2\mathtt{d})} A_\ell \otimes \Theta(r_\ell) \\ &= (I_\mathcal{H} \otimes V)^* \left(\sum_{\ell=1}^{N(2\mathtt{d})} A_\ell \otimes \pi_\Theta(r_\ell) \right) (I_\mathcal{H} \otimes V) \\ &= (I_\mathcal{H} \otimes V)^* (\mathrm{id}_{B(\mathcal{H})} \otimes \pi_\Theta)(A)(I_\mathcal{H} \otimes V).\end{aligned}$$

The representation π_Θ is finite-dimensional. Hence the assumed positivity of A implies

$$(\mathrm{id}_{B(\mathcal{H})} \otimes \pi_\Theta)(A) \succeq 0.$$

Therefore $\Phi_A^{(k)}(X) \succeq 0$. Since k was arbitrary, Φ_A is completely positive. □

By Lemma 4.1, Φ_A is completely positive on the operator system $\mathcal{O}_\mathtt{d} \subseteq M_{N(\mathtt{d})}(\mathbb{C})$. By Arveson's Extension Theorem [Pau02, Theorem 7.5], Φ_A extends to a completely positive map

$$\widetilde{\Phi}_A : M_{N(\mathtt{d})}(\mathbb{C}) \to B(\mathcal{H}).$$

Let $E_{\alpha,\beta}$ denote the matrix units in $M_{N(\mathtt{d})}(\mathbb{C})$. The Choi matrix

$$C = \left(\widetilde{\Phi}_A(E_{\alpha,\beta})\right)_{\alpha,\beta=1}^{N(\mathtt{d})} \in M_{N(\mathtt{d})}(B(\mathcal{H}))$$

is positive. As a result, there exist operators

$$B_1, \ldots, B_{N(\mathtt{d})} \in B(\mathcal{H}, \mathcal{E}), \qquad \mathcal{E} = \mathcal{H} \otimes \mathbb{C}^{N(\mathtt{d})},$$

such that

$$\widetilde{\Phi}_A(E_{\alpha,\beta}) = B_\alpha^* B_\beta.$$

Define

$$B = \sum_{\alpha=1}^{N(\mathtt{d})} B_\alpha \otimes q_\alpha \in B(\mathcal{H}, \mathcal{E}) \otimes \mathfrak{A}_\mathtt{d}.$$

Then

$$B^*B = \sum_{\alpha,\beta=1}^{N(\mathtt{d})} B_\alpha^* B_\beta \otimes q_\alpha^* q_\beta.$$

Let $\lambda \in \mathfrak{A}_{2\mathtt{d}}^*$. Applying $\mathrm{id} \otimes \lambda$ gives

$$\begin{aligned}(\mathrm{id} \otimes \lambda)(B^*B) &= \sum_{\alpha,\beta=1}^{N(\mathtt{d})} B_\alpha^* B_\beta \lambda(q_\alpha^* q_\beta) \\ &= \sum_{\alpha,\beta=1}^{N(\mathtt{d})} \widetilde{\Phi}_A(E_{\alpha,\beta}) \lambda(q_\alpha^* q_\beta) \\ &= \widetilde{\Phi}_A(T^\lambda) \\ &= \Phi_A(T^\lambda) \\ &= (\mathrm{id} \otimes \lambda)(A).\end{aligned}$$

Since the linear functionals on $\mathfrak{A}_{2\mathtt{d}}$ separate points, the maps $\mathrm{id} \otimes \lambda$ separate points of the algebraic tensor product $B(\mathcal{H}) \otimes \mathfrak{A}_{2\mathtt{d}}$. Hence $A = B^*B$.

To establish the bound given in Remark 1.3, in the case that $\mathcal{H}$ has dimension K, it suffices to check that Φ_A is K-positive [Pau02, Theorem 6.1]. In that case, take $k = K$; the representation π_Θ is on a space of dimension at most D from equation (3.7).

4.2. **Proof of Corollary 1.2.** For each finite group $\mathbb{G}_j$, take

$$\mathcal{A}_j = \mathbb{C}[\mathbb{G}_j]$$

and equip $\mathbb{C}[\mathbb{G}_j]$ with its canonical trace

$$\tau_j(g) = \begin{cases} 1, & g = \mathtt{1}, \\ 0, & g \neq \mathtt{1}. \end{cases}$$

Then

$$\mathbb{C}[\mathbb{G}_j] = \mathbb{C}\mathtt{1} \oplus \mathbb{C}[\mathbb{G}_j]^\circ, \qquad \mathbb{C}[\mathbb{G}_j]^\circ = \ker \tau_j,$$

and the set

$$\mathbb{G}_j \setminus \{\mathtt{1}\}$$

is an orthonormal basis of $\mathbb{C}[\mathbb{G}_j]^\circ$ with respect to the inner product

$$\langle a, b \rangle_j = \tau_j(b^* a).$$

Moreover,

$$\mathbb{C}[\mathbb{G}] \;\cong\; \mathbb{C}[F_r] * \mathbb{C}[\mathbb{G}_1] * \cdots * \mathbb{C}[\mathbb{G}_s].$$

A unitary representation of $\mathbb{G}$ is the same thing as a unital $*$-representation of the group algebra $\mathbb{C}[\mathbb{G}]$. Hence the positivity assumption in the corollary is precisely the positivity assumption appearing in Theorem 1.1. With these choices of bases, $\mathfrak{A}_\mathtt{d}$ is precisely the span of the group elements of extent at most $\mathtt{d}$. Applying Theorem 1.1 gives a Hilbert space $\mathcal{E}$ and an element

$$B \in B(\mathcal{H}, \mathcal{E}) \otimes \mathfrak{A}_\mathtt{d}$$

such that

$$A = B^*B.$$

4.3. **Proof of Corollary 1.7.** Since $\mathfrak{H}$ is finite, its group algebra is a finite-dimensional C^*-algebra. Writing $\widehat{\mathfrak{H}}$ for the set of irreducible unitary representations ρ of $\mathfrak{H}$, and $n_\rho = \dim \rho$, the Fourier transform gives a $*$-isomorphism

$$\mathbb{C}[\mathfrak{H}] \cong \bigoplus_{\rho \in \widehat{\mathfrak{H}}} M_{n_\rho}(\mathbb{C}), \qquad y \longmapsto \bigoplus_{\rho \in \widehat{\mathfrak{H}}} \rho(y).$$

