MATRIX CONVEX HULLS OF FREE SEMIALGEBRAIC SETS

J. WILLIAM HELTON[1], IGOR KLEP[2], AND SCOTT MCCULLOUGH[3]

Abstract. This article resides in the realm of the noncommutative (free) analog of real algebraic geometry – the study of polynomial inequalities and equations over the real numbers – with a focus on matrix convex sets $\mathcal{C}$ and their projections $\hat{\mathcal{C}}$. A free semialgebraic set which is convex as well as bounded and open can be represented as the solution set of a Linear Matrix Inequality (LMI), a result which suggests that convex free semialgebraic sets are rare. Further, Tarski's transfer principle fails in the free setting: The projection of a free convex semialgebraic set need not be free semialgebraic. Both of these results, and the importance of convex approximations in the optimization community, provide impetus and motivation for the study of the matrix convex hull of free semialgebraic sets.

This article presents the construction of a sequence $\mathcal{C}^{(d)}$ of LMI domains in increasingly many variables whose projections $\hat{\mathcal{C}}^{(d)}$ are successively finer outer approximations of the matrix convex hull of a free semialgebraic set $\mathcal{D}_p = \{X : p(X) \succeq 0\}$. It is based on free analogs of moments and Hankel matrices. Such an approximation scheme is possibly the best that can be done in general. Indeed, natural noncommutative transcriptions of formulas for certain well-known classical (commutative) convex hulls do not produce the convex hulls in the free case. This failure is illustrated here on one of the simplest free nonconvex $\mathcal{D}_p$.

A basic question is which free sets $\hat{\mathcal{S}}$ are the projection of a free semialgebraic set $\mathcal{S}$? Techniques and results of this paper bear upon this question which is open even for convex sets.

Date: February 3, 2014.

2010 *Mathematics Subject Classification.* Primary 46L07, 14P10, 90C22; Secondary 13J30, 46L89.

Key words and phrases. convex hull, linear matrix inequality (LMI), LMI domain, spectrahedron, spectrahedrop, semialgebraic set, free real algebraic geometry, noncommutative polynomial.

[1]Research supported by the National Science Foundation (NSF) grant DMS 1201498, and the Ford Motor Co.

[2]Supported by the Faculty Research Development Fund (FRDF) of the University of Auckland (project no. 3701119). Partially supported by the Slovenian Research Agency grant P1-0222.

[3]Research supported by the NSF grant DMS 1101137.

1. Introduction

This article resides within the realm of the recently emerging area of noncommutative (free) real algebraic geometry. As such it concerns free noncommutative polynomials $p(x) = p(x_1, \dots, x_g)$, and their associated free semialgebraic sets $\mathcal{D}_p$ (resp. $\mathfrak{P}_p$) consisting of those g-tuples of self-adjoint matrices X of the same size for which $p(X)$ is positive semidefinite (resp. definite). The case of (matrix) convex $\mathcal{D}_p$ is important in applications and also serves as an entrée to basic general aspects of free real algebraic geometry.

From the main result of [HM12], a bounded and open free semialgebraic set that is convex can be represented as the set of solutions to a Linear Matrix Inequality (LMI), called a free spectrahedron. This result is decidedly negative from the viewpoint of systems engineering, since it means that convex free semialgebraic sets are rare. It also motivates the theme of this article, the challenging problem of understanding the convex hull of a free semialgebraic set $\mathcal{D}_p$.

While formal definitions occur later, we now give the basic flavor of our results. The main classical approach for producing the convex hull of a basic semialgebraic set $\mathcal{D} \subseteq \mathbb{R}^g$ is to cleverly construct a spectrahedron $\mathcal{C}$ in a bigger space whose projection onto $\mathbb{R}^g$ is the convex hull of $\mathcal{D}$. In the literature the set $\mathcal{C}$ goes by several names. Here we will refer to these as an **LMI lift** or **spectrahedral lift** of the convex hull of $\mathcal{D}$. Developing the free analog of a theorem due to Lasserre for classical semialgebraic sets [Las09a], under modest hypotheses on $\mathcal{D}_p$, *we construct a sequence $\mathcal{C}^{(d)}$ of free spectrahedra in larger and larger spaces whose projections close down on the free convex hull of $\mathcal{D}_p$*. See Corollary 6.2.

We remark that solutions sets of LMIs play a prominent role in the theory of completely positive maps and operator systems [Arv72, Pau02] as well as quantum information theory (see for instance [JKPP11]). Moreover, their projections are related to recent advances in the theory of quotients of operator systems for which [FP12] is one of several recent references. A natural approach to understanding convexity in the free setting is through the study of free analogs of extreme points. One such is Arveson's [Arv72] notion of a *boundary representation* as a noncommutative analog of a peak point for a uniform algebra. As an emphatic culmination of a spate of recent activity, the article [DK+] validates Arveson's vision that an operator system has sufficiently many boundary representations to generate its C^*-envelope. For matrix convex hulls of free semialgebraic sets other notions of extreme points occur naturally (see for instance [Far04, WW99, Kls+]) and are treated in the forthcoming article [HKM+].

Beyond this point the news is bad. An approximation scheme, like that found here, is possibly the best that can be done in general. As evidence, we study thoroughly a $\mathcal{D}_p$ which has a strong claim to the title of simplest nonconvex free semialgebraic set. The free analogs of two different classical spectrahedral lifts $\mathcal{C}$ for $\mathcal{D}_p$ each have the property that the projection $\hat{\mathcal{C}}$ of $\mathcal{C}$ is convex and contains $\mathcal{D}_p$ and, at the scalar (commutative) level $\hat{\mathcal{C}}(1) = \mathcal{D}_p(1)$. However, in both cases, $\hat{\mathcal{C}}$ is not the free convex hull of $\mathcal{D}_p$; that is, $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p \subsetneq \hat{\mathcal{C}}$. See Example 4.4, Subsection 6.3 and Section 7.

A cornerstone of classical real algebraic geometry (RAG) is Tarski's transfer principle: the projection of a semialgebraic set is again semialgebraic. In free RAG the corresponding assertion is false even for convex sets, see [HM12]. Thus a basic question, on which this article bears and which is perhaps the most accessible path to understanding the class of sets closed with respect to projections, and containing the free semialgebraic sets, is which free sets are the projection of a free spectrahedron.

1.1. **Context and Perspective.**

The standard reference on classical RAG is [BCR98]. Two more tailored to our purposes are [Las09b] and [Lau09].

The construction of lifts used here is analogous to one introduced by Lasserre [Las09a] and Parrilo [Par06] independently. It involves positivity for multivariable moment matrices, studied systematically by Curto and Fialkow in a series of articles (see for example [CF08]), as well as their duals which are algebraic certificates of positivity for polynomials, called Positivstellensätze. Lasserre's key idea was to use a Positivstellensatz representation of linear functionals ℓ delineating the convex hull of the set $\mathcal{D}$ under study. When a nice Positivstellensatz exists for *all* such ℓ, one gets that a suitable spectrahedron $\mathcal{C}$, whose projection equals $\mathcal{D}$, exists. In fact, a related idea is that of the theta body introduced earlier to combinatorial optimization by Lovász in [Lov79]; see also [GLS93]. The recent survey [GT12] of Gouveia and Thomas ties these subjects together. See also their papers with Laurent and Parrilo [GLPT12, GPT10, GPT12]. LMI lifts of convex sets appeared in the book of Nesterov and Nemirovskii [NN94] at the outset of SDP. In their examples of sets with LMI representations – see Chapter 6 – rather than representing the sets, they gave representations for the lifts.

Returning to free lifts we mention that they are used in linear systems engineering to obtain free convex envelopes of sets. In the absence of any systematic theory, the literature consists of clever constructions (cf. [OGB02, GO10]). Moment matrix positivity in a free noncommutative context was studied in [PNA10], in connection with noncommutative sums of squares, following [HM04] and focusing on computational aspects; see also [HKM12].

While the setup of this paper is complex, that is, we work with self-adjoint complex matrices, the results carry over with little change to a combination of real symmetric and skew-symmetric matrices, cf. Remark 6.4.

We thank Cory Stone and Špela Špenko for many helpful comments on early versions of this manuscript.

1.2. **Guide to the Paper.**

- Section 2 contains basic definitions, including that of free polynomials, free semialgebraic sets, free convexity, and the matrix and operator convex hull of a free semialgebraic set.

- Section 3 concerns linear pencils and their relation to matrix convex hulls.
- Basic properties of projections of free spectrahedra are presented in Section 4.
- For a given free semialgebraic set $\mathcal{D}_p$, the construction of Section 5, based upon free analogs of moment sequences and Hankel matrices, produces an infinite free spectrahedron $\mathcal{C}$ together with a projection from $\mathcal{C}$ onto the operator convex hull $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$ of $\mathcal{D}_p$.
- In Section 6, truncations of the free Hankel matrices from Section 5 which in turn produce a sequence $(\mathcal{C}^{(d)})_d$ of (finite) free spectrahedra together with projections π_d are introduced. It is shown that $\pi_d(\mathcal{C}^{(d)})$ produces successively better outer approximations to $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$ and, in the limit, converges to $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$.
- Examples appear in Section 7.

2. Free Sets and Free Polynomials

Fix a positive integer g. For a positive integer n, let $\mathbb{S}_n^g$ denote the set of g-tuples of complex self-adjoint $n \times n$ matrices and let $\mathbb{S}^g$ denote the sequence $(\mathbb{S}_n^g)_n$. A **subset** Γ of $\mathbb{S}^g$ is a sequence $\Gamma = (\Gamma(n))_n$ where $\Gamma(n) \subseteq \mathbb{S}_n^g$ for each n. The subset Γ is **closed with respect to direct sums** if $A = (A_1, \dots, A_g) \in \Gamma(n)$ and $B = (B_1, \dots, B_g) \in \Gamma(m)$ implies

$$A \oplus B := \left(\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \dots, \begin{pmatrix} A_g & 0 \\ 0 & B_g \end{pmatrix} \right) \in \Gamma(n+m). \tag{2.1}$$

It is closed with respect to **(simultaneous) unitary conjugation** if for each n, each $A \in \Gamma(n)$ and each $n \times n$ unitary matrix U,

$$U^*AU = (U^*A_1U, \dots, U^*A_gU) \in \Gamma(n).$$

The set Γ is a **free set** if it is closed with respect to direct sums and simultaneous unitary conjugation. We refer the reader to [Voi04, Voi10, KVV+, MS11, Poe10, AM+, BB07] for a systematic study of free sets and free function theory.

We call a free set Γ **(uniformly) bounded** if there is a $C \in \mathbb{R}_{>0}$ such that $C - \sum X_j^2 \succeq 0$ for all $X \in \Gamma$.

2.1. Free Polynomials.

2.1.1. *Words and free polynomials.* We write $\langle x \rangle$ for the monoid freely generated by $x = (x_1, \dots, x_g)$, i.e., $\langle x \rangle$ consists of **words** in the g noncommuting letters $x_1, \dots, x_g$ (including the **empty word** $\varnothing$ which plays the role of the identity). Let $\mathbb{C}\langle x \rangle$ denote the associative $\mathbb{C}$-algebra freely generated by x, i.e., the elements of $\mathbb{C}\langle x \rangle$ are polynomials in the freely noncommuting variables x with coefficients in $\mathbb{C}$. Its elements are called **free polynomials**. Endow $\mathbb{C}\langle x \rangle$ with the natural **involution** * which extends the complex conjugation on $\mathbb{C}$, fixes x, reverses the order of words, and acts $\mathbb{R}$-linearly on polynomials. Polynomials fixed under this involution are **symmetric**. The length of the longest word in a free polynomial $f \in \mathbb{C}\langle x \rangle$ is the **degree** of f and is denoted by $\deg(f)$ or $|f|$ if $f \in \langle x \rangle$. The set of all words of degree at most k is $\langle x\rangle_k$, and $\mathbb{C}\langle x\rangle_k$ is the vector space of all free polynomials of degree at most k.

Fix positive integers ν and ℓ. **Free matrix polynomials** – elements of $\mathbb{C}^{\ell\times\nu}\langle x\rangle = \mathbb{C}^{\ell\times\nu}\otimes\mathbb{C}\langle x\rangle$; i.e., $\ell\times\nu$ matrices with entries from $\mathbb{C}\langle x\rangle$ – will play a role in what follows. Elements of $\mathbb{C}^{\ell\times\nu}\langle x\rangle$ are represented as

$$P=\sum_{w\in\langle x\rangle} B_w w\in\mathbb{C}^{\ell\times\nu}\langle x\rangle, \tag{2.2}$$

where $B_w\in\mathbb{C}^{\ell\times\nu}$, and the sum is finite. The involution * extends to matrix polynomials by

$$P^*=\sum_w B_w^* w^*\in\mathbb{C}^{\nu\times\ell}\langle x\rangle.$$

If $\nu=\ell$ and $P^*=P$, we say P is **symmetric**.

2.1.2. *Polynomial evaluations.* If $p\in\mathbb{C}\langle x\rangle$ is a free polynomial and $X\in\mathbb{S}_n^g$, then the evaluation $p(X)\in\mathbb{C}^{n\times n}$ is defined in the natural way by replacing x_i by X_i and sending the empty word to the appropriately sized identity matrix. Such evaluations produce finite dimensional $*$-representations of the algebra of free polynomials and vice versa.

Polynomial evaluations extend to matrix polynomials by evaluating entrywise. That is, if P is as in (2.2), then

$$P(X)=\sum_{w\in\langle x\rangle} B_w\otimes w(X)\in\mathbb{C}^{\ell n\times\nu n},$$

where $\otimes$ denotes the (Kronecker) tensor product. Note that if $P\in\mathbb{C}^{\ell\times\ell}\langle x\rangle$ is symmetric, and $X\in\mathbb{S}_n^g$, then $P(X)\in\mathbb{C}^{\ell n\times\ell n}$ is a self-adjoint matrix.

2.2. **Free Semialgebraic Sets.**

A symmetric free polynomial and even a symmetric matrix polynomial p in free variables naturally determine free sets [dOHMP09] via

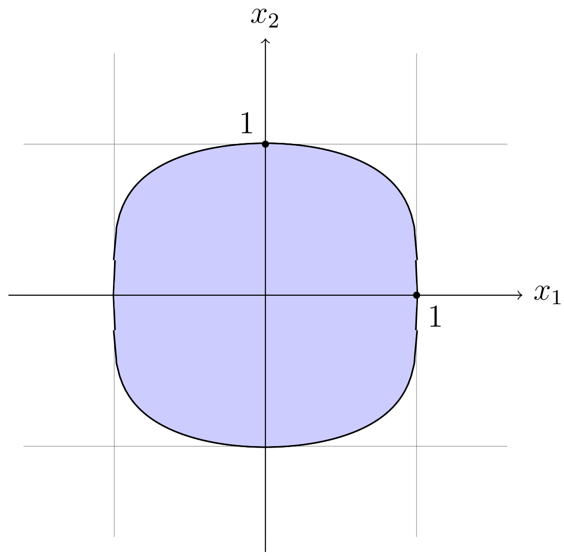
$$\mathcal{D}_p(n):=\{X\in\mathbb{S}_n^g : p(X)\succeq 0\},\qquad \mathcal{D}_p:=(\mathcal{D}_p(n))_n.$$

By analogy with real algebraic geometry [BCR98], we will refer to these as **free (basic closed) semialgebraic sets**.

Example 2.1. Consider

$$p=1-x_1^2-x_2^4. \tag{2.3}$$

In this case p is symmetric with $p(0)=1>0$. The free semialgebraic set $\mathcal{D}_p$ is called the **bent free TV screen**, or (bent) TV screen for short. We shall use this example at several places to illustrate the developments in this paper.



Bent TV screen $\mathcal{D}_p(1) = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1^2 - x_2^4 \geq 0\}$.

A subset Γ of $\mathbb{S}^g$ is **closed with respect to restriction to reducing subspaces** if $A \in \mathbb{S}_n^g$ and $H \subseteq \mathbb{C}^n$ is an invariant (reducing) subspace for A implies that A restricted to H is in Γ.

Lemma 2.2.

(1) For each n, the set $\mathcal{D}_p(n)$ is a semialgebraic subset of $\mathbb{S}_n^g$.
(2) The free semialgebraic set $\mathcal{D}_p$ is a free set. Moreover, it is closed with respect to restriction to reducing subspaces.

Proof. Fix n. There are scalar commutative polynomials $p_{i,j}$ in gn^2 variables such that $p(X) = (p_{i,j}(X))$ for $X \in \mathbb{S}_n^g$. By Sylvester's criterion, $p(X) \succeq 0$ if and only if all the principal minors of $p(X)$ are nonnegative. Since these minors are all polynomials, it follows that $\mathcal{D}_p(n)$ is a semialgebraic set.

