FREE CONVEX ALGEBRAIC GEOMETRY

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1. Introduction

This article is a tutorial on techniques and results in *free convex algebraic geometry* and *free positivity*. As such it also serves as a point of entry into the larger field of *free real algebraic geometry* (*free RAG*), and makes contact with noncommutative real algebraic geometry [Hel02, HKM10c, HKM+, HKM++, HM+, KS08a, KS08b, McC01, PNA10, Smü05, Smü09], free analysis and free probability (lying at its origins of free analysis, cf. [SV06]), free analytic function theory and free harmonic analysis [HKM10a, HKM10b, HKMS09, MS+, Pop06, Voi04, Voi10, KVV-].

The term free here refers to the central role played by algebras of (free, or noncommuting) polynomials $\mathbb{R}\langle x \rangle$ in free (freely noncommuting) variables $x = (x_1, \dots, x_g)$. A striking difference between the free and classical settings is the following Positivstellensatz.

Theorem 1.1 (Helton [Hel02])**.** *A nonnegative* (*suitably defined*) *free polynomial is a sum of squares.*

The subject of free RAG flows in two branches. One, free positivity is an analog of classical real algebraic geometry, a theory of polynomial inequalities embodied in Positivstellensätze. As is the case with the sum of squares result above (Theorem 1.1), generally free Positivstellensätze have cleaner statements than do their commutative counterparts; see e.g. [McC01, Hel02, HMP04, HKM++] for a sample. Free

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convexity, the second branch of free RAG, arose in an effort to unify a torrent of ad hoc techniques which came on the linear systems engineering scene in the mid 1990's. We soon give a quick sketch of the engineering motivation, based on the slightly more complete sketch given in the survey article [dOHMP09]. Mathematically, much as in the commutative case, free convexity is connected with free positivity through the second derivative: A free polynomial is convex if and only if its Hessian is positive.

The tutorial proper starts with Section 2. In the remainder of this introduction, motivation for the study of free positivity and convexity arising in Linear Systems Engineering, Quantum Phenomena, and other subjects such as Free Probability is provided, as are some suggestions for further reading.

1.1. **Motivation.** While the theory is both mathematically pleasing and natural, much of the excitement of free convexity and positivity stems from its applications. Indeed, the fact that a large class of linear systems engineering problems naturally lead to free inequalities provided the main force behind the development of the subject. In this motivational section, we describe in some detail the linear systems point of view. We also give a brief introduction to other applications.

1.1.1. *Linear Systems Engineering.* The layout of a linear systems problem is typically specified by a signal flow diagram. Signals go into boxes and other signals come out. The boxes in a linear system contain linear differential equations which are specified entirely by matrices (the coefficients of the differential equations). Often many boxes appear and many signals transmit between them. In a typical problem some boxes are given and some we get to design subject to the condition that the L^2 norm of various signals must compare in a prescribed way, e.g. the input to the system has L^2 norm bigger than the output. The signal flow diagram itself and corresponding problems do not specify the size of matrices involved. So any algorithms derived ideally apply to matrices of all sizes. Hence the problems are called *dimension free*.

An empirical observation is that system problems of this type convert to inequalities on polynomials in matrices, the form of the polynomials being determined entirely by the signal flow layout (and independent of the matrices involved). Thus the systems problem naturally leads to free polynomials and free positivity conditions.

Now we give more details.

1.1.2. *Linear systems.* A *linear system* $\mathfrak{F}$ is given by the linear differential equations

$$\frac{dx}{dt} = Ax + Bu,$$
$$y = Cx,$$

with the vector

- $x(t)$ at each time t being in the vector space $\mathcal{X}$ called the *state space*,
- $u(t)$ at each time t being in the vector space $\mathcal{U}$ called the *input space*,
- $y(t)$ at each time t being in the vector space $\mathcal{Y}$ called the *output space*,

and A, B, C being linear maps on the corresponding vector spaces.

1.1.3. *Connecting linear systems.* Systems can be connected in incredibly complicated configurations. We describe a simple connection and this goes a long way toward illustrating the general idea. Given two linear systems $\mathfrak{F}$, $\mathfrak{G}$, we describe the formulas for connecting them in feedback.