Write

$$A = \sum_{r \in \mathcal{W}_{2\mathtt{d}}} \sum_{y \in \mathfrak{H}} A_{r,y} \otimes r \otimes y.$$

For each $\rho \in \widehat{\mathfrak{H}}$, define

$$A_\rho = \sum_{r \in \mathcal{W}_{2\mathtt{d}}} \left(\sum_{y \in \mathfrak{H}} A_{r,y} \otimes \rho(y) \right) \otimes r \in B(\mathcal{H}) \otimes M_{n_\rho}(\mathbb{C}) \otimes \mathfrak{A}_{2\mathtt{d}}.$$

Let $\sigma : \mathfrak{A} \to B(\mathcal{F})$ be a finite-dimensional unital $*$-representation. Then

$$\pi_{\sigma,\rho}(a \otimes y) = \sigma(a) \otimes \rho(y)$$

is a finite-dimensional unital $*$-representation of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$. After interchanging the tensor factors $\mathcal{F}$ and $\mathbb{C}^{n_\rho}$, the assumed positivity of A gives

$$(\mathrm{id} \otimes \sigma)(A_\rho) \succeq 0.$$

That is, A_ρ satisfies the hypotheses of Theorem 1.1. Put $\mathcal{E} = \mathcal{H} \otimes \mathbb{C}^{N(\mathtt{d})}$. Applying Theorem 1.1 with coefficient Hilbert space $\mathcal{H} \otimes \mathbb{C}^{n_\rho}$, and identifying

$$(\mathcal{H} \otimes \mathbb{C}^{n_\rho}) \otimes \mathbb{C}^{N(\mathtt{d})} \cong \mathcal{E} \otimes \mathbb{C}^{n_\rho},$$

we obtain

$$B_\rho = \sum_{p \in \mathcal{W}_{\mathtt{d}}} B_{\rho,p} \otimes p \in B(\mathcal{H} \otimes \mathbb{C}^{n_\rho}, \mathcal{E} \otimes \mathbb{C}^{n_\rho}) \otimes \mathfrak{A}_{\mathtt{d}}$$

such that $A_\rho = B_\rho^* B_\rho$.

The Fourier transform induces a linear isomorphism

$$B(\mathcal{H}, \mathcal{E}) \otimes \mathbb{C}[\mathfrak{H}] \longrightarrow \bigoplus_{\rho \in \widehat{\mathfrak{H}}} B(\mathcal{H} \otimes \mathbb{C}^{n_\rho}, \mathcal{E} \otimes \mathbb{C}^{n_\rho}).$$

Consequently, for each $p \in \mathcal{W}_{\mathtt{d}}$, there are unique operators $B_{p,y} \in B(\mathcal{H}, \mathcal{E})$, indexed by $y \in \mathfrak{H}$, such that

$$B_{\rho,p} = \sum_{y \in \mathfrak{H}} B_{p,y} \otimes \rho(y), \qquad \rho \in \widehat{\mathfrak{H}}.$$

Define

$$B = \sum_{p \in \mathcal{W}_{\mathtt{d}}} \sum_{y \in \mathfrak{H}} B_{p,y} \otimes p \otimes y \in B(\mathcal{H}, \mathcal{E}) \otimes \mathfrak{A}_{\mathtt{d}} \otimes \mathbb{C}[\mathfrak{H}].$$

With respect to the Fourier transform, the ρ-block of B^*B is

$$B_\rho^* B_\rho = A_\rho,$$

which is exactly the ρ-block of A. Hence

$$A = B^* B. \tag{4.1}$$

Finally, suppose $\dim \mathcal{H} = K$, and set

$$D_{\mathfrak{H}} = \operatorname{lcm}\Big\{1, 2, \dots, K|\mathfrak{H}|N(\mathtt{d}), \dim \mathcal{A}_1, \dots, \dim \mathcal{A}_s\Big\}.$$

For $\rho \in \widehat{\mathfrak{H}}$, Remark 1.3 shows that it suffices to test A_ρ on representations of $\mathfrak{A}$ of dimension at most

$$D_\rho = \operatorname{lcm}\Big\{1, 2, \dots, Kn_\rho N(\mathtt{d}), \dim \mathcal{A}_1, \dots, \dim \mathcal{A}_s\Big\}.$$

Since $\mathtt{d} \geq 1$, we have $\dim \mathcal{A}_j \leq N(\mathtt{d})$, and hence

$$D_\rho = \operatorname{lcm}\{1, \dots, Kn_\rho N(\mathtt{d})\}.$$

Also $n_\rho^2 \leq |\mathfrak{H}|$, so

$$n_\rho D_\rho = \operatorname{lcm}\{n_\rho, 2n_\rho, \dots, Kn_\rho^2 N(\mathtt{d})\} \quad \text{divides} \quad D_{\mathfrak{H}}.$$

The representations $\pi_{\sigma,\rho}$ needed in the preceding argument therefore have dimension at most $D_{\mathfrak{H}}$. This proves the asserted testing bound.

5. Proofs for tensor products

We prove Theorem 1.4 and deduce Corollary 1.5. We also provide an example delineating the distinction between spatial and maximal tensor-product positivity.