It is evident that $\mathcal{D}_p$ is a free set. Suppose H reduces $A \in \mathcal{D}_p(n)$. In this case, $A = A^1 \oplus A^2$ for $A^j \in \mathbb{S}_{n_j}^g$ with $n_1 + n_2 = n$. Since $0 \preceq p(A) = p(A^1) \oplus p(A^2)$, it follows that $p(A^j) \succeq 0$ for each j. Hence $A \in \mathcal{D}_p(n_1)$ and $\mathcal{D}_p$ is closed with respect to restrictions to reducing subspaces. ∎

2.3. **Free Convexity.** A set $\Gamma = (\Gamma(n))_n \subseteq \mathbb{S}^g$ is **matrix convex** or **freely convex** if it is closed under direct sums and **(simultaneous) isometric conjugation**; i.e., if for each $m \leq n$, each $A = (A_1, \dots, A_g) \in \Gamma(n)$, and each isometry $V : \mathbb{C}^m \to \mathbb{C}^n$,

$$V^*AV := (V^*A_1V, \dots, V^*A_gV) \in \Gamma(m)$$

In particular, a matrix convex set is a free set.

In the case that Γ is matrix convex, it is easy to show that each $\Gamma(n)$ is itself convex. Indeed, given real numbers s, t with $s^2 + t^2 = 1$ and $X, Y \in \Gamma(n)$, let

$$V = \begin{pmatrix} sI_n \\ tI_n \end{pmatrix}$$

and observe that

$$V^* \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} V = s^2 X + t^2 Y \in \Gamma(n). \tag{2.4}$$

More generally, if $A^\ell = (A_1^\ell, \dots, A_g^\ell)$ are in $\Gamma(n_\ell)$, then $A = \bigoplus_\ell A^\ell \in \Gamma(n)$, where $n = \sum n_\ell$. Hence, if

$$V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{pmatrix}$$

is an isometry and V_ℓ are $n_\ell \times m$ matrices (for some m), then

$$V^* A V = \sum_{\ell=1}^k V_\ell^* A^\ell V_\ell \in \Gamma(m) \qquad \text{where} \quad \sum_{\ell=1}^k V_\ell^* V_\ell = I \tag{2.5}$$

A sum as in (2.5) is a **matrix convex combination** of the g-tuples $\{A^\ell : \ \ell = 1, \dots, k\}$.

Lemma 2.3. Suppose Γ is a free subset of $\mathbb{S}^g$.

(1) If Γ is closed with respect to restriction to reducing subspaces, then the following are equivalent:
   (i) Γ is matrix convex;
   (ii) each $\Gamma(n)$ is convex in the classical sense of taking scalar convex combinations.
(2) If Γ is (nonempty and) matrix convex, then $0 \in \Gamma(1)$ if and only if Γ is closed with respect to (simultaneous) conjugation by contractions.

Proof. Evidently (i) implies (ii). The implication (ii) implies (i) is proved in [HM04, §2]. For item (2), if Γ is closed with respect to conjugation by a contraction, then given an $A \in \Gamma(n)$, letting $z : \mathbb{C} \to \mathbb{C}^n$ be the zero mapping, gives $z^* A z = 0 \in \mathbb{C}^g$. Hence, $0 \in \Gamma(1)$. Conversely, suppose $0 \in \Gamma(1)$. In this case for each n the zero tuple 0_n is in $\Gamma(n)$ as Γ is closed with respect to direct sums. Given an $n \times n$ contraction F, and $X \in \Gamma(n)$ observe that $X \oplus 0_n \in \Gamma(2n)$, form the isometry

$$V^* = \begin{pmatrix} F^* & (I - F^*F)^{\frac{1}{2}} \end{pmatrix}$$

and compute

$$F^* X F = V^* \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} V \in \Gamma. \qquad \blacksquare$$

Remark 2.4. Combining the second items of Lemmas 2.2 and 2.3 it follows that the free semialgebraic set $\mathcal{D}_p$ is matrix convex if and only if each $\mathcal{D}_p(n)$ is convex.

Example 2.5. Consider the TV screen given by $p = 1 - x_1^2 - x_2^4$ introduced in Example 2.1. While $\mathcal{D}_p(1)$ is convex (see Example 2.1 for a picture), it is known that $\mathcal{D}_p$ is not matrix convex, see [DHM07] or [BPR13, Chapter 8]. Indeed, already $\mathcal{D}_p(2)$ is not a convex set.

2.4. **The Matrix Convex Hull.** The **matrix convex hull** of a subset $\Gamma = (\Gamma(n))_n$ of $\mathbb{S}^g$, denoted $\mathrm{co}^{\mathrm{mat}}\Gamma$, is the smallest matrix convex set containing Γ. As usual, the intersection of matrix convex sets is matrix convex, so the notion of a hull is well defined. Further, there is a simple description of the matrix convex hull of a free set.

For positive integers n let

$$\mathcal{C}(n) := \bigcup_{m\in\mathbb{N}} \{X \in \mathbb{S}_n^g : X = V^*ZV \text{ for some isometry } V \in \mathbb{C}^{m\times n} \text{ and } Z \in \Gamma(m)\}. \tag{2.6}$$

In the case that Γ is closed with respect to direct sums, it is straightforward to verify that $\mathcal{C} = \big(\mathcal{C}(n)\big)_n$ is a matrix convex set which contains Γ. On the other hand, $\mathcal{C}$ must be contained in any matrix convex set containing Γ. Hence we conclude:

Proposition 2.6. If Γ is closed with respect to direct sums, then $\mathcal{C}$ is its matrix convex hull.

2.5. **Topological Properties of the Matrix Convex Hull.** A natural norm on $\mathbb{S}_n^g$ is given by

$$\|X\|^2 = \sum_{j=1}^{g} \|X_j\|^2$$

for $X = (X_1, \dots, X_g) \in \mathbb{S}_n^g$.

The subset $\mathcal{S} = (\mathcal{S}(n))_n$ of $\mathbb{S}^g$ is **open** if each $\mathcal{S}(n)$ is open.

Lemma 2.7. If the open set $\mathcal{S} \subseteq \mathbb{S}^g$ is closed with respect to direct sums, then $\mathrm{co}^{\mathrm{mat}}\mathcal{S}$ is open.

Proof. To show that $\mathrm{co}^{\mathrm{mat}}\mathcal{S}(n)$ is open, let $X \in \mathrm{co}^{\mathrm{mat}}\mathcal{S}(n)$ be given. By Proposition 2.6, there exists an $m \in \mathbb{N}$, a $Z \in \mathcal{S}$ and an isometry $V : \mathbb{C}^n \to \mathbb{C}^m$ such that $X = V^*ZV$. Because $\mathcal{S}(m)$ is open, there exists an $\varepsilon > 0$ such that if $\|W - Z\| < \varepsilon$, then $W \in \mathcal{S}(m)$. Now suppose $Y \in \mathbb{S}_n^g$ and $\|Y - X\| < \varepsilon$. Writing,

$$Z = \begin{pmatrix} X & \beta \\ \beta^* & \delta \end{pmatrix}$$

with respect to the decomposition of $\mathbb{C}^m$ as the range of V direct sum its orthogonal complement, let

$$W = \begin{pmatrix} Y & \beta \\ \beta^* & \delta \end{pmatrix}.$$

Thus $W \in \mathcal{S}(m)$ and, by another application of Proposition 2.6, $Y = V^*WV \in \mathrm{co}^{\mathrm{mat}}\mathcal{S}(n)$. Hence $\mathrm{co}^{\mathrm{mat}}\mathcal{S}(n)$ is open. ∎

Let $\overline{\mathrm{co}}^{\mathrm{mat}}\Gamma$ denote the closure of the convex hull of the free set Γ, i.e.,

$$\overline{\mathrm{co}}^{\mathrm{mat}}\Gamma = \left(\overline{\mathrm{co}^{\mathrm{mat}}\Gamma(n)}\right)_n.$$

Lemma 2.8. If $K = (K(n))_n$ is a matrix convex set, then $\overline{K} = (\overline{K(n)})_n$ is also matrix convex. Here $\overline{K(n)}$ is the closure of $K(n)$ in $\mathbb{S}_n^g$.

Proof. To see that $\overline{K}$ is closed with respect to direct sums, suppose $X \in \overline{K(n)}$ and $Y \in \overline{K(m)}$. There exists sequences (X^ℓ) and (Y^ℓ) from $K(n)$ and $K(m)$ converging to X and Y respectively. It follows that $X^\ell \oplus Y^\ell \in K(n+m)$ converges to $X \oplus Y$ and thus $X \oplus Y \in \overline{K(n+m)}$.

To see that $\overline{K}$ is closed with respect to simultaneous isometric conjugation, suppose $X \in \overline{K(n)}$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry. There exists a sequence (X^ℓ) from $K(n)$ which converges to X. Thus, the sequence $(V^*X^\ell V)$ lies in $K(m)$ and converges to V^*XV. Thus $V^*XV \in \overline{K(m)}$ and the proof is complete. ∎

Lemma 2.9. Suppose Γ is a free set. If each $\Gamma(n)$ is compact, then for each m, $\mathrm{co}^{\mathrm{mat}}\Gamma(m)$ is naturally a nested increasing union of compact convex sets.

Proof. For each $n \geq m$, let

$$P_n(m) = \{V^*XV \mid V : \mathbb{C}^m \to \mathbb{C}^n \text{ is an isometry, } X \in \Gamma(n)\} \subseteq \mathrm{co}^{\mathrm{mat}}\Gamma(m).$$

Let $E_n(m) \subseteq \mathrm{co}^{\mathrm{mat}}\Gamma(m)$ denote the (ordinary) convex hull of $P_n(m)$. By Caratheodory's convex hull theorem [Bar02, Theorem I.2.3], $E_n(m)$ is a subset of $P_{n(\alpha+1)}(m)$ (where α is the dimension of $\mathbb{S}_m$). Since $P_n(m)$ is compact (being the image of the compact set $\{m \times n$ isometries$\} \times \Gamma(n)$ under the continuous map $(V, X) \mapsto V^*XV$), then so is $E_n(m)$. We have,

$$\mathrm{co}^{\mathrm{mat}}\Gamma(m) = \bigcup_{n \geq m} P_n(m) = \bigcup_{n \geq m} E_n(m).$$

Thus, $\mathrm{co}^{\mathrm{mat}}\Gamma(m)$ is the nested increasing union of a canonical sequence of compact convex sets. ∎

2.6. **Basic Definitions. Operator Level.** All the notions discussed above have natural counterparts on infinite-dimensional Hilbert spaces.

Fix a separable Hilbert space $\mathscr{K}$ and let $\mathrm{Lat}(\mathscr{K})$ denote the **lattice of subspaces of** $\mathscr{K}$. For a $K \in \mathrm{Lat}(\mathscr{K})$, let $\mathbb{S}_K^g$ denote g-tuples $X = (X_1, \dots, X_g)$ of self-adjoint operators on K. A collection $\Gamma = (\Gamma(K))_K$ where $\Gamma(K) \subseteq \mathbb{S}_K^g$ for each $K \leq \mathscr{K}$, is a **free operator set** if it is closed under direct sums and with respect to simultaneous conjugation by unitary operators. If in addition it is closed with respect to simultaneous conjugation by isometries $V : H \to K$, where $H, K \in \mathrm{Lat}(\mathscr{K})$, then Γ is **operator convex**.

Note that $(\mathbb{S}^g_K)_K$ is itself a free operator set which will be henceforth denoted by $\mathbb{S}^g_{\text{oper}}$. Given a symmetric free matrix polynomial p with $p(0) \succ 0$, let

$$\mathcal{D}_p^\infty = \{X \in \mathbb{S}^g_{\text{oper}} : p(X) \succeq 0\}$$

be the **operator free semialgebraic set** defined by p. It is easy to see that $\mathcal{D}_p^\infty$ is a free operator set. For $K \in \text{Lat}(\mathscr{K})$, we write

$$\mathcal{D}_p^\infty(K) = \{X \in \mathbb{S}^g_K : p(X) \succeq 0\}.$$

A free operator semialgebraic set Γ is **uniformly bounded** if there is a $C \in \mathbb{R}_{>0}$ such that $C - \sum X_j^2 \succeq 0$ for all $X \in \Gamma$.

2.7. **The Operator Convex Hull.** Each free polynomial p gives rise to two operator convex hulls. The **operator convex hull of $\mathcal{D}_p$** is the sequence of sets $\text{co}^{\text{oper}}\mathcal{D}_p = (\text{co}^{\text{oper}}\mathcal{D}_p(n))_n$ where $X \in \mathbb{S}^g_n$ is in $\big(\text{co}^{\text{oper}}\mathcal{D}_p\big)(n)$ if there exists a $Z \in \mathcal{D}_p^\infty$ (acting on a Hilbert space $\mathscr{K}$) and an isometry $V : \mathbb{C}^n \to \mathscr{K}$ such that $X = V^*ZV$.

The notion of the **(operator) convex hull of $\mathcal{D}_p^\infty$** is defined similarly. Thus $\text{co}\,\mathcal{D}_p^\infty$ is the sequence of sets $(\text{co}\,\mathcal{D}_p^\infty(K))_K$ where, for $K \in \text{Lat}(\mathscr{K})$, the tuple $X \in \mathbb{S}^g_K$ is in $\big(\text{co}\,\mathcal{D}_p^\infty\big)(K)$ if there exists a $Z \in \mathcal{D}_p^\infty$ (acting on the Hilbert space $\mathscr{K}$) and an isometry $V : K \to \mathscr{K}$ such that $X = V^*ZV$.

Later we will see in Theorem 5.4 that $\text{co}^{\text{oper}}\mathcal{D}_p$ is closed.

3. Linear Pencils and Matrix Convex Hulls

Classical convex sets in $\mathbb{R}^g$ are defined as intersections of half-spaces and are thus described by linear functionals. Matrix convex sets are defined analogously by linear pencils; cf. [EW97, HM12]. This section surveys some basic facts about convex hulls and their associated linear pencils.

3.1. **Linear Pencils.** Given $k \times k$ self-adjoint matrices $A_0, \dots, A_g$, let

$$L(x) = A_0 + \sum_{j=1}^g A_j x_j \in \mathbb{S}_k\langle x\rangle$$

denote the corresponding **(affine) linear pencil** of **size** k. In the case that $A_0 = 0$; i.e., $A = (A_1, ..., A_g) \in \mathbb{S}^g_k$, let

$$\Lambda_A(x) = \sum_{j=1}^g A_j x_j$$

denote the corresponding **homogeneous (truly) linear pencil** and

$$L_A = I + \Lambda_A$$

the associated **monic linear pencil**.

The linear pencil can of course be evaluated at a point $x \in \mathbb{R}^g$ in the obvious way, producing the Linear Matrix Inequality, $L(x) \succeq 0$. The solution set to this inequality is known as a **spectrahedron** or **LMI domain** and is obviously a convex semialgebraic set.

The pencil L is a free object too as it is naturally evaluated on $X \in \mathbb{S}_n^g$ using (Kronecker's) tensor product

$$L(X) := A_0 \otimes I + \sum_{j=1}^{g} A_j \otimes X_j. \tag{3.1}$$

The free semialgebraic set $\mathcal{D}_L$ is easily seen to be matrix convex. We will refer to $\mathcal{D}_L$ as a **free spectrahedron** or **free LMI domain** and say that a free set Γ is **freely LMI representable** if there is a linear pencil L such that $\Gamma = \mathcal{D}_L$. In particular, if Γ is freely LMI representable with a monic L_A, then 0 is in the interior of Γ. Note too that $\mathcal{D}_L(1) \subseteq \mathbb{R}^g$ is a spectrahedron.

Later we shall also use linear pencils which are based on infinite-dimensional operators A_i and the associated pencil $L(x) = A_0 + \sum A_j x_j$. In this case the free set $\mathcal{D} = (\mathcal{D}(n))$, where $\mathcal{D}(n) = \{X \in \mathbb{S}_n^g : L(X) \succeq 0\}$ is an **infinite spectrahedron**. We emphasize that the unmodified term free spectrahedron always requires the A_i to act on a finite-dimensional space.

The following is a special case (see [HM12, §6]) of a Hahn-Banach separation theorem due to Effros and Winkler [EW97].

Theorem 3.1. *If $\mathcal{C} = (\mathcal{C}(n))_{n\in\mathbb{N}} \subseteq \mathbb{S}^g$ is a closed matrix convex set containing 0 and $Y \in \mathbb{S}_m^g$ is not in $\mathcal{C}(m)$, then there is a monic linear pencil L of size m such that $L(X) \succeq 0$ for all $X \in \mathcal{C}$, but $L(Y) \not\succeq 0$.*

Proof. From [EW97, Theorem 5.4], there exist $m \times m$ matrices $A_1, \dots, A_g \in \mathbb{C}^{m\times m}$ such that

$$I - \frac{1}{2}\Big(\sum A_j \otimes X_j + \big(\sum A_j \otimes X_j\big)^*\Big) \succeq 0$$

for all n and $X \in \mathcal{C}(n)$, but at the same time

$$I - \frac{1}{2}\Big(\sum A_j \otimes Y_j + \big(\sum A_j \otimes Y_j\big)^*\Big) \not\succeq 0$$

Note however, that since $X_j^* = X_j$, it follows that $A_j \otimes X_j + A_j^* \otimes X_j^* = (A_j + A_j^*) \otimes X_j$. Thus, it can be assumed that $A \in \mathbb{S}_m^g$. ■

Though linear matrix inequalities appear special, the following result from [HM12] says that they actually account for matrix convexity of free semialgebraic sets.