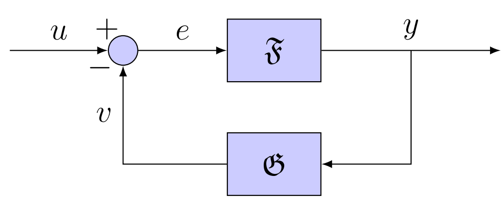
The systems $\mathfrak{F}$ and $\mathfrak{G}$ themselves are respectively given by the linear differential equations

$$\frac{dx}{dt} = Ax + Be, \qquad \frac{d\xi}{dt} = Q\,\xi + R\,w,$$
$$y = Cx, \qquad v = S\,\xi.$$

Feedback connection of them is described by the algebraic statements

$$w = y \qquad \text{and} \qquad e = u - v.$$

This set up is typically described by the diagram



called a *signal flow diagram.* The *closed loop system* is a new system whose differential equations are

$$\frac{dx}{dt} = Ax - BS\xi + Bu,$$
$$\frac{d\xi}{dt} = Q\,\xi + R\,y = Q\,\xi + R\,Cx,$$
$$y = Cx.$$

In matrix form this is

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & -BS \\ RC & Q \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix}, \tag{1.1}$$

where the state space of the closed loop systems is the direct sum $\mathcal{X} \oplus \mathcal{Y}$ of the state spaces $\mathcal{X}$ of $\mathfrak{F}$ and $\mathcal{Y}$ of $\mathfrak{G}$. The moral of the story is:

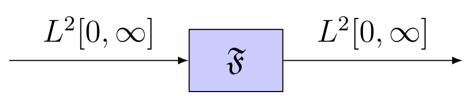
System connections produce a new system whose coefficients are matrices with entries which are polynomials or at worst "rational expressions" in the coefficients of the component systems.

Complicated signal flow diagrams give complicated matrices of polynomials or rationals. Note in what was said the dimensions of vector spaces and matrices never entered explicitly; the algebraic form of (1.1) is completely determined by the flow diagram. Thus, such linear systems lead to *dimension free* problems.

1.1.4. *Energy dissipation.* We have a system $\mathfrak{F}$ and want a condition which checks whether

$$\int_0^\infty |u|^2 dt \geq \int_0^\infty |\mathfrak{F}u|^2 dt, \qquad x(0) = 0,$$

holds for all input functions u, where $\mathfrak{F}u = y$ in the above notation. If this holds $\mathfrak{F}$ is called a *dissipative system.*



The energy dissipative condition is formulated in the language of analysis, but it converts to algebra (or at least an algebraic inequality) because of the following construction, which assumes the existence of a "potential energy"-like function V on the state space. A function V which satisfies $V \geq 0$, $V(0) = 0$, and

$$V(x(t_1)) + \int_{t_1}^{t_2} |u(t)|^2 dt \quad \geq \quad V(x(t_2)) + \int_{t_1}^{t_2} |y(t)|^2 dt$$

for all input functions u and initial states x_1 is called a *storage function.* The displayed inequality is interpreted physically as *potential energy now + energy in ≥ potential energy then + energy out.*

Assuming enough smoothness of V, we can differentiate this integral condition and use $\frac{dx}{dt}(t_1) = \; Ax(t_1) + Bu(t_1)$ to obtain a differential inequality

$$0 \;\geq\; \nabla V(x)(Ax + Bu) \;+\; |Cx|^2 - |u|^2, \tag{1.2}$$

on what is called the "reachable set" (which we do not need to define here).

In the case of linear systems, V can be chosen quadratic. So it has the form $V(x) = \langle Ex, x\rangle$ with $E \succeq 0$ and $\nabla V(x) = 2Ex$.

Theorem 1.2. *The linear system A, B, C is dissipative if inequality (1.2) holds for all $u \in \mathcal{U}, x \in \mathcal{X}$. Conversely, if A, B, C is "reachable", then dissipativity implies inequality (1.2) holds for all $u \in \mathcal{U}$, $x \in \mathcal{X}$.*

In the linear case, we may substitute $\nabla V(x) = 2Ex$ in (1.2) to obtain

$$0 \geq 2(Ex)^\intercal(Ax + Bu) + |Cx|^2 - |u|^2,$$

for all u, x. Then maximize in x to get

$$0 \geq x^\intercal[EA + A^\intercal E + EBB^\intercal E + C^\intercal C]x.$$

Thus the classical *Riccati matrix inequality*

$$0 \succeq \; EA + A^\intercal E + EBB^\intercal E + C^\intercal C \quad \text{with} \qquad E \succeq 0 \tag{1.3}$$

ensures dissipativity of the system; and, it turns out, is also implied by dissipativity when the system is reachable.

1.1.5. *Schur Complements and Linear Matrix Inequalities.* Using Schur complements[1], the Riccati inequality of equation (1.3) is equivalent to the inequality

$$L(E) := \begin{bmatrix} EA + A^\intercal E + C^\intercal C & EB \\ B^\intercal E & -I \end{bmatrix} \preceq 0.$$

Here A, B, C describe the system and E is an unknown matrix. If the system is reachable, then A, B, C is dissipative if and only if $L(E) \preceq 0$ and $E \succeq 0$.