5.1. Proof of Theorem 1.4.

Proof. By (1.4), since $B(\mathcal{H}) \otimes_{\max} C_u^*(\mathfrak{A}) \otimes_{\max} C^*(\mathfrak{H})$ is a C^*-algebra, there exists $\varepsilon > 0$ such that $A \succeq \varepsilon I$.

We use the separation and GNS argument of [HM04, Propositions 3.3 and 4.1], with the truncated representation construction of Theorem 3.1. Put

$$\mathcal{V}_j = B(\mathcal{H}) \otimes \mathfrak{A}_j \otimes \mathbb{C}[\mathfrak{H}] \quad (j = \mathtt{d}, 2\mathtt{d}), \qquad \Sigma_{\mathtt{d}} = \left\{ \sum_{k=1}^{m} F_k^* F_k : F_k \in \mathcal{V}_{\mathtt{d}},\ m \geq 1 \right\},$$

where tensor products without subscripts are algebraic.

First, I is an algebraic order unit for $\Sigma_{\mathtt{d}}$ in $(\mathcal{V}_{2\mathtt{d}})_{\mathrm{sa}}$, i.e.,

$$\text{for every } Z = Z^* \in \mathcal{V}_{2\mathtt{d}} \text{ there is } R > 0 \text{ with } RI \pm Z \in \Sigma_{\mathtt{d}}. \tag{5.1}$$

To see this, let $F = b \otimes w \otimes g$, with $w = w_1 \cdots w_m \in \mathcal{W}_{\mathtt{d}}$ a reduced word. Put $c_i = \|w_i\|^2$ and $t_i = (c_i \mathtt{1} - w_i^* w_i)^{1/2}$. Each t_i belongs to the same finite-dimensional factor as w_i, or is zero when w_i belongs to a free-group factor. The telescoping identity

$$\left(\prod_i c_i\right) \mathtt{1} - w^* w = \sum_i \left(\prod_{h<i} c_h\right) (t_i w_{i+1} \cdots w_m)^* (t_i w_{i+1} \cdots w_m)$$

shows that the difference on the left is a sum of squares of elements of $\mathfrak{A}_{\mathtt{d}}$. Since g is unitary and $\|b\|^2 I - b^* b$ is a square in $B(\mathcal{H})$, it follows that $RI - F^* F \in \Sigma_{\mathtt{d}}$ for some $R > 0$. The identity

$$m \sum_i F_i^* F_i - \left(\sum_i F_i\right)^* \left(\sum_i F_i\right) = \sum_{i<j} (F_i - F_j)^* (F_i - F_j)$$

extends this assertion to every $F \in \mathcal{V}_{\mathtt{d}}$. Polarization and Lemma 2.3 now give (5.1).

Fix $0 < \delta < \varepsilon$ and suppose that $A - \delta I \notin \Sigma_{\mathtt{d}}$. By algebraic order-unit separation, as in [HM04, Proposition 3.3], there is a complex-linear functional Λ on $\mathcal{V}_{2\mathtt{d}}$ such that

$$\Lambda(F^*F) \geq 0 \quad (F \in \mathcal{V}_{\mathtt{d}}), \qquad \Lambda(I) = 1, \qquad \Lambda(A - \delta I) \leq 0. \tag{5.2}$$

Equip $\mathcal{V}_{\mathtt{d}}$ with the positive semidefinite form $\langle F, G\rangle_\Lambda = \Lambda(G^*F)$, and let $\mathcal{K}_0$ be the Hilbert-space completion of the quotient by its null space. Denote the classes by $[F]$ and put $\xi = [I]$. Left multiplication in the two outer factors gives commuting representations

$$\kappa_0(b)[F] = [(b \otimes \mathtt{1} \otimes \mathrm{e})F], \qquad \upsilon_0(g)[F] = [(I_{\mathcal{H}} \otimes \mathtt{1} \otimes g)F].$$

Here κ_0 is a unital $*$-representation of $B(\mathcal{H})$: its boundedness follows from the square root of $\|b\|^2 I - b^*b$. Also υ_0 is a unitary representation of $\mathfrak{H}$. Set

$$G = \mathcal{U}(B(\mathcal{H})) \times \mathfrak{H}, \qquad \rho(u, g) = \kappa_0(u)\upsilon_0(g),$$

where $\mathcal{U}(B(\mathcal{H}))$ is the group of unitary operators on $\mathcal{H}$. Regard G as a discrete group. Since the unitaries linearly span $B(\mathcal{H})$, the canonical unital $*$-homomorphism

$$\mathbb{C}[G] \longrightarrow B(\mathcal{H}) \otimes \mathbb{C}[\mathfrak{H}], \qquad (u, g) \longmapsto u \otimes g,$$

is surjective.

To apply Theorem 3.1, take $k = 1$, take the monoid to be G itself, and define the complex-linear map

$$\Theta : \mathfrak{A}_{2\mathtt{d}} \otimes \mathbb{C}[G] \longrightarrow \mathbb{C}, \qquad \Theta(a, (u, g)) = \Lambda(u \otimes a \otimes g).$$

The positivity of Λ implies that Θ is a positive truncated kernel. Under the canonical identification of $\mathcal{H}_\Theta$ with $\mathcal{K}_0$, the representation τ_Θ is ρ.