Theorem 3.2. *Fix p a symmetric real matrix polynomial. If $p(0) \succ 0$ and the strict positivity set $\mathfrak{P}_p = \{X : p(X) \succ 0\}$ of p is bounded, then $\mathfrak{P}_p$ is matrix convex if and only if there is a monic linear pencil L such that $\mathfrak{P}_p = \mathfrak{P}_L = \{X : L(X) \succ 0\}$.*

3.2. **Pencils and Hulls.**

Lemma 3.3. Let $\mathcal{C}$ be a matrix convex set. If L is a pencil of size k, then L is positive semidefinite on $\mathcal{C}$ if and only if L is positive semidefinite on $\mathcal{C}(k)$.

Proof. Suppose L is positive semidefinite on $\mathcal{C}(k)$ and let m and $X \in \mathcal{C}(m)$ be given. Fix a vector $v \in \mathbb{C}^k \otimes \mathbb{C}^m$. Letting $\{e_1, \dots, e_k\}$ denote the standard orthonormal basis for $\mathbb{C}^k$, there exist vectors $v_1, \dots, v_k \in \mathbb{C}^m$ such that

$$v = \sum_{j=1}^{k} e_j \otimes v_j.$$

Let H denote the span of $\{v_1, \dots, v_k\}$ and let $V : H \to \mathbb{C}^m$ denote the inclusion mapping. It follows that

$$\begin{aligned} \langle L(X)v, v\rangle &= \langle L(X) I \otimes V v, I \otimes V v\rangle \\ &= \langle L(V^*XV)v, v\rangle. \end{aligned}$$

Since $V^*XV \in \mathcal{C}(k)$, it follows that $\langle L(X)v, v\rangle \geq 0$. Hence $L(X) \succeq 0$ and the proof is complete. ∎

Proposition 3.4. *Let L be a $\mu \times \mu$ linear pencil. Then*

$$L|_{\mathcal{D}_p} \succeq 0 \quad \iff \quad L|_{\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(\mu)} \succeq 0.$$

Of course, the downside of Proposition 3.4 is that it does not give bounds on the isometries needed in $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(\mu)$ (as they appear in Equation (2.6)).

Proof. Evidently L is positive semidefinite on $\mathcal{D}_p$ if and only if L is positive semidefinite on $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$. An application of Lemma 3.3 completes the proof. ∎

Just like the closed convex hull of a subset C of $\mathbb{R}^g$ can be written as an intersection of half-spaces containing C, closed matrix convex hulls are intersections of free spectrahedra.

Corollary 3.5. *Let p be a symmetric free polynomial (with as usual $p(0) \succ 0$). For $n \in \mathbb{N}$, the set $\overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p(n)$ consists of all g-tuples $Z \in \mathbb{S}_n^g$ satisfying $L(Z) \succeq 0$ for all $n \times n$ monic linear pencils L with $\mathcal{D}_L \supseteq \mathcal{D}_p$ (equivalently $L|_{\mathcal{D}_p} \succeq 0$).*

Proof. This corollary is a version of the matricial Hahn-Banach Theorem 3.1. Indeed, if $Z \notin \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p(n)$, then by these matricial Hahn-Banach theorems there is an $n \times n$ pencil L with $L(Z) \not\succeq 0$ and $L|_{\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(n)} \succeq 0$. The latter implies by Proposition 3.4 that $L|_{\mathcal{D}_p} \succeq 0$, that is, $\mathcal{D}_L \supseteq \mathcal{D}_p$.

To prove the reverse inclusion, suppose L is $n \times n$ with $L|_{\mathcal{D}_p} \succeq 0$. If $Z \in \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p(n)$ there is a sequence $Z_i \in \mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(n)$ converging to Z. Such Z_i must have the form $Z_i = V_i^* X_i V_i$, with $X_i \in \mathcal{D}_p$ and V_i is an isometry. Thus

$$L(V_i^* X_i V_i) = (I \otimes V_i)^* L(X_i)(I \otimes V_i) \succeq 0.$$

Since $L(Z_i) \to L(Z)$, we have $L \succeq 0$ on $\overline{\mathrm{co}}^{\mathrm{mat}} \mathcal{D}_p(n)$, and so we are done. ∎

Corollary 3.6. *Suppose ℓ is an affine linear function, and L is a linear pencil. Then*

$$\ell|_{\mathcal{D}_L(1)} \geq 0 \quad \Longleftrightarrow \quad \ell|_{\mathcal{D}_L} \succeq 0.$$

Proof. While this is an obvious corollary of Proposition 3.4, let us present a short and independent self-contained argument. The implication $(\Leftarrow)$ is obvious. For the converse assume $X \in \mathcal{D}_L(n)$ with $\ell(X) \not\succeq 0$. Let v be a unit eigenvector of $\ell(X)$ with negative eigenvalue. For $v^*Xv := (v^*X_1v, \dots, v^*X_gv) \in \mathbb{R}^g$ we have

$$L(v^*Xv) = (I \otimes v)^* L(X)(I \otimes v) \succeq 0,$$

i.e., $v^*Xv \in \mathcal{D}_L(1)$, and

$$\ell(v^*Xv) = v^*\ell(X)v < 0.$$ ∎

4. Projections of Free Spectrahedra: Free Spectrahedrops

Let L be a linear pencil in the variables $(x_1, \dots, x_g; y_1, \dots, y_h)$,

$$L = D + \sum_{j=1}^{g} A_j x_j + \sum_{\ell=1}^{h} B_\ell y_\ell.$$

The set

$$\operatorname{proj}_x \mathcal{D}_L(1) = \{x \in \mathbb{R}^g : \exists\, y \in \mathbb{R}^h \text{ such that } L(x,y) \succeq 0\}$$

is known as a **spectrahedral shadow** or is a **semidefinite programming (SDP) representable set** [BPR13] and the representation afforded by L is an **SDP representation**. SDP representable sets are evidently convex and lie in a middle ground between LMI representable sets and general convex sets. They play an important role in convex optimization. In the case that $S \subseteq \mathbb{R}^g$ is closed semialgebraic and with some mild additional hypothesis, it is proved in [HN10] based upon the Lasserre–Parrilo construction ([Las09a, Par06]) that the convex hull of S is SDP representable.

Given a linear pencil L, let $\operatorname{proj}_x \mathcal{D}_L$ denote the free set

$$\operatorname{proj}_x \mathcal{D}_L = \bigcup_{n \in \mathbb{N}} \{X \in \mathbb{S}_n^g : \exists\, Y \in \mathbb{S}_n^h \text{ such that } L(X,Y) \succeq 0\}.$$

We will call a set of the form $\operatorname{proj}_x \mathcal{D}_L$ a **free spectrahedrop** or a **freely SDP representable set** or even a **free spectrahedral shadow**. Thus a free spectrahedrop is a coordinate projection of a free spectrahedron.

Proposition 4.1. *Free spectrahedrops are matrix convex. In particular, they are closed with respect to restrictions to reducing subspaces.*

Example 4.2. The second half of Proposition 4.1 fails for projections of general free semi-algebraic sets. As an example, consider

$$q = yx^2y + zx^2z - 1 \tag{4.1}$$

and the projection $\mathcal{S}$ of $\mathcal{D}_q$ onto x. Thus,

$$\mathcal{S} = \{X \in \mathbb{S} : \exists\ (Y, Z) \in \mathbb{S}^2 \text{ such that } q(X, Y, Z) \succeq 0\}.$$

It is easy to show $I_3 \oplus 0_3$ is in $\mathcal{S}$, but of course 0_3 is not. Incidentally, this gives a simple example of a free semialgebraic set whose projection is not semialgebraic, in sharp contrast to Tarski's transfer principle in classical real algebraic geometry [BCR98].

On the other hand, Proposition 4.1 implies that, for a linear pencil L, projections of $\mathcal{D}_L$ are closed with respect to restrictions to reducing subspaces. Nevertheless, a projection of $\mathcal{D}_L$ need not be semialgebraic, cf. [HM12, §9].

4.1. **Free Spectrahedrops and Monic Lifts.** Recall a free set $\mathcal{K}$ is a free spectrahedrop if it is a (coordinate) projection of a free spectrahedron, $\mathcal{D}_\Lambda$. The next lemma shows that even when Λ is not a monic pencil, if 0 is in the interior of $\mathcal{K}$, then $\mathcal{K}$ admits a *monic* LMI lift.

Lemma 4.3. If $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_\Lambda$ is a free spectrahedrop containing 0 in its interior, then there exists a monic linear pencil $L(x, y)$ such that

$$\mathcal{K} = \operatorname{proj}_x \mathcal{D}_L = \{X \in \mathbb{S}^g \mid \exists Y \in \mathbb{S}^h : \ L(X, Y) \succeq 0\}. \tag{4.2}$$

If $\mathcal{D}_\Lambda$ is bounded, then we may further ensure $\mathcal{D}_L$ is bounded.

Proof. Suppose

(i) $\Lambda(x, y)$ is an affine linear pencil,

$$\Lambda(x, y) = \Lambda_0 + \sum_{j=1}^{g} \Lambda_j x_j + \sum_{k=1}^{h} \Omega_k y_k;$$

(ii) $\mathcal{K}$ is the projection of $\mathcal{D}_\Lambda$ onto x-space. Thus, $\mathcal{K} = \{X \in \mathbb{S}^g : \exists Y \in \mathbb{S}^h : \ \Lambda(X, Y) \succeq 0\}$.

Without loss of generality, it may be assumed the number h of y-variables is the smallest possible with respect to the the properties (i) and (ii).

Let $\mathring{\mathcal{D}}_\Lambda(1)$ denote the interior of $\mathcal{D}_\Lambda(1)$ and suppose first that this interior is empty. In this case the convex subset $\mathcal{D}_\Lambda(1)$ of $\mathbb{R}^{g+h}$ lies in a proper affine subspace of $\mathbb{R}^{g+h}$. That is, there is an affine linear functional (with real coefficients)

$$\ell(x, y) = \ell_0 + \sum_{j=1}^{g} \ell_j x_j + \sum_{k=1}^{h} \omega_k y_k$$

such that $\ell = 0$ on $\mathcal{D}_\Lambda(1)$. Equivalently, $\ell = 0$ on $\mathcal{D}_\Lambda$, cf. Corollary 3.6. At least one ω_k is nonzero as otherwise ℓ would produce a nontrivial affine linear map vanishing on $\mathcal{K}$, contradicting the assumption that $\mathcal{K}$ has nonempty interior. Without loss of generality, $\omega_h = 1$. Consider the pencil $\tilde{\Lambda}$ in the variables $(x, \tilde{y}) = (x_1, \dots, x_g, y_1, \dots, y_{h-1})$,

$$\tilde{\Lambda}(x, \tilde{y}) = \ell_0(\Lambda_0 - \Omega_h) + \sum_{j=1}^{g}(\Lambda_j - \ell_j\Omega_h)x_j + \sum_{k=1}^{h-1}(\Omega_k - \omega_k\Omega_h)y_k = \Lambda(x, y) - \Omega_h\ell(x, y).$$

Given $X \in \mathcal{K}(n)$, there is a $Y \in \mathbb{S}_n^h$ such that $\Lambda(X, Y) \succeq 0$. Letting $\tilde{Y} = (Y_1, \dots, Y_{h-1})$,

$$\tilde{\Lambda}(X, \tilde{Y}) = \Lambda(X, Y) - \Omega_h \otimes \ell(X, Y) = \Lambda(X, Y) \succeq 0.$$

On the other hand, if there is a $\tilde{Y} = (Y_1, \dots, Y_{h-1})$ such that $\tilde{\Lambda}(X, \tilde{Y}) \succeq 0$, then with

$$Y_h = -(\ell_0 I + \sum_{j=1}^{g} \ell_j X_j + \sum_{k=1}^{h-1} \omega_k Y_k),$$

and $Y = (\tilde{Y}, Y_h)$, it follows that $\ell(X, Y) = 0$. Hence,

$$\Lambda(X, Y) = \tilde{\Lambda}(X, \tilde{Y}) + \Omega_h \otimes \ell(X, Y) = \tilde{\Lambda}(X, \tilde{Y}) \succeq 0.$$

It follows that $\tilde{\Lambda}$ satisfies conditions (i) and (ii), contradicting the minimality assumption on the number of y-variables. Hence $\mathcal{D}_\Lambda(1)$ has a nontrivial interior.

The projection $\operatorname{proj}_x : \mathring{\mathcal{D}}_\Lambda(1) \to \mathcal{K}(1)$ is continuous, so the preimage of a small ball $B_\varepsilon \subseteq \mathbb{R}^g$ around $0 \in \mathring{\mathcal{K}}(1)$ is an open subset of $\mathcal{D}_\Lambda(1)$. At least one of these points will have its x-component equal to 0, say $(0, \hat{y}) \in \mathring{\mathcal{D}}_\Lambda(1)$. By replacing $\Lambda(x, y)$ with $L(x, y) = \Lambda(x, y - \hat{y})$ we obtain a linear pencil L such that $\operatorname{proj}_x \mathcal{D}_L = \operatorname{proj}_x \mathcal{D}_\Lambda$ but now the free spectrahedron $\mathcal{D}_L$ has $(0, 0)$ as an interior point. Hence a standard reduction shows we may take L to be monic (cf. [HV07]). It is clear that $\mathcal{D}_L$ is bounded if $\mathcal{D}_\Lambda$ is bounded. ■

4.2. **Convex Hulls and Spectrahedrops.** Given a free semialgebraic set $\mathcal{D}_p$, a goal is to determine when its convex hull, or closed convex hull, or its operator convex hull is a free spectrahedrop. When this can be done it provides a potentially useful approximation to $\mathcal{D}_p$.

Example 4.4. Recall the polynomial $p = 1 - x_1^2 - x_2^4$ from Example 2.1. That the set $\mathcal{D}_p(1)$ has an LMI lift is well known and is given as follows. Let

$$\Lambda(x_1, x_2, y) = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y \\ x_1 & y & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & x_2 \\ x_2 & y \end{pmatrix}.$$

It is readily checked that $\operatorname{proj}_x \mathcal{D}_\Lambda(1) = \mathcal{D}_p(1)$. Further, Lemma 4.3 implies that Λ can be replaced by a monic linear pencil L, cf. Subsection 7.1.

Proposition 4.5. *Assume $\mathcal{D}_p(1)$ is bounded and L is a monic linear pencil. If* $\operatorname{co}(\mathcal{D}_p(1))$*, the ordinary convex hull of $\mathcal{D}_p(1) \subseteq \mathbb{R}^g$, admits an LMI lift to $\mathcal{D}_L(1)$ and $\mathcal{D}_p \subseteq \operatorname{proj}_x \mathcal{D}_L$, then* $\operatorname{co}(\mathcal{D}_p(1)) = \operatorname{co}^{\mathrm{mat}}\mathcal{D}_p(1)$*.*

Proof. Suppose ℓ is an affine linear function nonnegative on $\mathcal{D}_p(1)$. Then $\ell|_{\mathcal{D}_L(1)} \geq 0$ and hence by Corollary 3.6, $\ell|_{\mathcal{D}_L} \succeq 0$. Since $\operatorname{co}^{\mathrm{mat}}\mathcal{D}_p \subseteq \operatorname{proj}_x \mathcal{D}_L$, this implies $\ell|_{\operatorname{co}^{\mathrm{mat}}\mathcal{D}_p(1)} \geq 0$. As $\mathcal{D}_p(1) \subseteq \operatorname{co}^{\mathrm{mat}}\mathcal{D}_p(1)$, this shows $\overline{\operatorname{co}\mathcal{D}_p(1)} = \operatorname{co}^{\mathrm{mat}}\mathcal{D}_p(1)$. As $\mathcal{D}_p(1)$ is compact, its convex hull is closed, so we are done. ∎

Remark 4.6. Later in Section 5 we shall give a procedure for constructing a family of L with the property

$$\operatorname{co}^{\mathrm{mat}}\mathcal{D}_p \subseteq \operatorname{proj}_x \mathcal{D}_L. \tag{4.3}$$

While for many p the ordinary convex hull of $\mathcal{D}_p(1)$, admits an LMI lift to $\mathcal{D}_L(1)$, the property (4.3) is not always satisfied. Indeed, the conclusion of Proposition 4.5 can fail.

Example 4.7. Returning to the polynomial $p(x_1, x_2) = 1 - x_1^2 - x_2^4$ of Example 4.4, Proposition 4.5 implies that $\mathcal{D}_p(1) = \operatorname{co}^{\mathrm{mat}}\mathcal{D}_p(1)$. Since, as noted in Example 2.5, $\mathcal{D}_p(2)$ is not convex, $\mathcal{D}_p$ is not a free spectrahedrop. We do not know if the closed matrix convex hull of $\mathcal{D}_p$ is a free spectrahedrop, but Theorem 5.4 below says it almost is.

5. Construction of the Free Lift

Classically, given a commutative semialgebraic set $\mathcal{D}_p(1) \subseteq \mathbb{R}^g$, a construction proposed by Lasserre [Las09a] (see also Parrilo [Par06]) produces a sequence of spectrahedra $(D^{(n)})$ and projections (π_n) such that $\pi_n(D^{(n)}) \supseteq \mathcal{D}_p(1)$ is a nested decreasing sequence of semialgebraic sets approximating the convex hull of $\mathcal{D}_p(1)$. Under mild hypotheses, this *sequence of relaxations* actually terminates and presents $\operatorname{co}\big(\mathcal{D}_p(1)\big)$ as a projection of a spectrahedron; i.e., there is an m such that $\operatorname{co}\big(\mathcal{D}_p(1)\big) = \pi_m(\mathcal{D}^{(m)})$ [HN09, HN10]. For a substantial recent advance, see Scheiderer's complete solution in two dimensions [Sce11, Sce+]. We refer to [DKL11, Hen11, NPS10] for further results in this direction.

There are two parts to our free analog of the Lasserre–Parrilo construction. The first, described in this section, constructs for a given $\mathcal{D}_p$, via free analogs of moment sequences and Hankel matrices, an infinite free spectrahedron $\mathcal{L}_p$, and a canonical projection of $\mathcal{L}_p$ onto the operator convex hull of $\mathcal{D}_p$.

The second part of the construction, appearing in Section 6, consists of a systematic procedure for passing from $\mathcal{L}_p$ to a sequence of finite free spectrahedra and corresponding projections onto increasingly finer outer approximations to the operator convex hull of $\mathcal{D}_p$.

5.1. Free Hankel matrices.

The key ingredient of the systematic method for constructing lifts presented here are the block free (multivariable) analogs of Hankel matrices. A Hankel matrix H is one that is constant on antidiagonals so that the entry $H_{i,j}$ depends only on the sum $i+j$. In particular, a sequence $(h_k)_k$ of self-adjoint $m \times m$ matrices determines a block Hankel matrix $H = (h_{i+j})_{i,j}$. The sequence (h_k) is often referred to as a moment sequence. In the case that H is positive semidefinite, the normalization $h_0 = I$ is typically harmless.

Free Hankel matrices have a description in terms of free moment sequences. Given a positive integer n, a sequence $W := (W_\alpha)_\alpha$ of $n \times n$ matrices W_α indexed by words α in the free symmetric variables $x = (x_1, \dots, x_g)$ is a **moment sequence** if it is symmetric in the sense that $W_{\alpha^*} = W_\alpha^*$ and is normalized by $W_\varnothing = I$. Note that the symmetry of W implies that each W_{x_j} is a self-adjoint matrix. The moment sequence (W_α) determines the **free Hankel** matrix

$$H(W) = \big(W_{\alpha^*\beta}\big)_{\alpha,\beta}.$$

For a positive integer d,

$$H_d(W) = \big(W_{\alpha^*\beta}\big)_{|\alpha|,|\beta|\leq d}.$$

is a **truncated free Hankel matrix** associated to W.

Let p be a $\ell \times \ell$-matrix valued polynomial of degree at most δ. Thus,

$$p(x) = \sum_{|\gamma|\leq\delta} p_\gamma \gamma$$

for some $\ell \times \ell$ matrices p_γ. The $p-$**localizing matrix** $H_p^{\Uparrow}(W)$ associated to $H(W)$ is the $n\ell \times n\ell$ (block) matrix with (α,β) entry

$$H_p^{\Uparrow}(W)_{\alpha,\beta} := \sum_{|\gamma|\leq\delta} p_\gamma \otimes W_{\alpha^*\gamma\beta}.$$

Of course, if $p=1$, then

$$H_1^{\Uparrow}(W) = H(W).$$

For $d \in \mathbb{N}$, the d**-truncated localizing matrix** of p is

$$H_{p,d}^{\Uparrow}(W) := \big(H_p^{\Uparrow}(W)_{\alpha,\beta}\big)_{|\alpha|,|\beta|\leq d}.$$

Note that if the word γ has length $2m-1$ or $2m$, then it can be written as a product $\gamma = \eta^*\sigma$ of words of length at most m. Hence, the truncated localizing matrix actually only depends upon the entries $W_{\alpha^*\beta}$ for $|\alpha|,|\beta| \leq d + \left\lceil \frac{1}{2}\deg(p)\right\rceil$. Here $\lceil\lrcorner\rceil$ denotes the "smallest integer not less than" function. The reader is encouraged to skip ahead temporarily to Subsection 6.3 to get a feel for the structure of these matrices.

An element $Z \in \mathcal{D}_p(m)$ (so acting on $\mathbb{C}^m$) along with an isometry $V : \mathbb{C}^n \to \mathbb{C}^m$ determines a moment sequence,

$$Y_\alpha = V^* Z^\alpha V. \tag{5.1}$$

For instance, if $\alpha = x_1x_2x_1$, then

$$Y_\alpha = V^* Z_1 Z_2 Z_1 V.$$

Note that the fact that $Z^{\varnothing} = I$ and the assumption that V is an isometry implies $Y_{\varnothing} = I$. Further, an easy calculation shows that this moment sequence satisfies

$$H(Y) \succeq 0 \quad \text{and} \quad H_p^{\Uparrow}(Y) \succeq 0. \tag{5.2}$$

Likewise, an element $Z \in \mathcal{D}_p^{\infty}(K)$ along with $n \in \mathbb{N}$ and an isometry $V : \mathbb{C}^n \to K$ determines a moment sequence $(Y_\alpha)_\alpha$ via (5.1) for which (5.2) holds.

5.2. **Riesz Maps.** Let $s \in \mathbb{N}$. To a moment sequence $W = (W_\alpha)_\alpha$ of $n \times n$ matrices there is the associated linear **Riesz mapping**

$$\Phi_W^s : \mathbb{C}^{s\times s}\langle x\rangle \to \mathbb{C}^{sn\times sn}, \quad \sum_{\alpha\in\langle x\rangle} B_\alpha \alpha \mapsto \sum_{\alpha\in\langle x\rangle} B_\alpha \otimes W_\alpha.$$

This linear map is symmetric in the sense that

$$\Phi_W^s(P^*) = \Phi_W^s(P)^*$$

for $P \in \mathbb{C}^{s\times s}\langle x\rangle$.

Similarly, to a truncated Hankel matrix $H_d(W)$, or the corresponding truncated moment sequence $W = (W_\alpha)_{|\alpha|\le 2d}$, we can associate a **Riesz map**

$$\Phi_W^s : \mathbb{C}^{s\times s}\langle x\rangle_{2d} \to \mathbb{C}^{sn\times sn}, \quad \sum_{\alpha\in\langle x\rangle_{2d}} B_\alpha \alpha \mapsto \sum_{\alpha\in\langle x\rangle_{2d}} B_\alpha \otimes W_\alpha.$$

Proposition 5.1. *Suppose W is a moment sequence and let $p \in \mathbb{C}^{\ell\times\ell}\langle x\rangle$ be a symmetric free matrix polynomial. For positive integers s and t,*

(1) *if $H(W) \succeq 0$, then $\Phi_W^s(P^*P) \succeq 0$ for all $P \in \mathbb{C}^{t\times s}\langle x\rangle$;*
(2) *if $H_d(W) \succeq 0$, then $\Phi_W^s(P^*P) \succeq 0$ for all $P \in \mathbb{C}^{t\times s}\langle x\rangle_d$;*
(3) *if $H_p^{\Uparrow}(W) \succeq 0$, then $\Phi_W^s(f^*(I_t \otimes p)f) \succeq 0$ for all $f \in \mathbb{C}^{t\ell\times s}\langle x\rangle$;*
(4) *if $H_{p,d}^{\Uparrow}(W) \succeq 0$, then $\Phi_W^s(f^*(I_t \otimes p)f) \succeq 0$ for all $f \in \mathbb{C}^{t\ell\times s}\langle x\rangle_d$.*

Proof. (1) Write $P = \sum_{\alpha\in\langle x\rangle} P_\alpha \alpha$. Then

$$\Phi_W^s(P^*P) = \sum_{\alpha,\beta} P_\alpha^* P_\beta \otimes W_{\alpha^*\beta}.$$

Let $\vec{P} = \big(P_v\big)_{v\in\langle x\rangle}$ be a column block-vector of coefficients of P. Then

$$\Phi_W^s(P^*P) = \big(\vec{P} \otimes I_n\big)^* \big(I_t \otimes H(W)\big)\big(\vec{P} \otimes I_n\big) \succeq 0 \tag{5.3}$$

since $I_t \otimes H(W) \succeq 0$ by assumption. For the proof of (2) simply replace $H(W)$ by $H_d(W)$ in (5.3).

The proofs of (3) and (4) are similar to those of (1) and (2) respectively. For (3), using the vector notation as in items (1) and (2),

$$\begin{aligned}\Phi_W^s\big(f^*(I_t\otimes p)f\big) &= \sum_\sigma \Big(\sum_{\alpha,\beta}\sum_{\gamma:\alpha^*\gamma\beta=\sigma} f_\alpha^* p_\gamma f_\beta\Big)\otimes W_\sigma \\ &= \sum_{\alpha,\beta}\big(f_\alpha^*\otimes I_n\big)\Big(\sum_\gamma I_t\otimes p_\gamma\otimes W_{\alpha^*\gamma\beta}\Big)\big(f_\beta\otimes I_n\big) \\ &= \big(\vec f\otimes I_n\big)^*\big(I_t\otimes H_p^{\Uparrow}(W)\big)\big(\vec f\otimes I_n\big).\end{aligned}$$

For (4) we use the corresponding truncated version

$$\Phi_W^s(f^*pf) = \big(\vec f\otimes I_n\big)^*\big(I_t\otimes H_{p,d}^{\Uparrow}(W)\big)\big(\vec f\otimes I_n\big),$$

where $\vec f$ is a block column vector consisting of coefficients of f. ■

5.3. **Lasserre–Parrilo Lift: Moment Relaxations.** Given a positive integer n, let

$$\mathcal{L}_p(n) := \{Y=(Y_\alpha)_\alpha : Y_\alpha\in\mathbb{C}^{n\times n},\quad Y_\varnothing = I,\quad Y_{\alpha^*}=Y_\alpha^*,\quad H(Y)\succeq 0,\quad H_p^{\Uparrow}(Y)\succeq 0\}$$

and let $\mathcal{L}_p$ denote the sequence $(\mathcal{L}_p(n))_n$. Implicitly, the Y in $\mathcal{L}_p$ are understood to be moment sequences. Moreover, let

$$\mathcal{L}_p^{\mathrm{fin}} := \{Y\in\mathcal{L}_p : \operatorname{rank} H(Y)<\infty\}.$$

In particular, the Y appearing in (5.1) is in $\mathcal{L}_p$ if $Z\in\mathcal{D}_p^\infty$, and is in $\mathcal{L}_p^{\mathrm{fin}}$ if $Z\in\mathcal{D}_p$,. Given $Y\in\mathcal{L}_p(n)$, let

$$\hat Y = (Y_{x_1},Y_{x_2},\dots,Y_{x_g})\in\mathbb{S}_n^g.$$

Theorem 5.2. *If $X\in\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$, then there is a $Y\in\mathcal{L}_p^{\mathrm{fin}}$ such that*

$$X=\hat Y.$$

Conversely, if $Y\in\mathcal{L}_p^{\mathrm{fin}}$, then $\hat Y\in\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$.

Proof. If X is in the matrix convex hull of $\mathcal{D}_p$, then there is an isometry Q and $Z\in\mathcal{D}_p$ such that $X=Q^*ZQ$. In this case the moment sequence $Y_\alpha = Q^*Z^\alpha Q$ satisfies the conclusion of the first part of the theorem.

To prove the converse, suppose (Y_α) is a moment sequence from $\mathcal{L}_p^{\mathrm{fin}}(n)$. Define, on the vector space $\mathbb{C}\langle x\rangle\otimes\mathbb{C}^n$, the sesquilinear form

$$[s,t]_Y = \sum_{\alpha,\beta}\langle Y_{\beta^*\alpha}s_\alpha, t_\beta\rangle \tag{5.4}$$

where $s=\sum\alpha\otimes s_\alpha$ and $t=\sum\beta\otimes t_\beta$. The assumption that $H(Y)\succeq 0$ implies that the form $[s,t]_Y$ is positive semidefinite. Let $\mathcal{E}_Y$ denote the (pre-)Hilbert space obtained by modding out the subspace

$$\mathcal{N} = \{s : [s,s]_Y = 0\}.$$

Note that $\operatorname{rank} H(Y) < \infty$ implies $\dim \mathcal{E}_Y < \infty$ and hence $\mathcal{E}_Y$ is a Hilbert space.

An important observation is the following: If $s \in \mathcal{N}$ and $1 \leq j \leq g$, then $r = x_j s \in \mathcal{N}$, i.e., $\mathcal{N}$ is a left $\mathbb{C}\langle x\rangle$-submodule of $\mathbb{C}\langle x\rangle \otimes \mathbb{C}^n$. To prove this observation, note that, because $H(Y)$ is positive semidefinite, if $s \in \mathcal{N}$ then

$$\sum_\alpha Y_{\beta^*\alpha} s_\alpha = 0$$

for each β (and conversely). In this case,

$$\sum_\gamma Y_{\beta^*\gamma} r_\gamma = \sum_\alpha Y_{\beta^* x_j \alpha} s_\alpha = \sum_\gamma Y_{(x_j\beta)^*\alpha} s_\alpha = 0$$

and hence $r \in \mathcal{N}$. It now follows that the mapping Z_j sending s to $x_j s$ is well defined on the finite-dimensional Hilbert space $\mathcal{E}_Y$. The computation above also shows that whether or not $s \in \mathcal{N}$,

$$\langle x_j s, t\rangle = \langle s, x_j t\rangle$$

and hence $Z_j^* = Z_j$.

Define $Q : \mathbb{C}^n \to \mathcal{E}_Y$ by

$$Qv = \varnothing \otimes v$$

and note that Q is an isometry. By construction, $Q^* Z^\alpha Q = Y_\alpha$.

Finally, to see $p(Z) = \sum p_\gamma \otimes Z^\gamma$ is positive definite, let $s = \sum e_j \otimes \alpha \otimes s_{\alpha,j}$ be given, where $\{e_1, \ldots, e_\ell\}$ is the standard orthonormal basis for $\mathbb{C}^\ell$ (the space that the p_γ act on) and $s_{\alpha,j} \in \mathbb{C}^n$. Then,

$$\begin{aligned}
\langle p(Z)s, s\rangle &= \sum_{\alpha,\beta,\gamma,j,k} \langle p_\gamma \otimes Z^\gamma e_j \otimes \alpha \otimes s_{\alpha,j}, e_k \otimes \beta \otimes s_{\beta,k}\rangle \\
&= \sum \langle p_\gamma e_j, e_k\rangle \, \langle Z^\gamma \alpha \otimes s_{\alpha,j}, \beta \otimes s_{\beta,k}\rangle \\
&= \sum \langle p_\gamma e_j, e_k\rangle \, \langle Y_{\beta^*\gamma\alpha} s_{\alpha,j}, s_{\beta,k}\rangle \\
&= \sum_{\alpha,\beta} \Big\langle \big(\sum_\gamma p_\gamma \otimes Y_{\beta^*\gamma\alpha}\big) \sum_j e_j \otimes s_{\alpha,j}, \sum_k e_k \otimes s_{\beta,k} \Big\rangle \\
&= \langle H_p^{\Uparrow}(Y)\vec{s}, \vec{s}\rangle \geq 0,
\end{aligned}$$

where $\vec{s}$ is the vector $(s_\alpha)_\alpha$ for $s_\alpha = \sum_j e_j \otimes s_{\alpha,j}$. Thus the assumption that $H_p^{\Uparrow}(Y)$ is positive semidefinite implies $p(Z)$ is positive semidefinite. We conclude that $\hat{Y} = Q^* Z Q$ is in the matrix convex hull of $\mathcal{D}_p$. ∎

Definition 5.3. Given p, let

$$\hat{\mathcal{L}}_p := \{\hat{Y} : Y \in \mathcal{L}_p\} \quad \text{and} \quad \hat{\mathcal{L}}_p^{\text{fin}} := \{\hat{Y} : Y \in \mathcal{L}_p^{\text{fin}}\}.$$

Theorem 5.2 says that the matrix convex hull $\operatorname{co}^{\text{mat}} \mathcal{D}_p$ of $\mathcal{D}_p$ equals $\hat{\mathcal{L}}_p^{\text{fin}}$.