Put

$$\widetilde{\mathcal{K}} = \mathcal{K}_0 \otimes \ell^2(\mathbb{N}), \qquad \widetilde{\rho}(s) = \rho(s) \otimes I_{\ell^2(\mathbb{N})}, \qquad s \in G.$$

Identify $\mathcal{K}_0$ with the first summand of $\widetilde{\mathcal{K}}$ and continue to write $\xi = [I]$ for the vector $[I] \otimes e_0$. The canonical realization in Theorem 3.1 provides a unital $*$-representation

$$\pi : \mathfrak{A} \otimes \mathbb{C}[G] \longrightarrow B(\widetilde{\mathcal{K}})$$

such that

$$\Theta(p^*q, s^{-1}t) = \Big\langle \pi\Big((p^*q) \otimes (s^{-1}t)\Big)\xi, \xi \Big\rangle, \qquad p, q \in \mathfrak{A}_{\mathtt{d}}, \quad s, t \in G,$$

and

$$\pi(\mathtt{1} \otimes s) = \widetilde{\rho}(s), \qquad s \in G.$$

Let $e_G = (I_{\mathcal{H}}, \mathrm{e})$ be the identity of G, and define

$$\sigma : \mathfrak{A} \longrightarrow B(\widetilde{\mathcal{K}}), \qquad \sigma(a) = \pi(a \otimes e_G).$$

Since the two tensor factors commute in $\mathfrak{A} \otimes \mathbb{C}[G]$, we have

$$\sigma(a)\widetilde{\rho}(u, g) = \widetilde{\rho}(u, g)\sigma(a), \qquad a \in \mathfrak{A}, \quad (u, g) \in G.$$

Furthermore, (3.4) gives

$$\sigma(q)\widetilde{\rho}(u, g)\xi = [u \otimes q \otimes g], \qquad q \in \mathfrak{A}_{\mathtt{d}}, \quad (u, g) \in G,$$

where the class on the right is viewed in the first summand of $\widetilde{\mathcal{K}}$.

Set

$$\kappa(b) = \kappa_0(b) \otimes I_{\ell^2(\mathbb{N})}, \qquad \upsilon(g) = \upsilon_0(g) \otimes I_{\ell^2(\mathbb{N})}, \qquad b \in B(\mathcal{H}), \quad g \in \mathfrak{H}.$$

These representations have commuting ranges, and

$$\widetilde{\rho}(u, g) = \kappa(u)\upsilon(g) = \upsilon(g)\kappa(u), \qquad (u, g) \in G.$$

Since σ commutes with $\widetilde{\rho}$, taking (u, e) and $(I_{\mathcal{H}}, g)$ shows that $\sigma(a)$ commutes with every $\kappa(u)$, u unitary, and every $\upsilon(g)$. The unitaries linearly span $B(\mathcal{H})$, so $\sigma(\mathfrak{A})$ commutes with $\kappa(B(\mathcal{H}))$ and $\upsilon(\mathfrak{H})$.

The representation σ extends to $C^*_u(\mathfrak{A})$, and υ extends to $C^*(\mathfrak{H})$; their ranges still commute with each other and with the range of κ. By the universal property of the maximal tensor product,

$$\Pi(b \otimes a \otimes g) = \kappa(b)\sigma(a)\upsilon(g)$$

therefore extends to a unital $*$-representation of $B(\mathcal{H}) \otimes_{\max} C^*_u(\mathfrak{A}) \otimes_{\max} C^*(\mathfrak{H})$ on $\widetilde{\mathcal{K}}$.

By linearity in the unitary coefficient u, the vector identity supplied by the theorem becomes

$$\sigma(q)\kappa(b)\upsilon(g)\xi = [b \otimes q \otimes g], \qquad b \in B(\mathcal{H}), \quad q \in \mathfrak{A}_{\mathtt{d}}, \quad g \in \mathfrak{H}.$$

In particular, for $b, c \in B(\mathcal{H})$, $p, q \in \mathfrak{A}_{\mathtt{d}}$ and $g, h \in \mathfrak{H}$,

$$\begin{aligned}
&\left\langle \Pi(b^*c \otimes p^*q \otimes g^{-1}h)\xi, \xi \right\rangle \\
&\qquad = \left\langle \kappa(c)\sigma(q)\upsilon(h)\xi, \kappa(b)\sigma(p)\upsilon(g)\xi \right\rangle \\
&\qquad = \left\langle [c \otimes q \otimes h], [b \otimes p \otimes g] \right\rangle_{\Lambda} \\
&\qquad = \Lambda(b^*c \otimes p^*q \otimes g^{-1}h).
\end{aligned}$$

Taking $b = I_{\mathcal{H}}$ and $g = \mathrm{e}$, and using Lemma 2.3, shows that the elements $b^*c \otimes p^*q \otimes g^{-1}h$ span $\mathcal{V}_{2\mathtt{d}}$. Hence

$$\Lambda(Z) = \langle \Pi(Z)\xi, \xi \rangle, \qquad Z \in \mathcal{V}_{2\mathtt{d}}.$$

Since $\|\xi\|^2 = \Lambda(I) = 1$, the inequality $A \succeq \varepsilon I$ in the maximal tensor product implies

$$\Lambda(A - \delta I) = \langle \Pi(A - \delta I)\xi, \xi \rangle \geq (\varepsilon - \delta)\|\xi\|^2 = \varepsilon - \delta > 0,$$

contrary to (5.2). Therefore $A - \delta I \in \Sigma_{\mathtt{d}}$.

Because $A - \delta I \in \Sigma_{\mathtt{d}}$,

$$A - \delta I = \sum_{j=1}^{m} F_j^* F_j, \qquad F_j \in \mathcal{V}_{\mathtt{d}}.$$

Stacking the F_j in a column produces a finite polynomial with coefficients in $B(\mathcal{H}, \mathcal{H}^{\oplus m})$ whose square is $A - \delta I$. Appending the constant row $\sqrt{\delta} I_{\mathcal{H}}$ gives

$$B = \begin{bmatrix} F_1 \\ \vdots \\ F_m \\ \sqrt{\delta} I \end{bmatrix}, \qquad B^*B = A.$$

Every row has $\mathfrak{A}$-extent at most $\mathtt{d}$, and their group supports have finite union. Thus, with $n = m + 1$, this is a factorization of the asserted form.

For the final assertion, let $\mathcal{F}$ be the finite group support of the factor B. The common-left-denominator argument from Lemma 3.2 gives $c \in \mathcal{Y}$ with $c\mathcal{F} \subseteq \mathcal{Y}$. Replacing every group label g by cg leaves the square unchanged, because $(cg)^{-1}(ch) = g^{-1}h$. Enlarging M' if necessary gives $c\mathcal{F} \subseteq \mathcal{Y}_{\leq M'}$. □

5.2. **Proof of Corollary 1.5.** It turns out, building on work of Kirchberg [Kir93, Kir94], results of Pisier [Pis96, Theorems 0.1 and 0.2] say that the min and max tensor products agree in the context of Corollary 1.5 allowing for an application of Theorem 1.4. The Hilbert spaces in [Pis96] are assumed separable, a restriction that Lemma 5.1 below says can be relaxed in our setting.

Lemma 5.1. *If C is a unital C^*-algebra such that*

$$B(\ell^2) \otimes_{\min} C = B(\ell^2) \otimes_{\max} C,$$

then, for every Hilbert space $\mathcal{H}$, without any separability assumption,

$$B(\mathcal{H}) \otimes_{\min} C = B(\mathcal{H}) \otimes_{\max} C.$$

Proof. The case $\mathcal{H} = \{0\}$ is immediate, so assume that $\mathcal{H} \neq \{0\}$. Let

$$x = \sum_{j=1}^{m} T_j \otimes c_j \in B(\mathcal{H}) \otimes C,$$

where the tensor product in this display is algebraic, and put

$$\mathcal{B}_0 = C^*(I_{\mathcal{H}}, T_1, \ldots, T_m) \subseteq B(\mathcal{H}).$$

Since $\mathcal{B}_0$ is separable, it admits a faithful unital $*$-representation

$$\iota : \mathcal{B}_0 \longrightarrow B(\ell^2).$$

The inverse map

$$\iota^{-1} : \iota(\mathcal{B}_0) \longrightarrow \mathcal{B}_0 \subseteq B(\mathcal{H})$$

is a unital $*$-homomorphism. By Arveson's extension theorem [Pau02, Theorem 7.5], it extends to a unital completely positive map

$$\Phi : B(\ell^2) \longrightarrow B(\mathcal{H}).$$

In particular, $\Phi(\iota(b)) = b$ for every $b \in \mathcal{B}_0$.

Set

$$y = \sum_{j=1}^{m} \iota(T_j) \otimes c_j \in B(\ell^2) \otimes C.$$

Functoriality of the maximal tensor product for completely positive contractions [Pis96, Remark 12] implies that

$$\Phi \otimes \mathrm{id}_C : B(\ell^2) \otimes_{\max} C \longrightarrow B(\mathcal{H}) \otimes_{\max} C$$

is a unital completely positive map, and hence is contractive. Since $(\Phi \otimes \mathrm{id}_C)(y) = x$, we obtain

$$\begin{aligned}\|x\|_{B(\mathcal{H})\otimes_{\max} C} &\leq \|y\|_{B(\ell^2)\otimes_{\max} C} \\ &= \|y\|_{B(\ell^2)\otimes_{\min} C} \\ &= \|x\|_{B(\mathcal{H})\otimes_{\min} C}.\end{aligned}$$

The first equality follows from the hypothesis. The last equality follows from injectivity of the minimal tensor product, applied to the faithful maps $\mathcal{B}_0 \hookrightarrow B(\mathcal{H})$ and $\iota : \mathcal{B}_0 \hookrightarrow B(\ell^2)$. Since the reverse norm inequality always holds, the minimal and maximal norms agree on the algebraic tensor product. Passing to the completions proves the assertion. □

Proof of Corollary 1.5. Put

$$C = C_u^*(\mathfrak{A}) = C^*(F_r) * \mathcal{A}_1 * \cdots * \mathcal{A}_s, \qquad D = C \otimes_{\max} C^*(\mathfrak{H}).$$

The finite-dimensional factors $\mathcal{A}_j$ are nuclear. Thus Pisier's theorems for free groups and unital free products [Pis96, Theorems 0.1 and 0.2] give

$$B(\ell^2) \otimes_{\min} C = B(\ell^2) \otimes_{\max} C.$$

Applying Lemma 5.1 gives

$$B(\mathcal{H}) \otimes_{\min} C = B(\mathcal{H}) \otimes_{\max} C. \tag{5.3}$$

Since $\mathfrak{H}$ is amenable, $C^*(\mathfrak{H})$ is nuclear. In particular,

$$D = C \otimes_{\min} C^*(\mathfrak{H}).$$

Using associativity, nuclearity, and (5.3), we obtain

$$\begin{aligned}B(\mathcal{H}) \otimes_{\max} D &= (B(\mathcal{H}) \otimes_{\max} C) \otimes_{\max} C^*(\mathfrak{H}) \\ &= (B(\mathcal{H}) \otimes_{\min} C) \otimes_{\min} C^*(\mathfrak{H}) \\ &= B(\mathcal{H}) \otimes_{\min} \big(C \otimes_{\min} C^*(\mathfrak{H})\big) \\ &= B(\mathcal{H}) \otimes_{\min} D.\end{aligned}$$

By (1.5), there is an $\varepsilon > 0$ such that

$$A \succeq \varepsilon I \quad \text{in} \quad B(\mathcal{H}) \otimes_{\min} D.$$

The preceding equality therefore gives

$$A \succeq \varepsilon I \quad \text{in} \quad B(\mathcal{H}) \otimes_{\max} C_u^*(\mathfrak{A}) \otimes_{\max} C^*(\mathfrak{H}).$$

Theorem 1.4 now yields the asserted factorization, including the half-extent bound and the support restriction $u \in \mathcal{Y}_{\leq M'}$ for some $M' \geq M$. The target space can be taken to be a finite ampliation of $\mathcal{H}$, and is therefore separable whenever $\mathcal{H}$ is separable. If $\mathcal{H}$ is infinite-dimensional, this ampliation is unitarily isomorphic to $\mathcal{H}$. □

5.3. **Limitations of spatial positivity.** The conclusion of Remark 1.6 does not, in general, extend to $B(\mathcal{H})$-coefficients under the same uniform strict positivity hypothesis.