Next we turn to operator convex hulls. To obtain a good lifting theorem we make a boundedness assumption which replaces the rank $H(Y)$ finite condition used in Theorem 5.2.

Given $K \in \mathbb{R}_{>0}$, the matrix polynomial p is **K-archimedean** if there exist matrix polynomials s_j and f_j such that

$$K^2 - \sum_j x_j^2 = \sum s_j^* s_j + \sum f_j^* p f_j, \tag{5.5}$$

and p is **archimedean** if it is K-archimedean for some $K > 0$.

Theorem 5.4. *If p is archimedean, then $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p = \hat{\mathcal{L}}_p$. Moreover, $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p(n)$ is closed and bounded and contains $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(n)$ for each n.*

5.3.1. *Proof of Theorem 5.4.* The proof begins with several lemmas.

Lemma 5.5. If p is archimedean, then $\mathcal{D}_p^\infty$ is uniformly bounded.

Proof. If p is archimedean, then by (5.5) there is $N \in \mathbb{N}$ with $N - \sum_j x_j^2|_{\mathcal{D}_p^\infty} \succeq 0$. Hence $\mathcal{D}_p^\infty \subseteq \big\{X \in \mathbb{S}^g_{\mathrm{oper}} : \|X\|^2 \leq N\big\}$. ∎

Lemma 5.6. If $Y \in \mathcal{L}_p(n)$, then there exist

(i) a Hilbert space $\mathscr{H}$;
(ii) a dense subset $\mathscr{P}$ of $\mathscr{H}$;
(iii) a tuple $Z = (Z_1, \dots, Z_g)$ such that each $Z_j : \mathscr{P} \to \mathscr{P}$ is self-adjoint in the sense that $\langle Z_j p, q\rangle = \langle p, Z_j q\rangle$ for each $p, q \in \mathscr{P}$; and
(iv) an isometry $V : \mathbb{C}^n \to \mathscr{P}$

such that

(a) $p(Z) : \mathscr{P} \to \mathscr{P}$ is positive semidefinite;
(b) $\hat{Y} = V^* Z V$; and
(c) if in addition p is K-archimedean, then each Z_j is a bounded operator (and so extends to all of $\mathscr{H}$) with $K^2 - \sum Z_j^2 \succeq 0$. Hence the $\hat{Y}$ from (b) is in $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$.

Proof. Following the proof of Theorem 5.2, given a moment sequence $(Y_\alpha) \in \mathcal{L}_p(n)$, define the pre-inner product $[\_, \_]$ on $\mathscr{R} = \mathbb{C}\langle x\rangle \otimes \mathbb{C}^n$, as in Equation (5.4). Let

$$\mathcal{N} = \{f \in \mathbb{C}\langle x\rangle \otimes \mathbb{C}^n : [f, f] = 0\}.$$

A standard argument shows that $\mathcal{N}$ is a subspace of $\mathscr{R}$ and that the form

$$[f, g] = [f + \mathcal{N}, g + \mathcal{N}]$$

is well defined and positive definite on the quotient $\mathscr{P}$ of $\mathscr{R}$ by $\mathcal{N}$.

The operators Z_j of multiplication by x_j are as before (see the proof of Theorem 5.2) well defined relative to this pre-inner product; i.e., each $Z_j : \mathscr{P} \to \mathscr{P}$. Moreover,

$$p(Z) \succeq 0 \tag{5.6}$$

on $\mathscr{P}$ too. Define $V:\mathbb{C}^n \to \mathscr{R}$ by

$$Vh = (\varnothing \otimes h) + \mathcal{N}.$$

Then V is an isometry (since $Y_\varnothing = I$) and $V^* Z V = \hat{Y}$.

Let us show that the Z_j are bounded under the archimedean hypothesis. By K-archimedeanity of p,

$$K^2 - \sum_j x_j^2 = \sum_i f_i^* f_i + \sum_k r_k^* p r_k$$

for some free polynomials f_i, r_k. It is now clear that (5.6) implies $K^2 - \sum_j Z_j^2 \succeq 0$, i.e., $\|Z\|^2 \le K^2$ so Z is bounded. Then by definition, $\hat{Y} \in \mathrm{co}^{\mathrm{oper}}\mathcal{D}_p(n)$. ∎

The proof of the moreover statement in Theorem 5.4 will use the following lemma.

Lemma 5.7. If p is archimedean, then for each α there is a constant C_α such that if $Y \in \mathcal{L}_p$, then $\|Y_\alpha\| \le C_\alpha$. Further, if $(Y^j)_j = ((Y^j_\alpha)_\alpha)_j$ is a sequence from $\mathcal{L}_p(n)$ satisfying for each α there is a Y_α such that $(Y^j_\alpha)_j$ converges to Y_α, then $Y = (Y_\alpha)_\alpha \in \mathcal{L}_p(n)$.

Proof. Suppose p is K-archimedean. By Lemma 5.6, given $Y \in \mathcal{L}_p(n)$ there exists an operator tuple Z acting on a Hilbert space $\mathscr{H}$ with $K^2 - \sum Z_j^2 \succeq 0$ and $p(Z) \succeq 0$ as well as an isometry $V:\mathbb{C}^n \to \mathscr{H}$ such that $Y_\alpha = V^* Z^\alpha V$. Letting $|\alpha|$ denote the length of the word α, it is immediate that

$$\|Y_\alpha\| \le K^{|\alpha|}.$$

For the second part of the lemma, note that for d fixed, each $H_d(Y^j)$ is positive semidefinite. Since $H_d(Y^j)$ is a (finite) matrix and depends only upon $|\alpha| \le 2d$, it follows that $(H_d(Y^j))_j$ converges to $H_d(Y)$. Thus $H_d(Y)$ is positive semidefinite. Since $H_d(Y)$ is positive semidefinite for all d, it follows that $H(Y)$ is also positive semidefinite.

In a similar manner, each $H^{\Uparrow}_{p,d}(Y^j)$ is positive semidefinite and, for d fixed, $(H^{\Uparrow}_{p,d}(Y^j))_j$ converges to $H^{\Uparrow}_{p,d}(Y)$ and thus $H^{\Uparrow}_{p,d}(Y)$ is positive semidefinite. It follows that $H^{\Uparrow}_p(Y)$ is positive semidefinite. Thus $Y \in \mathcal{L}_p$. ∎

Proof of Theorem 5.4. From Lemma 5.6 it follows that $\hat{\mathcal{L}}_p \subseteq \mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$. The reverse inclusion follows along the lines of the proof of the similar statement in Theorem 5.2. Namely, simply observe that if $Z \in \mathcal{D}_p^\infty$ and V is an isometry from $\mathbb{C}^n$ into the space $\mathscr{H}$ that Z acts on, then $Y_\alpha = V^* Z^\alpha V$ defines a moment sequence $Y \in \mathcal{L}_p(n)$.

The inclusion $\mathcal{D}_p \subseteq \mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$ is evident. Likewise the archimedean hypothesis and Lemma 5.5 readily imply the boundedness of $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$. Thus, to finish the proof of the moreover statement, it remains to show each $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p(n)$ is closed. Accordingly, suppose $(X^{(j)})$ is a sequence from $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(n)$ which converges to some $X \in \mathbb{S}_n^g$. In particular, the $X^{(j)}$ act on $\mathbb{C}^n$ and for each k, the sequence $(X_k^{(j)})_j$ converges to X_k.

For each j there is a tuple $Z^{(j)} \in \mathcal{D}_p^\infty$ acting on a Hilbert space K_j and an isometry $V_j : \mathbb{C}^n \to K_j$ such that

$$X^{(j)} = V_j^* Z^{(j)} V_j.$$

The moment sequence, $(Y_\alpha^{(j)})$ coming from the pairs $(Z^{(j)}, V_j)$ via

$$Y_\alpha^{(j)} = V_j^* (Z^{(j)})^\alpha V_j$$

is in $\mathcal{L}_p$.

For fixed α, the hypothesis and Lemma 5.7 together imply that the sequence $(Y_\alpha^{(j)})$ is bounded and thus has a convergent subsequence. Thus, by passing to a subsequence (using the usual diagonalization argument) we can assume that, for each α, there is a Y_α to which $Y_\alpha^{(j)}$ converges. By the second part of Lemma 5.7, this moment sequence (Y_α) belongs to $\mathcal{L}_p$. Hence the corresponding operator Z from Lemma 5.6 satisfies $p(Z) \succeq 0$. Thus $Z \in \mathcal{D}_p^\infty$. By construction,

$$V_0^* Z V_0 = (Y_{x_1}, \dots, Y_{x_g}) = X.$$

Hence $X \in \mathrm{co}^{\mathrm{oper}} \mathcal{D}_p(n)$ and therefore $\mathrm{co}^{\mathrm{oper}} \mathcal{D}_p(n)$ is closed. ∎

Remark 5.8. Note that the reverse inclusion,

$$\mathrm{co}^{\mathrm{oper}} \mathcal{D}_p(n) \subseteq \overline{\mathrm{co}}^{\mathrm{mat}} \mathcal{D}_p(n),$$

holds exactly when matrices - and not operators - suffice in the [HM04] Positivstellensatz. Indeed, $\mathrm{co}^{\mathrm{oper}} \mathcal{D}_p$ is the intersection of all $\mathcal{D}_L$ for monic L such that $L(Z) \succeq 0$ for all $Z \in \mathcal{D}_p^\infty$. On the other hand, $\overline{\mathrm{co}}^{\mathrm{mat}} \mathcal{D}_p$ is the intersection of all $\mathcal{D}_L$ for monic L such that $L(Z) \succeq 0$ for all $Z \in \mathcal{D}_p$. This theme was discussed in Corollary 3.5 above; see also Subsection 6.5.2.

6. Truncated Moments - Approximations of the Matrix Convex Hull

This section presents the second part of the Lasserre–Parrilo construction in the free setting. It consists of a sequence of truncations of the lift $\mathcal{L}_p$ of $\mathcal{D}_p$ from Section 5 to a sequence of finite free spectrahedra and corresponding projections onto increasingly finer outer approximations to the operator convex hull of $\mathcal{D}_p$. Alternately, the construction can be thought of as producing a sequence of approximate free spectrahedral lifts of a given free semialgebraic set $\mathcal{D}_p$ to LMI domains in increasingly many variables.

Whether this construction produces the matrix convex hull at a finite stage is a basic question. In Subsection 7.4 we give examples where the answer is yes. In fact, in these the convex hulls involved require no lifts, they are themselves free spectrahedra. In the other direction, for the TV screen the construction does not produce the matrix convex hull at the first stage, cf. Section 7.

6.1. **Main Formulas for Lifts.** To state the main result of this paper precisely, for $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$, let

$$\mathcal{L}_p(n;d) := \big\{(Y_\alpha)_\alpha : |\alpha| \le 2d + \deg p + 1,\ Y_\alpha \in \mathbb{C}^{n\times n},\ Y_\varnothing = I,\ Y_{\alpha^*} = Y_\alpha^*, \\ H_{d+\lceil \frac{1}{2}\deg p\rceil}(Y) \succeq 0,\ H^{\Uparrow}_{p,d}(Y) \succeq 0\big\}.$$

The sequence $\mathcal{L}_p(\text{␣};d) = (\mathcal{L}_p(n;d))_n$ is a free convex set and, as before, $\hat{\mathcal{L}}_p(n;d)$ denotes the image of the projection

$$\mathcal{L}_p(n;d) \ni Y \mapsto \hat{Y} = (Y_{x_1},\dots,Y_{x_g}) \in \mathbb{S}_n^g.$$

Theorem 6.1 (Clamping down theorem)**.** *If p is archimedean, then for each n,*

$$\bigcap_{d=0}^{\infty} \hat{\mathcal{L}}_p(n;d) = \hat{\mathcal{L}}_p(n).$$

Corollary 6.2. *If p is archimedean, then $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p(n) = \hat{\mathcal{L}}_p(n)$ for each $n \in \mathbb{N}$. Hence the sets $\hat{\mathcal{L}}_p(n;d)$ close down on $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p(n)$. Further, for each d there exists a linear pencil L_d such that $\hat{\mathcal{L}}_p(\text{␣};d)$ lifts to $\mathcal{D}_{L_d}$. Thus $\hat{\mathcal{L}}_p(\text{␣};d)$ is a sequence of free spectrahedrops which are outer approximations to $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$ and which converge monotonically to $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$ as d tends to infinity.*

Corollary 6.2 is an immediate consequence of Theorems 6.1 and 5.4, and the fact that there exists a linear pencil L_d such that $\mathrm{proj}_x\, \mathcal{D}_{L_d} = \hat{\mathcal{L}}_p(\text{␣};d)$ as we now explain.

6.2. $\hat{\mathcal{L}}_p(\text{␣};d)$ **are free spectrahedrops.** The free Lasserre–Parrilo construction produces the approximate lifts $\mathcal{D}_\Delta = \mathcal{L}_p(\text{␣};d)$ of $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$, in which $\mathcal{D}_\Delta$ is the positivity set of a linear matrix polynomial

$$\Delta(x,y) = A_0 + \sum_{j=1}^{g} A_j x_j + \sum_{\ell=1}^{h} \big(B_\ell y_\ell + B_\ell^* y_\ell^*\big) \tag{6.1}$$

where the coefficients are $k\times k$ self-adjoint matrices $A_0,\dots,A_g \in \mathbb{S}_k$, and $k\times k$ matrices $B_1,\dots,B_h \in \mathbb{C}^{k\times k}$. This Δ can be naturally evaluated at tuples $(X,Y) \in \mathbb{S}_n^g \times (\mathbb{C}^{n\times n})^h$ by

$$\Delta(X,Y) = A_0 \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j + \sum_{\ell=1}^{h} \big(B_\ell \otimes Y_\ell + B_\ell^* \otimes Y_\ell^*\big) \in \mathbb{S}_{nk}.$$

While the coefficients A_j are self-adjoint and the variables x_j are symmetric, the coefficients and variables B_ℓ and y_ℓ are not. Hence Δ is not a linear pencil according to the terminology in this article. The following lemma shows that the coefficients B_ℓ and variables y_ℓ can be replaced with self-adjoint coefficients and symmetric variables in such a way as to obtain a linear pencil L such that $\mathrm{proj}_x\, \mathcal{D}_\Delta = \mathrm{proj}_x\, \mathcal{D}_L$.

Lemma 6.3. *Given a linear matrix polynomial $\Delta(x,y)$ as in (6.1) in g symmetric and h free variables, there exists a linear pencil $L(x,w)$ in $g+2h$ variables such that*

$$\operatorname{proj}_x \mathcal{D}_\Delta = \operatorname{proj}_x \mathcal{D}_L.$$

Proof. Write $B_\ell = C_\ell + iD_\ell$ for $C_\ell, D_\ell \in \mathbb{S}_k$, and let $y_\ell = w_\ell + iw_{-\ell}$, where $w = (w_{-h}, \dots, w_{-1}, w_1, \dots, w_h)$ are free symmetric variables. Then

$$\begin{aligned} B_\ell y_\ell + B_\ell^* y_\ell^* &= (C_\ell + iD_\ell)(w_\ell + iw_{-\ell}) + (C_\ell - iD_\ell)(w_\ell - iw_{-\ell}) \\ &= 2(C_\ell w_\ell - D_\ell w_{-\ell}). \end{aligned}$$

Let

$$L(x,w) = A_0 + \sum_{j=1}^{g} A_j x_j + 2\sum_{\ell=1}^{h} \big(C_\ell w_\ell - D_\ell w_{-\ell}\big) \in \mathbb{S}_k\langle x, w\rangle. \tag{6.2}$$

This is a linear pencil in symmetric variables with self-adjoint coefficients. By construction,

$$\operatorname{proj}_x \mathcal{D}_L = \operatorname{proj}_x \mathcal{D}_\Delta. \qquad \blacksquare$$

Proof of Corollary 6.2. There is a linear matrix polynomial Δ as in (6.1) such that

$$\mathcal{L}_p(\text{␣}; d) = \mathcal{D}_\Delta$$

and thus

$$\hat{\mathcal{L}}_p(\text{␣}; d) = \operatorname{proj}_x \mathcal{D}_\Delta.$$

Hence Lemma 6.3 yields a linear pencil L_d with

$$\operatorname{proj}_x \mathcal{D}_{L_d} = \operatorname{proj}_x \mathcal{D}_\Delta = \hat{\mathcal{L}}_p(\text{␣}; d). \qquad \blacksquare$$

Remark 6.4. *Free Sets in Real Variables.* The first part of Corollary 6.2 holds when complex scalars, and thus complex self-adjoint matrices as well as complex polynomials, are replaced by real scalars, symmetric matrices and real polynomials. Call a linear matrix polynomial of the type in Equation (6.1) a pencil in **mixed variables**. If the notion of a spectrahedrop is relaxed to include the projection of the positivity set $\mathcal{D}_\Delta$ a pencil Δ in mixed variables onto the x (symmetric) variables, then the second part of Corollary 6.2 holds over $\mathbb{R}$ too.