Example 5.2. Let $\mathcal{H} = \ell^2$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, and choose a finite symmetric generating set for Γ, so that its generated semigroup is Γ itself.

By [ISW20, Corollary B], $C^*(\Gamma)$ does not have the local lifting property. Kirchberg's tensor-product characterization of this property, stated in [Oza01, Theorem 1.3], therefore gives

$$X \in B(\mathcal{H}) \otimes \mathbb{C}[\Gamma], \qquad \|X\|_{\min} < \|X\|_{\max}.$$

Here the norms are induced by $B(\mathcal{H}) \otimes_{\min} C^*(\Gamma)$ and $B(\mathcal{H}) \otimes_{\max} C^*(\Gamma)$, respectively. The element X may be chosen with group-algebra coefficients by approximating its finitely many $C^*(\Gamma)$-coefficients while preserving the strict norm inequality.

Choose

$$\|X\|_{\min}^2 < t < \|X\|_{\max}^2, \qquad A_0 = tI - X^*X,$$

and let A be obtained from A_0 by inserting the identity in the $\mathfrak{A}$ factor. Its finite group support satisfies the support condition of Corollary 1.5 for some M, and its $\mathfrak{A}$-extent is zero. For every unital $*$-representation π of $\mathfrak{A} \otimes \mathbb{C}[\Gamma]$,

$$(\mathrm{id}_{B(\mathcal{H})} \otimes \pi)(A) \succeq \left(t - \|X\|_{\min}^2\right) I.$$

However, A_0 is not positive in $B(\mathcal{H}) \otimes_{\max} C^*(\Gamma)$, since positivity there would imply $\|X\|_{\max}^2 \leq t$.

Suppose that $A = B^*B$ for a finite polynomial

$$B = \sum_{q,u} B_{q,u} \otimes (q \otimes u), \qquad B_{q,u} \in B(\mathcal{H}, \mathcal{E}),$$

where the q are reduced words. Apply $\mathrm{id} \otimes \tau_{\mathfrak{A}} \otimes \mathrm{id}$, where $\tau_{\mathfrak{A}}(p^*q) = \delta_{p,q}$ by Lemma 2.1. This gives

$$A_0 = \sum_{u,v} Q_{u,v} \otimes u^{-1}v, \qquad Q_{u,v} = \sum_{q} B_{q,u}^* B_{q,v}.$$

The finite matrix $Q = [Q_{u,v}]_{u,v}$ is positive over $B(\mathcal{H})$. Therefore, with $V = (I_{\mathcal{H}} \otimes u)_u$ viewed as a column,

$$A_0 = V^*(Q \otimes 1)V \succeq 0 \quad \text{in } B(\mathcal{H}) \otimes_{\max} C^*(\Gamma),$$

a contradiction. Thus the obstruction persists even if no extent bound is imposed on the factor.

We conclude with the following example that shows the uniform lower bound in Corollary 1.5 cannot be replaced by strict positivity of the quadratic form in every spatial evaluation.

Example 5.3. Take $\mathfrak{A} = \mathbb{C}$, $\mathfrak{H} = \mathbb{Z}^3$, and $\mathcal{H} = \ell^2(\mathbb{N})$. Let $p \geq 0$ be a trigonometric polynomial on $\mathbb{T}^3$ that is not a sum of hermitian squares. Put

$$D = \mathrm{diag}(1, 1/2, 1/3, \ldots), \qquad A = I_{\mathcal{H}} \otimes p + D \otimes 1.$$

For every unital $*$-representation $\pi : \mathbb{C}[\mathbb{Z}^3] \to B(\mathcal{K})$ and every nonzero $\xi \in \mathcal{H} \otimes \mathcal{K}$,

$$\left\langle (\mathrm{id}_{B(\mathcal{H})} \otimes \pi)(A)\xi, \xi \right\rangle \geq \left\langle (D \otimes I_{\mathcal{K}})\xi, \xi \right\rangle > 0.$$

Nevertheless, A admits no factorization $A = B^*B$ with B an operator-valued polynomial.

Indeed, suppose that $A = B^*B$, where

$$B = \sum_{g \in F} B_g \otimes g, \qquad B_g \in B(\mathcal{H}, \mathcal{E}),$$

and $F \subseteq \mathbb{Z}^3$ is finite. Choose a state φ on $B(\mathcal{H})$ annihilating compact operators. Then $\varphi(D) = 0$, and applying φ coefficientwise gives

$$p = \sum_{g,h \in F} \varphi(B_g^* B_h)\, g^{-1} h.$$

Since

$$\left[\varphi(B_g^* B_h)\right]_{g,h \in F} \succeq 0,$$

this expresses p as a finite sum of hermitian squares, a contradiction. □

References

[BT07] M. Bakonyi and D. Timotin, Extensions of positive definite functions on free groups, J. Funct. Anal. 246 (2007) 31–49.

[BS25] B. K. Berntson and C. Sünderhauf, *Complementary polynomials in quantum signal processing*, Commun. Math. Phys. 406 (2025), article 161.

[BCR98] J. Bochnak, M. Coste, M.-F. Roy, *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 36. Springer, ix, 430 p. (1998).

[BO08] N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics 88, American Mathematical Society (2008).

[Coh23] S. Gonen Cohen, Algebraic Certificates for Positivity and Kazhdan's Property (T), MSc thesis, ETH Zürich (2023). https://people.math.ethz.ch/~sgonen/Thesis.pdf

[CvS21] A. Connes and W. D. van Suijlekom, *Spectral truncations in noncommutative geometry and operator systems*, Commun. Math. Phys. 383 (2021) 2021–2067.