In the real setting, the construction of Lemma 6.3 expresses $B_\ell = C_\ell + D_\ell$ where C_ℓ is a symmetric matrix and D_ℓ is a skew-symmetric matrix. Thus, a mixed variable pencil can be replaced by a mixed variable pencil which is the sum of a linear pencil in symmetric coefficients and variables and a homogeneous linear polynomial in skew-symmetric coefficients and variables.

6.3. **Examples.** Here we explicitly write down the first Lasserre–Parrilo lift for the bent TV screen. For convenience, a word $x_{i_1}x_{i_2}\cdots x_{i_k}$ will be denoted by $i_1\, i_2\, \dots\, i_k$ and the corresponding moment by $Y_{i_1\, i_2\, \dots\, i_k}$.

6.3.1. *The $d = 0$ relaxation.* $\hat{\mathcal{L}}_p(\llcorner\!\lrcorner;0)$. We first apply the lifting construction to $d = 0$, and $p = 1 - x_1^2 - x_2^4$. Since $\deg p = 4$, the lift can be written as

$$
H_2(Y) = \begin{pmatrix}
1 & X_1 & X_2 & Y_{11} & Y_{12} & Y_{21} & Y_{22} \\
X_1 & Y_{11} & Y_{12} & Y_{111} & Y_{112} & Y_{121} & Y_{122} \\
X_2 & Y_{21} & Y_{22} & Y_{211} & Y_{212} & Y_{221} & Y_{222} \\
Y_{11} & Y_{111} & Y_{112} & Y_{1111} & Y_{1112} & Y_{1121} & Y_{1122} \\
Y_{21} & Y_{211} & Y_{212} & Y_{2111} & Y_{2112} & Y_{2121} & Y_{2122} \\
Y_{12} & Y_{121} & Y_{122} & Y_{1211} & Y_{1212} & Y_{1221} & Y_{1222} \\
Y_{22} & Y_{221} & Y_{222} & Y_{2211} & Y_{2212} & Y_{2221} & Y_{2222}
\end{pmatrix} \succeq 0, \tag{6.3}
$$
$$
H_{p,0}^{\Uparrow}(Y) = 1 - Y_{11} - Y_{2222} \succeq 0.
$$

It is well known (see e.g. [HN08]) that (6.3) is exact at the scalar level, meaning that $(X_1, X_2) \in \mathcal{D}_p(1)$ if and only if (6.3) has a solution.

Consider the following cut-down of (6.3):

$$
\check{H}_2(Y) = \begin{pmatrix}
1 & X_1 & X_2 & Y_{22} \\
X_1 & Y_{11} & Y_{12} & Y_{122} \\
X_2 & Y_{21} & Y_{22} & Y_{222} \\
Y_{22} & Y_{221} & Y_{222} & Y_{2222}
\end{pmatrix} \succeq 0, \tag{6.4}
$$
$$
H_{p,0}^{\Uparrow}(Y) = 1 - Y_{11} - Y_{2222} \succeq 0.
$$

We shall see later that the lifts given by (6.3) and (6.4) are equivalent, and are equivalent to the standard LMI lift for the TV screen; see Section 7 below for details.

6.3.2. *The $d = 1$ relaxation.* $\hat{\mathcal{L}}_p(\llcorner\!\lrcorner;1)$. Here is the next Lasserre–Parrilo relaxation:

$$
H_3(Y) = \begin{pmatrix}
1 & X_1 & X_2 & Y_{11} & Y_{12} & Y_{21} & Y_{22} & Y_{111} & Y_{112} & Y_{121} & Y_{122} & Y_{211} & Y_{212} & Y_{221} & Y_{222} \\
X_1 & Y_{11} & Y_{12} & Y_{111} & Y_{112} & Y_{121} & Y_{122} & Y_{1111} & Y_{1112} & Y_{1121} & Y_{1122} & Y_{1211} & Y_{1212} & Y_{1221} & Y_{1222} \\
X_2 & Y_{21} & Y_{22} & Y_{211} & Y_{212} & Y_{221} & Y_{222} & Y_{2111} & Y_{2112} & Y_{2121} & Y_{2122} & Y_{2211} & Y_{2212} & Y_{2221} & Y_{2222} \\
Y_{11} & Y_{111} & Y_{112} & Y_{1111} & Y_{1112} & Y_{1121} & Y_{1122} & Y_{11111} & Y_{11112} & Y_{11121} & Y_{11122} & Y_{11211} & Y_{11212} & Y_{11221} & Y_{11222} \\
Y_{21} & Y_{211} & Y_{212} & Y_{2111} & Y_{2112} & Y_{2121} & Y_{2122} & Y_{21111} & Y_{21112} & Y_{21121} & Y_{21122} & Y_{21211} & Y_{21212} & Y_{21221} & Y_{21222} \\
Y_{12} & Y_{121} & Y_{122} & Y_{1211} & Y_{1212} & Y_{1221} & Y_{1222} & Y_{12111} & Y_{12112} & Y_{12121} & Y_{12122} & Y_{12211} & Y_{12212} & Y_{12221} & Y_{12222} \\
Y_{22} & Y_{221} & Y_{222} & Y_{2211} & Y_{2212} & Y_{2221} & Y_{2222} & Y_{22111} & Y_{22112} & Y_{22121} & Y_{22122} & Y_{22211} & Y_{22212} & Y_{22221} & Y_{22222} \\
Y_{111} & Y_{1111} & Y_{1112} & Y_{11111} & Y_{11112} & Y_{11121} & Y_{11122} & Y_{111111} & Y_{111112} & Y_{111121} & Y_{111122} & Y_{111211} & Y_{111212} & Y_{111221} & Y_{111222} \\
Y_{211} & Y_{2111} & Y_{2112} & Y_{21111} & Y_{21112} & Y_{21121} & Y_{21122} & Y_{211111} & Y_{211112} & Y_{211121} & Y_{211122} & Y_{211211} & Y_{211212} & Y_{211221} & Y_{211222} \\
Y_{121} & Y_{1211} & Y_{1212} & Y_{12111} & Y_{12112} & Y_{12121} & Y_{12122} & Y_{121111} & Y_{121112} & Y_{121121} & Y_{121122} & Y_{121211} & Y_{121212} & Y_{121221} & Y_{121222} \\
Y_{221} & Y_{2211} & Y_{2212} & Y_{22111} & Y_{22112} & Y_{22121} & Y_{22122} & Y_{221111} & Y_{221112} & Y_{221121} & Y_{221122} & Y_{221211} & Y_{221212} & Y_{221221} & Y_{221222} \\
Y_{112} & Y_{1121} & Y_{1122} & Y_{11211} & Y_{11212} & Y_{11221} & Y_{11222} & Y_{112111} & Y_{112112} & Y_{112121} & Y_{112122} & Y_{112211} & Y_{112212} & Y_{112221} & Y_{112222} \\
Y_{212} & Y_{2121} & Y_{2122} & Y_{21211} & Y_{21212} & Y_{21221} & Y_{21222} & Y_{212111} & Y_{212112} & Y_{212121} & Y_{212122} & Y_{212211} & Y_{212212} & Y_{212221} & Y_{212222} \\
Y_{122} & Y_{1221} & Y_{1222} & Y_{12211} & Y_{12212} & Y_{12221} & Y_{12222} & Y_{122111} & Y_{122112} & Y_{122121} & Y_{122122} & Y_{122211} & Y_{122212} & Y_{122221} & Y_{122222} \\
Y_{222} & Y_{2221} & Y_{2222} & Y_{22211} & Y_{22212} & Y_{22221} & Y_{22222} & Y_{222111} & Y_{222112} & Y_{222121} & Y_{222122} & Y_{222211} & Y_{222212} & Y_{222221} & Y_{222222}
\end{pmatrix} \succeq 0,
$$

$$H^{\Uparrow}_{p,1}(Y) = \begin{pmatrix} 1-Y_{11}-Y_{2222} & X_1-Y_{111}-Y_{22221} & X_2-Y_{112}-Y_{22222} \\ X_1-Y_{111}-Y_{12222} & Y_{11}-Y_{1111}-Y_{122221} & Y_{12}-Y_{1112}-Y_{122222} \\ X_2-Y_{211}-Y_{22222} & Y_{21}-Y_{2111}-Y_{222221} & Y_{22}-Y_{2112}-Y_{222222} \end{pmatrix} \succeq 0.$$

6.4. **Proof of Theorem 6.1.** The following lemma generalizes Lemma 5.7.

Lemma 6.5. If p is archimedean, then there is a natural number ν and a positive number C such that if

$$Y \in \mathcal{L}_p(n;d)$$

and α is a word with length $|\alpha|$ at most $2(d-\nu)$, then

$$\|Y_\alpha\| \leq C^{|\alpha|}.$$

The proof of this lemma uses the following variant of the Gelfand-Naimark-Segal (GNS) construction. A Hankel matrix $Y \in \mathcal{L}_p(n;d)$ with a truncated positive semidefiniteness property generates a pre-Hilbert space as follows. Assuming that $H_{d+\lceil \frac{1}{2}\deg p\rceil}(Y)$ and $H^{\Uparrow}_{p,d}(Y)$ are positive semidefinite, define the sesquilinear form on $\mathbb{C}^n \otimes \mathbb{C}\langle x\rangle_{d+\lceil \frac{1}{2}\deg(p)\rceil}$ by

$$\langle h\otimes\alpha, k\otimes\beta\rangle = \langle Y_{\beta^*\alpha}h, k\rangle.$$

Positivity of $H^{\Uparrow}_{p,d}(Y)$ is then equivalent to the condition,

$$\Big\langle \sum h_\alpha \otimes p\alpha, \sum h_\beta\otimes\beta\Big\rangle \geq 0$$

for all $h \in \mathbb{C}^n \otimes \mathbb{C}\langle x\rangle_{d+\lceil \frac{1}{2}\deg(p)\rceil}$ of the form

$$h = \sum_{|\alpha|\leq d} h_\alpha\otimes\alpha.$$

In particular, if f is a polynomial of degree ν and

$$q = f^*pf,$$

then for $h = \sum_{|\alpha|\leq d-\nu} h_\alpha \otimes \alpha$,

$$\big\langle h_\alpha\otimes f^*pf\alpha, \sum_\beta h_\beta\otimes\beta\big\rangle = \big\langle h_\alpha\otimes pf\alpha, \sum_\beta h_\beta\otimes f\beta\big\rangle \geq 0.$$

Hence the localizing matrix $H^{\Uparrow}_{q,d-\nu}(Y)$ is positive semidefinite. An analogous statement is true for a polynomial $q = s^*s$ when s has degree at most $\nu + \lceil \frac{1}{2}\deg(p)\rceil$.

Proof of Lemma 6.5. By the archimedean hypothesis, there exist a constant C and a natural number μ such that for each j there exist polynomials $s_{j,1},\ldots,s_{j,\mu}$ and $f_{j,1},\ldots,f_{j,\mu}$ with

$$q_j = C^2 - x_j^2 = \sum_k s^*_{j,k}s_{j,k} + \sum_\ell f^*_{j,\ell}pf_{j,\ell}. \tag{6.5}$$

Choose ν such that $\deg(f_{j,\ell}) \le \nu$ for all j, ℓ and $\deg(s_{j,k}) \le \nu + \left\lceil \frac{1}{2}\deg(p) \right\rceil$ for all j,k. Fixing j and letting $S_k = s_{j,k}^* s_{j,k}$ and $F_k = f_{j,\ell}^* p f_{j,\ell}$ it follows that

$$H^{\Uparrow}_{q_j,d-\nu}(Y) = \sum H^{\Uparrow}_{S_k,d-\nu}(Y) + \sum H^{\Uparrow}_{F_\ell,d-\nu}(Y).$$

Hence, by the discussion above, the localizing matrix $H^{\Uparrow}_{q_j,d-\nu}(Y)$ is positive semidefinite.

Given a β with $|\beta| < d-\nu$, positivity of the localizing matrix for $C^2 - x_j^2$ implies that

$$C^2 Y_{\beta^*\beta} \succeq Y_{x_j^*\beta^*\beta x_j}.$$

Thus an induction argument on $|\beta|$ gives, for any $|\beta| \le d-\nu$ that

$$Y_{\beta^*\beta} \preceq C^{2|\beta|} I.$$

Now suppose α is a word with $|\alpha| \le 2(d-\nu)$. There exist words β and γ of length at most $d-\nu$ such that $\alpha = \beta^*\gamma$. From the fact that $H(Y) \succeq 0$, it follows that

$$\begin{pmatrix} Y_{\beta^*\beta} & Y_{\beta^*\gamma} \\ Y_{\gamma^*\beta} & Y_{\gamma^*\gamma} \end{pmatrix}$$

is positive semidefinite. Thus,

$$Y_{\gamma^*\beta} Y_{\beta^*\gamma} \preceq C^{2\left(|\beta|+|\gamma|\right)} I.$$

The desired inequality follows. ■

Proof of Theorem 6.1. It is obvious that

$$\bigcap_d \mathcal{L}_p(n;d) = \mathcal{L}_p(n)$$

in the sense that a moment sequence $(Y_\alpha)_\alpha$ all of whose truncations satisfy the positive semidefiniteness of the Hankel matrices $H_{d+\left\lceil \frac{1}{2}\deg p \right\rceil}(Y)$, and $H^{\Uparrow}_{p,d}(Y(n))$, is in $\mathcal{L}_p(n)$, i.e., makes the infinite Hankel matrices $H(Y)$ and $H^{\Uparrow}_p(Y)$ positive semidefinite.

Suppose the moment sequence $Z \in \bigcap_d \hat{\mathcal{L}}_p(n;d)$. In this case, for each d there is a (truncated) moment sequence $Y^{(d)} = (Y^{(d)}_\alpha) \in \hat{\mathcal{L}}_p(n;d)$ such that

$$(Y^{(d)}_{x_1}, \dots, Y^{(d)}_{x_g}) = Z.$$

By construction, for each α the sequence $(Y^{(d)}_\alpha)_{2d \ge |\alpha|+\deg(p)}$ is bounded. Since we have countably many such sequences, there is a subsequence (d_k) with the property that $(Y^{(d_k)}_\alpha)$ converges termwise (in α with k tending to ∞). This limit moment sequence Y will be in $\mathcal{L}_p(n)$ and moreover,

$$Z = (Y_{x_1}, \dots, Y_{x_g}) = \hat{Y}$$

so that $Z \in \hat{\mathcal{L}}_p(n)$. ■

6.5. **Truncated Quadratic Modules and the BPCP.** Given $\alpha,\beta,\mu\in\mathbb{N}$, and an $\ell\times\ell$ free matrix polynomial p, set

$$M^\mu_{\alpha,\beta}(p) := \Sigma^\mu_\alpha + \Big\{ \sum_i^{\text{finite}} f_i^* p f_i : \ f_i \in \mathbb{C}^{\ell\times\mu}\langle x\rangle_\beta \Big\} \ \subseteq\ \mathbb{C}^{\mu\times\mu}\langle x\rangle_{\max\{2\alpha,2\beta+a\}}, \tag{6.6}$$

where $a=\deg(p)$ and Σ^μ_α denotes all $\mu\times\mu$ sums of squares of degree $\le 2\alpha$. Obviously, if $f\in M^\mu_{\alpha,\beta}(p)$ then $f|_{\mathcal{D}_p}\succeq 0$. We call $M^\mu_{\alpha,\beta}(p)$ the **truncated quadratic module** defined by p. For notational convenience, we write M_k for $M_{\alpha,\beta}$ with $k=\max\{2\alpha,2\beta+a\}$. We also introduce

$$M^\mu(p) := \bigcup_{\alpha,\beta} M^\mu_{\alpha,\beta}(p),$$

the **quadratic module** defined by p. If $\mu=1$ we shall often omit the superscript μ. Observe that p is archimedean if the convex cone $M^\mu(p)$ has an order unit, i.e., for all symmetric $\mu\times\mu$ matrix polynomials f there is $N\in\mathbb{N}$ with $N-f\in M^\mu(p)$. (This notion is easily seen to be independent of μ, cf. [HKM13, §6].)

Definition 6.6. Let $\mu,N\in\mathbb{N}$. We say that p has the (N,μ)**-bound positivity certificate property (BPCP)**, if for every $\mu\times\mu$ linear pencil L, we have

$$L|_{\mathcal{D}_p}\succeq 0 \quad\iff\quad L\in M^\mu_N(p).$$

If N can be chosen independently of μ, then we say p has the N**-BPCP**.

We refer the reader to [Scw04, NiS07] for the classical commutative study of degree bounds needed in Positivstellensatz certificates.

6.5.1. *A sufficient stopping criterion for the free Lasserre–Parrilo lift.*

Lemma 6.7. If, for a positive integer n, the set $\mathcal{L}_p(n)$ is bounded in the sense that for each α there exists a C_α such that $\|Y_\alpha\|\le C_\alpha$ for all $Y\in\mathcal{L}_p(n)$, then $\hat{\mathcal{L}}_p(n)$ is compact.

Proof. With the boundedness hypothesis, the set $\mathcal{L}_p(n)$, viewed as a subset of the product space $\prod_\alpha \mathbb{S}_n$ is entrywise bounded. It is also seen to be entrywise closed. Thus it is a product of compact sets and therefore compact. Consequently the projection $Y\mapsto\hat{Y}$ being the finite product of the projections determined by the x_j has compact range; i.e., $\hat{\mathcal{L}}_p(n)$ is compact. ∎

The next theorem says if N-BPCP holds, then one of the truncated Lasserre–Parrilo lifts gives exactly the free convex hull of $\mathcal{D}_p$.

Theorem 6.8. *If $\mathcal{D}_p$ is uniformly bounded, and p has the N-BPCP, then*

$$\overline{\text{co}}^{\text{mat}}\mathcal{D}_p = \hat{\mathcal{L}}_p\Big(\,\llcorner\!\lrcorner;\left\lceil\frac{N}{2}\right\rceil\Big).$$

Proof. Let $\eta = \lceil \frac{N}{2} \rceil$. Clearly, $\mathcal{D}_p \subseteq \hat{\mathcal{L}}_p(\llcorner;\eta)$, and since $\hat{\mathcal{L}}_p(\llcorner;\eta)$ is matrix convex, $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p \subseteq \hat{\mathcal{L}}_p(\llcorner;\eta)$. Since $\mathcal{D}_p$ is uniformly bounded and p has the N-BPCP, p is archimedean. Hence by Lemma 6.5, $\mathcal{L}_p(\llcorner;\eta)$ is compact (e.g. in the product topology), and hence $\overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p \subseteq \overline{\hat{\mathcal{L}}_p(\llcorner;\eta)} = \hat{\mathcal{L}}_p(\llcorner;\eta)$.

Now assume $Y \in \hat{\mathcal{L}}_p(\llcorner;\eta) \setminus \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p$, and choose $W \in \mathcal{L}_p(\llcorner;\eta)$ satisfying $\hat{W} = Y$. Suppose Y is a g-tuple of size $\mu \times \mu$ matrices. By the Hahn-Banach Theorem 3.1 there is a linear pencil L (of size μ) with $L|_{\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p} \succeq 0$ and $L(Y) \not\succeq 0$. By the N-BPCP property for p, we have that $L \in M_N^\mu(p)$, i.e.,

$$L = \sum_k h_k^* h_k + \sum_{i=1}^r f_i^* p f_i. \tag{6.7}$$

Here $\deg(h_k) \leq \lfloor \frac{N}{2} \rfloor$ and $2\deg(f_i) + \deg(p) \leq N$ for $i = 1, \ldots, r$. Now apply the Riesz map Φ_W^μ to (6.7):

$$\Phi_W^\mu(L) = \sum_k \Phi_W^\mu(h_k^* h_k) + \sum_{i=1}^r \Phi_W^\mu(f_i^* p f_i). \tag{6.8}$$

Since $H_\eta(W) \succeq 0$ and $H_{p,\eta}^{\Uparrow}(W) \succeq 0$, Proposition 5.1 implies the right hand side of (6.8) is positive semidefinite. On the other hand, since L is linear, $\Phi_W^\mu(L) = L(\hat{W}) = L(Y) \not\succeq 0$, a contradiction. ∎

6.5.2. *More on the Positivstellensatz.* The polynomial p has the **linear Positivstellensatz property** (**LPP**) if whenever L is a monic linear pencil positive semidefinite on $\mathcal{D}_p$, then for each $\varepsilon > 0$ there exists natural numbers n_s and n_f and matrix polynomials $s_1, \ldots, s_{n_s}$ and $f_1, \ldots, f_{n_f}$ such that

$$L + \varepsilon = \sum_{j=1}^{n_s} s_j^* s_j + \sum_{j=1}^{n_f} f_j^* p f_j.$$

(So $L + \varepsilon \in M^\mu(p)$.) Note that the LPP condition is weaker than the BPCP.

Proposition 6.9. *Suppose $\mathcal{D}_p$ is uniformly bounded and p has the LPP. Then $\overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p = \hat{\mathcal{L}}_p$.*

Proof. Observe that the uniform boundedness of $\mathcal{D}_p$ together with the LPP implies p is archimedean. Suppose L is positive semidefinite on $\mathcal{D}_p$. By the LPP,

$$\varepsilon + L = \sum s_j^* s_j + \sum f_j^* p f_j. \tag{6.9}$$

On the other hand, if $X \in \hat{\mathcal{L}}_p$, then by Lemma 5.6 there exists a $Z \in \mathcal{D}_p^\infty$ and an isometry V such that $X = V^* Z V$. Because $Z \in \mathcal{D}_p^\infty$, it follows from the representation (6.9), that $\varepsilon + L(Z) \succeq 0$. Since $\varepsilon > 0$ was arbitrary, this shows $L(Z) \succeq 0$. Hence by Corollary 3.5, $X \in \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p$. ∎

7. Examples

In this section we present a few examples, starting with a detailed study of the TV screen and its "classical" spectrahedral lifts, see Subsections 7.1 and 7.2. We show that, unlike in the commutative settings, the first Lasserre–Parrilo lift is not exact. Then in Subsection 7.3 we prove that the matrix convex hull of the TV screen is dense in its operator convex hull. Finally, Subsection 7.4 contains simple examples where the Lasserre–Parrilo lifts are exact.

7.1. **The Bent TV Screen.**

Recall the bent TV screen,

$$p = 1 - x^2 - y^4.$$

The corresponding free semialgebraic set $\mathcal{D}_p$ is called the **TV screen**.

Lemma 7.1. *p is $\frac{5}{4}$-archimedean.*

Proof. Simply note that

$$\frac{5}{4} - x^2 - y^2 = \left(y^2 - \frac{1}{2}\right)^2 + (1 - x^2 - y^4). \qquad \blacksquare$$

The usual lift of $\mathcal{D}_p(1) = \mathrm{co}\,\big(\mathcal{D}_p(1)\big)$ is given by $\mathcal{D}_\Lambda(1)$, where

$$\Lambda = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & w \\ x & w & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & y \\ y & w \end{pmatrix}.$$

However, Λ is not monic, so we modify the construction somewhat. Let

$$L_1(x,y,w) = \begin{pmatrix} 1 & \gamma y \\ \gamma y & w+\alpha \end{pmatrix}, \quad L_2(x,y,w) = \begin{pmatrix} 1 & 0 & \gamma^2 x \\ 0 & 1 & w \\ \gamma^2 x & w & 1-2\alpha w \end{pmatrix}$$

where $\alpha > 0$ and $1+\alpha^2 = \gamma^4$, and set $L = L_1 \oplus L_2$. While strictly speaking L is not monic, the free spectrahedron $\mathcal{D}_L$ contains 0 in its interior, so L can be easily modified to become monic. It is worth noting that

$$\Lambda(X,Y,W) \succeq 0 \qquad \Longleftrightarrow \qquad W \succeq Y^2 \ \text{ and } \ 1 - X^2 - W^2 \succeq 0 \tag{7.1}$$

as is easily seen by using Schur complements.

Let $\mathcal{C}$ denote the free spectrahedrop obtained as the projection of $\mathcal{D}_L$ onto the first two coordinates. Thus,

$$\mathcal{C} = \{(X,Y) \in \mathbb{S}^2 : \exists W \in \mathbb{S} \text{ such that } L(X,Y,W) \succeq 0\}. \tag{7.2}$$

It is easy to see $\mathcal{C} = \{(X,Y) \in \mathbb{S}^2 : \exists W \in \mathbb{S} \text{ such that } \Lambda(X,Y,W) \succeq 0\}$.

The main result of this section is:

Theorem 7.2. $\overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{D}_p = \mathrm{co}^{\mathrm{oper}}\mathcal{D}_p \subsetneq \mathcal{C}$.

We shall prove the equality in Subsection 7.3 below, and now proceed to establish the strict inclusion.

Lemma 7.3.

(1) *The projection* $\mathcal{C}(1)$ *of* $\mathcal{D}_L(1)$ *onto the* (x,y)*-space equals* $\mathcal{D}_p(1)$.
(2) $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p \subseteq \mathcal{C}$.

Proof. Given $(x,y) \in \mathcal{D}_p(1)$, let $w = \gamma^2 y^2 - \alpha$. This makes $L_1(x,y,w)$ positive semidefinite and singular. The Schur complement of the top 2×2 block of $L_2(x,y,w)$ is thus

$$1 - 2\alpha w - \gamma^4 x^2 - w^2 = 1 + \alpha^2 - \gamma^4 x^2 - \gamma^4 y^4 = \gamma^4(1 - x^2 - y^4) \geq 0$$

making $L_2(x,y,w) \succeq 0$.

Conversely, if $(x,y,w) \in \mathcal{D}_L(1)$, then $w \geq \gamma^2 y^2 - \alpha$. Again, by way of Schur complements,

$$\begin{aligned} 0 &\leq 1 - 2\alpha w - \gamma^4 x^2 - w^2 = 1 + \alpha^2 - (\alpha + w)^2 - \gamma^4 x \\ &\leq 1 + \alpha^2 - \gamma^4 y^4 - \gamma^4 x^2 = \gamma^4(1 - x^2 - y^4), \end{aligned}$$

showing $1 - x^2 - y^4 \geq 0$.

For (2), take $(X,Y) \in \mathcal{D}_p$. Thus $I - X^2 - Y^4 \succeq 0$. Set $W = \gamma^2 Y^2 - \alpha I$. This makes $L_1(X,Y,W) \succeq 0$. The Schur complement of the block top 2×2 block of $L_2(X,Y,W)$ is thus

$$1 - 2\alpha W - \gamma^4 X^2 - W^2 = 1 + \alpha^2 - \gamma^4 X^2 - \gamma^4 Y^4 = \gamma^4(1 - X^2 - Y^4) \succeq 0$$

making $L_2(X,Y,W) \succeq 0$. Since $\mathcal{C}$ is matrix convex, this establishes $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p \subseteq \mathcal{C}$. ∎

Lemma 7.4. $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p \subsetneq \mathcal{C}$.

Proof. For this strict inclusion we simply exhibit matrix tuples, namely, points in the projection $\mathcal{C}$ onto the (x,y)-space of $\mathcal{D}_L$ which are not in $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$. In terms of $\mu > 0$ specified below, let

$$Y = \sqrt{\mu}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Take

$$W = \mu\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Choose μ so that the norm of W is 1 and let

$$X^2 = 1 - W^2.$$

Then $1 - X^2 - W^2 = 0$ and at the same time $Y^2 \leq W$. Thus $(X,Y) \in \mathcal{C}$. On the other hand,

$$Y^4 - W^2 = \mu^2\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \not\preceq 0.$$

Hence $I - X^2 - Y^4 \not\succeq 0$, i.e., $(X,Y) \notin \mathcal{D}_p$.

We next show that $(X,Y) \notin \mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$. It suffices to show if $\tilde{X}, \tilde{Y}$ are of the form

$$\tilde{X} = \begin{pmatrix} X & \alpha \\ \alpha^* & * \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} Y & \beta \\ \beta^* & \nu \end{pmatrix}.$$

then $I - \tilde{X}^2 - \tilde{Y}^4 \not\succeq 0$. We argue by contradiction and accordingly assume $I - \tilde{X}^2 - \tilde{Y}^4 \succeq 0$. To do this the first step will be to show that $\beta = 0$. Next, $\beta = 0$ implies, projecting onto the top subspace,

$$0 \preceq I - (X^2 + \alpha\alpha^*) - Y^4 \preceq I - X^2 - Y^4.$$

But then, because $I - X^2 - Y^4 \not\succeq 0$, we get a contradiction.

Now to the attack on β. Note that

$$\tilde{Y}^2 = \begin{pmatrix} Y^2 + \beta\beta^* & \delta \\ \delta^* & * \end{pmatrix}. \tag{7.3}$$

for some δ and some $*$. Let $T := Y^2 + \beta\beta^* \succeq Y^2$. Further, note that

$$\tilde{Y}^4 = \begin{pmatrix} T^2 + \delta\delta^* & * \\ * & * \end{pmatrix}.$$

The upper left entry of $I - \tilde{X}^2 - \tilde{Y}^4$ equals

$$0 \preceq I - (X^2 + \alpha\alpha^*) - (T^2 + \delta\delta^*) \preceq I - X^2 - T^2 = W^2 - T^2. \tag{7.4}$$

Further, we have

$$Y^2 = \mu \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So after dividing (7.4) through by μ^2, we obtain,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2 \succeq \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\mu}\beta\beta^* \right)^2. \tag{7.5}$$

Since the square root function is operator monotone, (7.5) yields

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \succeq \frac{1}{\mu}\beta\beta^*, \tag{7.6}$$

or equivalently,

$$\beta = \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} b^*,$$

for some vector b with norm ≤ 1. Putting these back into (7.5) leads to

$$\begin{pmatrix} -2\|b\|^4 - 2\|b\|^2 + 4 & -2\|b\|^4 - \|b\|^2 + 3 \\ -2\|b\|^4 - \|b\|^2 + 3 & 2 - 2\|b\|^4 \end{pmatrix} \succeq 0.$$

Since the determinant of this matrix equals

$$-(\|b\|^2 - 1)^2,$$

we see $\|b\| = 1$. In particular, we have equality in (7.6) and (7.4). Hence $T = W$ so that $I - X^2 - T^2 = 0$.

Returning to the upper left hand entry of $I - \tilde{X}^2 - \tilde{Y}^4$, it follows from (7.3) and (7.4) that we have

$$I - X^2 - T^2 - \delta\delta^* \succeq 0.$$

Hence $\delta\delta^* = 0$ and so $\delta = 0$. Since δ is of the form $\delta = Y\beta + \beta\nu$, we have

$$\begin{aligned} 0 = Y\beta + \beta\nu &= \sqrt{\mu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\beta + \beta\nu \\ &= \mu\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}b^* + \sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix}b^*\nu \\ &= \mu\begin{pmatrix}1\\0\end{pmatrix}b^* + \sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix}b^*\nu, \end{aligned}$$

leading to

$$b^*\nu = 0 \quad \text{and} \quad b^*\nu + \sqrt{\mu}b^* = 0.$$

Hence $b^* = 0$. This implies $\beta = 0$, delivering the promised contradiction. ■

Proposition 7.5. $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(1) = \mathcal{D}_p(1)$.

Proof. This follows from Lemma 7.3 and Proposition 4.5. Alternately, use $\mathcal{D}_p \subseteq \mathrm{co}^{\mathrm{mat}}\mathcal{D}_p \subseteq \mathcal{C}$ together with item (1) of Lemma 7.3. ■

7.2. **Comparing the L-Lift with the Lasserre–Parrilo Relaxations.** Two Lasserre-Parrilo lifts of the bent TV screen were proposed in Subsection 6.3. The malicious point constructed in the proof of Lemma 7.4 serves to show that the lift $\mathcal{L}_p(\text{\textvisiblespace};0)$ based on $H^{\Uparrow}_{p,0}(Y) \oplus H_2(Y) \succeq 0$ is again inexact, i.e., its projection $\hat{\mathcal{L}}_p(\text{\textvisiblespace};0)$ is still strictly bigger than $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$. On the other hand, the second Lasserre–Parrilo relaxation $H^{\Uparrow}_{p,1}(Y) \oplus H_3(Y) \succeq 0$ does seem to separate the malicious point from $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$ – according to our computer experiments.

Proposition 7.6. *Let $p = 1 - x_1^2 - x_2^4$. Then $\hat{\mathcal{L}}_p(\text{\textvisiblespace};0) = \mathcal{C}$, while $\hat{\mathcal{L}}_p(2;1) \subsetneq \mathcal{C}(2)$.*

Proof. Let $\mathcal{L}'_p(\text{\textvisiblespace};0)$ denote the "reduced" lift obtained by using (6.4), and $\hat{\mathcal{L}}'_p(\text{\textvisiblespace};0)$ its projection. It is clear that $\hat{\mathcal{L}}'_p(\text{\textvisiblespace};0) \supseteq \hat{\mathcal{L}}_p(\text{\textvisiblespace};0)$. Next, assume $(X_1, X_2) \in \hat{\mathcal{L}}'_p(\text{\textvisiblespace};0)$, and take a feasible point Y for (6.4). Then with $W = Y_{22}$ we have $W \succeq X_2^2$ by considering the submatrix of $\check{H}_2(Y)$ spanned by columns and rows $1, 3$. Likewise, $Y_{11} \succeq X_1^2$ and $Y_{2222} \succeq W^2$. Hence

$$0 \preceq 1 - Y_{11} - Y_{2222} \preceq 1 - X_1^2 - W^2,$$

showing $\Lambda(X_1, X_2, W) \succeq 0$, i.e., $(X_1, X_2) \in \mathcal{C}$.