[Dix77] J. Dixmier, *C^*-algebras*, North-Holland Math. Library, Vol. 15, North-Holland Publishing Co., xiii+492 pp. (1977).

[Dri04] M. A. Dritschel, *On factorization of trigonometric polynomials,* Integral Equations Operator Theory 49 (2004) 11–42.

[DR10] M. A. Dritschel and J. Rovnyak, *The operator Fejér–Riesz theorem*, in *A Glimpse at Hilbert Space Operators*, Operator Theory: Advances and Applications 207, Birkhäuser, Basel, 2010, pp. 223–254.

[DW05] M. A. Dritschel, H. J. Woerdeman, *Outer factorizations in one and several variables*, Trans. Amer. Math. Soc. 357 (2005) 4661–4679.

[Dum17] B. Dumitrescu, *Positive Trigonometric Polynomials and Signal Processing Applications*, 2nd ed., Signals and Communication Technology, Springer (2017).

[GW04] J. S. Geronimo and H. J. Woerdeman, *Positive extensions, Fejér–Riesz factorization and autoregressive filters in two variables*, Ann. of Math. (2) 160 (2004) 839–906.

[Hel02] J. W. Helton, *"Positive" noncommutative polynomials are sums of squares*, Ann. of Math. (2) 156 (2002) 675–694.

[HM04] J. W. Helton, Scott McCullough, *A positivstellensatz for non-commutative polynomials*, Trans. Amer. Math. Soc. 356 (2004) 3721–3737.

[ISW20] A. Ioana, P. Spaas and M. Wiersma, *Cohomological obstructions to lifting properties for full C^*-algebras of property (T) groups*, Geom. Funct. Anal. 30 (2020) 1402–1438.

[JKM26a] A. Jindal, I. Klep, and S. McCullough, *Positive operator-valued noncommutative polynomials are squares*, Integral Equations Operator Theory 98, article 7 (2026).

[JKM26b] A. Jindal, I. Klep and S. McCullough, *Operator-valued Positivstellensätze on matrix convex sets and free products of finite abelian groups*, preprint, arXiv:2604.27062.

[Kir93] E. Kirchberg, *On non-semisplit extensions, tensor products and exactness of group C^*-algebras*, Invent. Math. 112 (1993) 449–489.

[Kir94] E. Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. Reine Angew. Math. 452 (1994) 39–77.

[KLM26] I. Klep, J. Levenson and S. McCullough, *Fejér–Riesz factorization for positive noncommutative trigonometric polynomials*, accepted for publication in Math. Ann.

[Mar08] M. Marshall, *Positive polynomials and sums of squares*, American Mathematical Society (AMS), xii, 187 p. (2008).

[McC01] S. McCullough, *Factorization of operator-valued polynomials in several non-commuting variables*, Linear Algebra Appl. 326 (2001) 193–203.

[NT13] T. Netzer and A. Thom, *Real closed separation theorems and applications to group algebras*, Pacific J. Math. 263 (2013) 435–452.

[Oza01] N. Ozawa, *On the lifting property for universal C^*-algebras of operator spaces*, J. Operator Theory 46 (2001) 579–591.

[Oza13] N. Ozawa, *About the Connes embedding conjecture: algebraic approaches*, Jpn. J. Math. 8 (2013) 147–183.

[Pau02] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press (2002).

[Pis96] G. Pisier, *A simple proof of a theorem of Kirchberg and related results on C^*-norms*, J. Operator Theory 35 (1996) no. 2, 317–335.

[Roa58] M. Rosenblatt, *A multi-dimensional prediction problem*, Ark. Mat. 3 (1958) 407–424.

[Rou68] M. Rosenblum, *Vectorial Toeplitz operators and the Fejér–Riesz theorem*, J. Math. Anal. Appl. 23 (1968) 139–147.

[Sce00] C. Scheiderer, *Sums of squares of regular functions on real algebraic varieties*, Trans. Amer. Math. Soc. 352 (2000) 1039–1069.

[Sce24] C. Scheiderer, *A course in real algebraic geometry. Positivity and sums of squares*, Graduate Texts in Mathematics 303. Springer, xviii, 404 p. (2024).

[VDN92] D.-V. Voiculescu, K. J. Dykema, and A. Nica, *Free Random Variables*, CRM Monograph Series 1, AMS (1992).

[WBV99] S.-P. Wu, S. Boyd, and L. Vandenberghe, *FIR filter design via spectral factorization and convex optimization*, in *Applied and Computational Control, Signals, and Circuits*, Vol. 1, B. N. Datta (ed.), Birkhäuser, 215–245 (1999).

Appendix A. An elementary Følner-set proof of Corollary 1.5

We also give a direct Følner-set proof of Corollary 1.5.

Proof. Choose $\varepsilon > 0$ such that

$$(\mathrm{id}_{B(\mathcal{H})} \otimes \pi)(A) \succeq \varepsilon I$$

for every unital $*$-representation π of $\mathfrak{A} \otimes \mathbb{C}[\mathfrak{H}]$; such an ε exists by (1.5).

Choose a finite symmetric set $G \subseteq \mathfrak{H}$ containing e and the group support of A, and write

$$A = \sum_{g \in G} A_g \otimes g, \qquad A_g = \sum_{a \in \mathcal{W}_{2\mathfrak{a}}} A_{a,g} \otimes a.$$

Coefficients outside the original support are understood to be zero. Since $A = A^*$, we have $A_{g^{-1}} = A_g^*$. For $a \in \mathfrak{A}$, let

$$\|a\|_u = \sup_\sigma \|\sigma(a)\|,$$

where the supremum is over all unital $*$-representations of $\mathfrak{A}$. This norm is finite, since the cyclic generators are represented by unitaries and the representations of the finite-dimensional factors are contractive. Put

$$C_0 = \sum_{g\in G}\sum_{a\in\mathcal{W}_{2\mathtt{d}}} \|A_{a,g}\|\,\|a\|_u,$$

and choose $0<\delta<1$ so that

$$\frac{\delta C_0}{1-\delta} < \frac{\varepsilon}{2}.$$

Since $\mathfrak{H}$ is amenable, there is a finite nonempty right Følner set $F\subseteq\mathfrak{H}$ such that

$$k_F(g) := \frac{\#\{(u,v)\in F^2 : u^{-1}v = g\}}{|F|} = \frac{|F\cap Fg^{-1}|}{|F|} > 1-\delta, \qquad g\in G.$$