Conversely, let $(X_1,X_2)\in\mathcal{C}$. Choose Y so that

$$\check{H}_2(Y)=\begin{pmatrix}1 & X_1 & X_2 & W\\ X_1 & X_1^2 & X_1X_2 & X_1W\\ X_2 & X_2X_1 & W & X_2W\\ W & WX_1 & WX_2 & W^2\end{pmatrix}.$$

Then

$$H^{\Uparrow}_{p,0}(Y)=1-X_1^2-W^2\succeq 0$$

by assumption. Furthermore,

$$\check{H}_2(Y)=\begin{pmatrix}1 & 0 & X_2 & W\\ 0 & 1 & 0 & 0\end{pmatrix}^*\begin{pmatrix}1 & X_1\\ X_1 & X_1^2\end{pmatrix}\begin{pmatrix}1 & 0 & X_2 & W\\ 0 & 1 & 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & W-X_2^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\succeq 0.$$

All this shows $(X_1,X_2)\in\hat{\mathcal{L}}'_p(\llcorner\!\lrcorner;0)$.

As a final step, we extend $\check{H}_2(Y)$ to a positive semidefinite $H_2(Y)$. Again, this is now straightforward. Using

$$Z=\begin{pmatrix}X_1^2 & X_1X_2 & 0\\ 0 & 0 & 0\\ 0 & 0 & X_1\\ 0 & 0 & 0\end{pmatrix}$$

we set

$$PH_2(Y)P=\begin{pmatrix}I_4 & Z\end{pmatrix}^*\check{H}_2(Y)\begin{pmatrix}I_4 & Z\end{pmatrix},$$

where P is the permutation matrix of the permutation $\begin{pmatrix}4 & 5 & 6 & 7\end{pmatrix}$. Hence $(X_1,X_2)\in\hat{\mathcal{L}}_p(\llcorner\!\lrcorner;0)$, concluding the first part of the proof.

The second statement of the proposition follows from numerical computer experiments; see the Mathematica notebook `TVlift.nb` available from `arxiv`. ■

7.3. **Matrix versus Operator Convex Hull: Bent TV Screen.** From Theorem 5.4, the closure of $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$ is contained in $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p$. While this inclusion is generally proper (e.g. there are examples of archimedean p with $\mathcal{D}_p=\varnothing\neq\mathcal{D}_p^\infty$), the proposition below says that these sets are the same in at least one non-trivial example. The proof uses spectral theory for bounded self-adjoint operators on a Hilbert space.

Proposition 7.7. *Let $p=1-x_1^2-x_2^4$. Then $\mathrm{co}^{\mathrm{oper}}\mathcal{D}_p(n)$ is the closure of $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p(n)$.*

Proof. Fix a point $X\in\mathbb{S}_m^2$ in the operator convex hull of the bent TV screen. Thus, there a Hilbert space $\mathscr{H}$ and a tuple $Y=(Y_1,Y_2)$ of bounded self-adjoint operators on $\mathscr{H}$ such that

$$I\succeq Y_1^2+Y_2^4,$$

and an isometry $V:\mathbb{C}^m \to \mathscr{H}$ such that $X = V^*YV$.

Since Y_2 is self-adjoint, it has a spectral decomposition,

$$Y_2 = \int_{-1}^{1} t\, dE(t),$$

for a spectral measure E on the interval $[-1,1]$. Given a positive integer N, let

$$\omega_j^N = \left[\frac{j}{N}, \frac{j+1}{N}\right)$$

for $-N \le j < N-1$ and let $\omega_{N-1}^N = [\frac{N-1}{N}, 1]$. For $0 \le j$, let $t_j = \frac{j}{N}$ and for $j < 0$, let $t_j = \frac{j+1}{N}$. Let

$$Z = \sum_{j=-N}^{N-1} t_j E(\omega_j^N)$$

and observe that Z and Y_2 commute. In particular,

$$Z^4 \preceq Y_2^4. \tag{7.7}$$

Consider finite dimensional subspaces

$$E(\omega_j^N)\mathscr{H} \supseteq \mathscr{H}_j = E(\omega_j^N)V\mathbb{C}^m.$$

Let $\mathscr{K} = \bigoplus_{j=-N}^{N-1} \mathscr{H}_j$. Thus, $\mathscr{K}$ is finite dimensional and contained in $\mathscr{H}$. Further, letting $W:\mathscr{K}\to\mathscr{H}$ denote the inclusion of $\mathscr{K}$ into $\mathscr{H}$,

$$\tilde{Y}_2 = W^*ZW$$

satisfies,

$$\tilde{Y}_2^4 = W^*Z^4W \preceq W^*Y_2^4W,$$

because of (7.7). Let $\tilde{Y}_1 = W^*Y_1W$. It follows that

$$\tilde{Y}_1^2 + \tilde{Y}_2^4 \preceq Y_1^2 + Y_2^4 \preceq I.$$

At the same time, by construction, V maps into $\mathscr{K}$ so that W^*V is an isometry and

$$X_1 = (W^*V)^*(W^*Y_1W)W^*V$$

Thus, the pair $(W^*V)^*\tilde{Y}W^*V = (X_1, (W^*V)^*\tilde{Y}_2W^*V)$ is in the bent TV screen.

Emphasizing the dependence of W on N, write $W_N = W$ and $Z^N = Z$. With this notation, observe that

$$\|Z^N - Y_2\| = \left\|\sum t_jE(\omega_j^N) - \int t\, dE(t)\right\| \le \frac{1}{N}.$$

Hence, Z^N converges in the strong operator topology to Y_2. Since $W_NW_N^*V = V$, it follows that

$$(W_N^*V)^*\tilde{Y}_2^NW_N^*V = (W_N^*V)^*W_N^*Z^NW_N(W_N^*V) = V^*Z^NV$$

converges to $V^*Y_2V = X_2$. The conclusion is that X is in the closure of the matrix convex hull of the bent TV screen. ■

7.4. **Examples where the Lasserre–Parrilo Lift is Exact.** Consider first $p = 1 - xy^2x$. Then

$$\mathcal{D}_p \supseteq \big(\{0\} \times \mathbb{S}\big) \cup \big(\mathbb{S} \times \{0\}\big),$$

so $\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p$ will equal $\mathbb{S}^2$. In particular, the first Lasserre–Parrilo lift $\hat{\mathcal{L}}_p(\text{␣}; 0)$ is exact.

For an example with a little different flavor, let $p = (1-2y^2+x^2) \oplus (1-2x^2+y^2)$. Then $\mathcal{D}_p$ given by

$$\mathcal{D}_p = \Big\{(X,Y) \in \mathbb{S}^2 : Y^2 - \frac{1}{2}X^2 \preceq \frac{1}{2},\ X^2 - \frac{1}{2}Y^2 \preceq \frac{1}{2}\Big\}$$

is bounded, and

$$\mathrm{co}^{\mathrm{mat}}\mathcal{D}_p = \big\{(X,Y) \in \mathbb{S}^2 : \|X\| \le 1,\ \|Y\| \le 1\big\}$$

is again the projection of the first Lasserre–Parrilo lift $\mathcal{L}_p(\text{␣}; 0)$.

References

[AM+] J. Agler, J.E. McCarthy: Global holomorphic functions in several non-commuting variables, *preprint* http://arxiv.org/abs/1305.1636 4

[Arv72] W. Arveson: Subalgebras of C*-algebras II, *Acta Math.* **128** (1972) 271–308. 2

[BB07] J.A. Ball, V. Bolotnikov: Interpolation in the noncommutative Schur-Agler class, *J. Operator Theory* **58** (2007) 83–126. 4

[Bar02] A. Barvinok: *A course in convexity*, Amer. Math. Soc., 2002. 9

[BPR13] G. Blekherman, P.A. Parrilo, R.R. Thomas (editors): *Semidefinite optimization and convex algebraic geometry*, MOS-SIAM Series on Optimization **13**, SIAM, 2013. 8, 13

[BCR98] J. Bochnack, M. Coste, M.-F. Roy: *Real algebraic geometry*, Springer, 1998. 3, 5, 14

[CF08] R.E. Curto, L. Fialkow, An analogue of the Riesz-Haviland theorem for the truncated moment problem, *J. Funct. Anal.* **255** (2008) 2709–2731. 3

[DK+] K.R. Davidson, M. Kennedy: The Choquet boundary of an operator system, *preprint* http://www.arxiv.org/abs/1303.3252 2

[DKL11] E. De Klerk, M. Laurent: On the Lasserre hierarchy of semidefinite programming relaxations of convex polynomial optimization problems, *SIAM J. Optimization* **21** (2011) 824–832. 16

[dOHMP09] M. de Oliveira, J.W. Helton, S. McCullough, M. Putinar: Engineering systems and free semi-algebraic geometry, in: *Emerging applications of algebraic geometry*, pp. 17–61, Springer, 2009. 5

[DHM07] H. Dym, J.W. Helton, S. McCullough: Irreducible noncommutative defining polynomials for convex sets have degree four or less, *Indiana Univ. Math. J.* **56** (2007) 1189–1232. 8

[EW97] E.G. Effros, S. Winkler: Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems, *J. Funct. Anal.* **144** (1997) 117–152. 10, 11

[Far04] D. Farenick: Pure matrix states on operator systems, *Linear Algebra Appl.* **393** (2004) 149–173. 2

[FP12] D. Farenick, V.I. Paulsen: Operator system quotients of matrix algebras and their tensor products *Math. Scand.* **111** (2012) 210–243. 2

[GO10] M.R. Graham, M. de Oliveira: Linear matrix inequality tests for frequency domain inequalities with affine multipliers, *Automatica* **46** (2010) 897–901. 3

[GLPT12] J. Gouveia, M. Laurent, P.A. Parrilo, R.R. Thomas: A new semidefinite programming hierarchy for cycles in binary matroids and cuts in graphs, *Math. Program.* **133** (2012) 203–225. 3

[GPT10] J. Gouveia, P.A. Parrilo, R.R. Thomas: Theta bodies for polynomial ideals, *SIAM J. Optimization* **20** (2010) 2097–2118. 3

[GPT12] J. Gouveia, P.A. Parrilo, R.R. Thomas: Lifts of convex sets and cone factorizations, *Math. Oper. Res.* **38** (2013) 248–264. 3

[GT12] J. Gouveia, R. Thomas: Convex hulls of algebraic sets, in: *Handbook on Semidefinite, Conic and Polynomial Optimization*, pp. 113–138, Springer, 2012. 3

[GLS93] M. Grötschel, L. Lovász, A. Schrijver: *Geometric Algorithms and Combinatorial Optimization*, Springer, 1993. 3

[HKM12] J.W. Helton, I. Klep, S. McCullough: The convex Positivstellensatz in a free algebra, *Adv. Math.* **231** (2012) 516–534. 3

[HKM13] J.W. Helton, I. Klep, S. McCullough: The matricial relaxation of a linear matrix inequality, *Math. Program.* **138** (2013) 401–445. 29

[HKM+] J.W. Helton, I. Klep, S. McCullough: Polar duals of matrix convex sets and free LMI lifts, *in preparation.* 2

[HM04] J.W. Helton, S. McCullough: A Positivstellensatz for noncommutative polynomials, *Trans. Amer. Math. Soc.* **356** (2004) 3721–3737. 3, 7, 23

[HM12] J.W. Helton, S. McCullough: Every free basic convex semi-algebraic set has an LMI representation, *Ann. of Math. (2)* **176** (2012) 979–1013. 2, 3, 10, 11, 14

[HN08] J.W. Helton, J. Nie: Structured semidefinite representation of some convex sets, in: *47th IEEE Conference on Decision and Control (CDC)* (2008) 4797–4800 26

[HN09] J.W. Helton, J. Nie: Sufficient and necessary conditions for semidefinite representability of convex hulls and sets, *SIAM J. Optimization* **20** (2009) 759–791. 16

[HN10] J.W. Helton, J. Nie: Semidefinite representation of convex sets, *Math. Program.* **122** (2010) 21–64. 13, 16

[HV07] J.W. Helton, V. Vinnikov: Linear matrix inequality representation of sets, *Commun. Pure Appl. Math.* **60** (2007) 654–674. 15

[Hen11] D. Henrion: Semidefinite representation of convex hulls of rational varieties, *Acta Appl. Math.* **115** (2011) 319–327. 16

[JKPP11] N. Johnston, D.W. Kribs, V.I. Paulsen, R. Pereira: Minimal and maximal operator spaces and operator systems in entanglement theory, *J. Funct. Anal.* **260** (2011) 2407–2423. 2

[KVV+] D. Kalyuzhnyi-Verbovetskiĭ, V. Vinnikov: Foundations of noncommutative function theory, *preprint* http://arxiv.org/abs/1212.6345 4

[Kls+] C. Kleski: Boundary representations and pure completely positive maps, to appear in *J. Operator Theory*, http://arxiv.org/abs/1110.4149 2

[Las09a] J.B. Lasserre: Convex sets with semidefinite representation, *Math. Program.* **120** (2009) 457–477. 2, 3, 13, 16

[Las09b] J.B. Lasserre: *Moments, positive polynomials and their applications*, World Scientific, 2009. 3

[Lau09] M. Laurent: Sums of squares, moment matrices and optimization over polynomials, in: *Emerging applications of algebraic geometry*, pp. 157–270. Springer, 2009. 3

[Lov79] L. Lovász: On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* **25** (1979) 1–7. 3

[MS11] P.S. Muhly, B. Solel: Progress in noncommutative function theory, *Sci. China Ser. A* **54** (2011) 2275–2294. 4

[NN94] Y. Nesterov A. Nemirovskii: *Interior-point polynomial algorithms in convex programming*, SIAM, 1994. 3

[NPS10] T. Netzer, D. Plaumann, M. Schweighofer: Exposed faces of semidefinitely representable sets, *SIAM J. Optimization* **20** (2010) 1944–1955. 16

[NiS07] J. Nie, M. Schweighofer: On the complexity of Putinar's Positivstellensatz, *J. Complexity* **23** (2007) 135–150. 29

[OGB02] M. de Oliveira, J. Geromel, J. Bernussou: Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems, *Internat. J. Control* **75** (2002) 666–679. 3

[Par06] P. Parrilo: Exact semidefinite representation for genus zero curves, Talk at the Banff workshop *Positive Polynomials and Optimization*, Banff, Canada, 2006. 3, 13, 16

[Pau02] V. Paulsen: *Completely bounded maps and operator algebras*, Cambridge University Press, 2002. 2

[PNA10] S. Pironio, M. Navascués, A. Acín: Convergent relaxations of polynomial optimization problems with noncommuting variables, *SIAM J. Optim.*, **20** (2010) 2157–2180. 3

[Poe10] G. Popescu: Free holomorphic automorphisms of the unit ball of $B(H)^n$, *J. reine angew. Math.* **638** (2010) 119–168. 4

[Sce11] C. Scheiderer: Convex hulls of curves of genus one, *Adv. Math.* **228** (2011) 2606–2622. 16

[Sce+] C. Scheiderer: Semidefinite representation for convex hulls of real algebraic curves, *preprint* http://arxiv.org/abs/1208.3865 16

[Scw04] M. Schweighofer: On the complexity of Schmüdgen's Positivstellensatz, *J. Complexity* **20** (2004) 529–543. 29

[Voi04] D.-V. Voiculescu: Free analysis questions I: Duality transform for the coalgebra of $\partial_{X:B}$, *International Math. Res. Notices* **16** (2004) 793–822. 4

[Voi10] D.-V. Voiculescu: Free analysis questions II: The Grassmannian completion and the series expansions at the origin, *J. reine angew. Math.* **645** (2010) 155–236. 4

[WW99] C. Webster, S. Winkler: The Krein-Milman theorem in operator convexity, *Trans. Amer. Math. Soc.* **351** (1999) 307–322. 2

J. William Helton, Department of Mathematics, University of California, San Diego

E-mail address: helton@math.ucsd.edu

Igor Klep, Department of Mathematics, The University of Auckland, New Zealand

E-mail address: igor.klep@auckland.ac.nz

Scott McCullough, Department of Mathematics, University of Florida, Gainesville

E-mail address: sam@math.ufl.edu

NOT FOR PUBLICATION

Contents