Notice that $k_F(g^{-1}) = k_F(g)$ and $0\le k_F(g)\le 1$. The common-left-denominator argument in the proof of Lemma 3.2 gives $c\in\mathcal{Y}$ such that $cF\subseteq\mathcal{Y}$. Since

$$(cu)^{-1}(cv) = u^{-1}v,$$

replacing F by cF leaves every $k_F(g)$ unchanged. We may therefore assume that

$$F\subseteq\mathcal{Y}_{\le M'}$$

for some integer $M'\ge M$.

Define

$$A'_g = \begin{cases} k_F(g)^{-1}A_g, & g\in G,\\ 0, & g\notin G,\end{cases} \qquad A' = \sum_{g\in G} A'_g\otimes g.$$

The symmetry of G and the identity $k_F(g^{-1}) = k_F(g)$ imply that $A' = A'^{*}$. For every unital $*$-representation π of $\mathfrak{A}\otimes\mathbb{C}[\mathfrak{H}]$,

$$\begin{aligned}\Big\|(\mathrm{id}_{B(\mathcal{H})}\otimes\pi)(A'-A)\Big\| &\le \sum_{g\in G}\sum_{a\in\mathcal{W}_{2\mathtt{d}}}\left|\frac{1}{k_F(g)}-1\right|\|A_{a,g}\|\,\|a\|_u\\ &\le \frac{\delta C_0}{1-\delta} < \frac{\varepsilon}{2}.\end{aligned}$$

Consequently,

$$(\mathrm{id}_{B(\mathcal{H})}\otimes\pi)(A') \succeq \frac{\varepsilon}{2}I. \tag{A.1}$$

Consider the finite operator matrix

$$T = \left[A'_{u^{-1}v}\right]_{u,v\in F} \in B\Big(\mathcal{H}\otimes\ell^2(F)\Big)\otimes\mathfrak{A}_{2\mathtt{d}}.$$

We claim that T is positive on every finite-dimensional unital $*$-representation of $\mathfrak{A}$. To see this, let $\sigma:\mathfrak{A}\to B(\mathcal{K})$ be such a representation and let R be the right regular representation of $\mathfrak{H}$,

$$R_g\delta_v = \delta_{vg^{-1}}.$$

Its (u,v) matrix entry is 1 precisely when $g = u^{-1}v$. Apply (A.1) to the representation

$$a\otimes g \longmapsto \sigma(a)\otimes R_g,$$

and compress the $\ell^2(\mathfrak{H})$ factor to $\ell^2(F)$. After the canonical permutation of tensor factors, the result is

$$\Big(\mathrm{id}_{B(\mathcal{H}\otimes\ell^2(F))}\otimes\sigma\Big)(T)\succeq\frac{\varepsilon}{2}I.$$

This proves the claim.

By Theorem 1.1, there exist a Hilbert space $\mathcal{E}$ and an element

$$D\in B\Big(\mathcal{H}\otimes\ell^2(F),\mathcal{E}\Big)\otimes\mathfrak{A}_{\mathtt{d}}\qquad\text{such that}\qquad D^*D=T.$$

We may take

$$\mathcal{E}=\Big(\mathcal{H}\otimes\ell^2(F)\Big)\otimes\mathbb{C}^{N(\mathtt{d})}.$$

For $u\in F$, let $J_u:\mathcal{H}\to\mathcal{H}\otimes\ell^2(F)$ be the isometry $J_uh=h\otimes\delta_u$, and set

$$D_u=D(J_u\otimes\mathtt{1})\in B(\mathcal{H},\mathcal{E})\otimes\mathfrak{A}_{\mathtt{d}}.$$

Then $D_u^*D_v=A'_{u^{-1}v}$. Define

$$B=\frac{1}{\sqrt{|F|}}\sum_{u\in F}D_u\otimes u.$$

Expanding each D_u in the basis $\mathcal{W}_{\mathtt{d}}$ shows that B has the asserted support, with zero coefficients for $u\in\mathcal{Y}_{\le M'}\setminus F$. Moreover,

$$\begin{aligned}B^*B&=\frac{1}{|F|}\sum_{u,v\in F}D_u^*D_v\otimes u^{-1}v\\&=\frac{1}{|F|}\sum_{u,v\in F}A'_{u^{-1}v}\otimes u^{-1}v\\&=\sum_{g\in G}k_F(g)A'_g\otimes g=A.\end{aligned}$$

The displayed choice of $\mathcal{E}$ is a finite ampliation of $\mathcal{H}$, and is therefore separable whenever $\mathcal{H}$ is separable. If $\mathcal{H}$ is infinite-dimensional, choose a unitary $U:\mathcal{E}\to\mathcal{H}$ and replace B by $(U\otimes\mathtt{1}\otimes\mathrm{e})B$ to obtain coefficients in $B(\mathcal{H})$ without changing B^*B. □

Abhay Jindal, Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
Email address: abhay.jindal@fmf.uni-lj.si

Igor Klep, Faculty of Mathematics and Physics, University of Ljubljana & Famnit, University of Primorska, Koper & Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
Email address: igor.klep@fmf.uni-lj.si

Jacob Levenson, Department of Mathematics, University of Florida, Gainesville
Email address: levenson.j@ufl.edu

Scott McCullough, Department of Mathematics, University of Florida, Gainesville
Email address: sam@math.ufl.edu