The key feature in this reformulation of the Riccati inequality is that $L(E)$ is linear in E, so the inequality $L(E) \preceq 0$ is a Linear Matrix Inequality (LMI) in E.

[1]The *Schur complement* of a matrix (with pivot δ^{-1}) is defined by

$$\text{SchurComp} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} := \alpha - \beta\delta^{-1}\gamma.$$

A key fact is: if δ is invertible, then the matrix is positive semidefinite if and only if $\gamma = \beta^\intercal$, $\delta \succ 0$ and its Schur complement is positive semidefinite.

1.1.6. *Putting it together.* We have shown two ingredients of linear system theory, connection laws (algebraic) and dissipation (inequalities), but have yet to put them together. It is in fact a very mechanical procedure and any trained engineer does it with no difficulty. After going through the procedure one sees that the problem a Matlab toolbox designer faces is this:

(GRAIL) Given a symmetric matrix of nc polynomials

$$p(a,x) = \Big[p_{ij}(a,x)\Big]_{i,j=1}^{k},$$

and a tuple of matrices A, provide an algorithm for finding X making $p(A,X) \succeq 0$ or better yet as large as possible.

Algorithms for doing this are based on numerical optimization or a close relative, so even if they find a local solution there is no guarantee that it is global. If p is convex in X, then these problems disappear.

Thus, systems problems described by signal flow diagrams produce messes of matrix inequalities with some matrices known and some unknown and the constraints that some polynomials are positive semidefinite. The inequalities can get very complicated as one might guess, since signal flow diagrams get complicated. We do not go into details but refer the reader to [dOHMP09, §4.1] for a classical simple example.

The engineer would like for these polynomial inequalities to be convex in the unknowns. Convexity guarantees that local optima are global optima (finding global optima is often of paramount importance) and facilitates numerics.

Hence the major issues in linear systems theory are:

(1) *Which problems convert to a convex matrix inequality? How does one do the conversion?*
(2) *Find numerics which will solve large convex problems. How do you use special structure, such as most unknowns are matrices and the formulas are all built of noncommutative rational functions?*
(3) *Are convex matrix inequalities more general than LMIs?*

The mathematics here aims toward helping an engineer who writes a toolbox which other engineers will use for designing systems, like control systems. What goes in such toolboxes is algebraic formulas with matrices A, B, C unspecified and reliable numerics for solving them when a user does specify A, B, C as matrices. A user who designs a controller for a helicopter puts in the mathematical systems model for his helicopter and puts in matrices, for example, A is a particular 8×8 real matrix etc. Another user who designs a satellite controller might have a 50 dimensional state space and of course would pick completely different A, B, C. Essentially any matrices of any compatible dimensions can occur. Any claim we make about our formulas should hold regardless of the size of the matrices plugged in.

The toolbox designer faces two completely different tasks. One is manipulation of algebraic inequalities; the other is numerical solutions. Often the first is far more daunting since the numerics is handled by some standard package (although for numerics problem size is a demon). Thus there is a great need for algebraic theory. Most of this chapter bears on questions like (3) above where the unknowns are matrices. *The last two questions will not be addressed.* Here we treat (3) when there are no a variables. When there are a variables see [HHLM08]. Thus we shall consider polynomials $p(x)$ in free noncommutative variables x and focus on their convexity on free semialgebraic sets.

1.1.7. *Quantum Phenomena.* Free Positivstellensätze - algebraic certificates for positivity - of which Theorem 1.1 is the grandad, have physical applications. Applications to quantum physics are explained by Pironio, Navascués, Acín [PNA10] who also consider computational aspects related to noncommutative sum of squares. How this pertains to operator algebras is discussed by Schweighofer and the second author in [KS08a]. The important Bessis-Moussa-Villani conjecture (BMV) from quantum statistical mechanics is tackled in [KS08b, CKP10]. Doherty, Liang, Toner, Wehner [DLTW08] employ noncommutative positivity and the Positivstellensatz [HM04b] of the first and the third author to consider the quantum moment problem and multi-prover games.

1.1.8. *Miscellaneous applications.* A number of other scientific disciplines use free analysis, though less systematically than in free real algebraic geometry.

Free probability. Striking is free probability. Voiculescu developed it to attack one of the purest of mathematical questions regarding von Neumann algebras. From the outset it was elegant and it came to have great depth. Subsequently, it was discovered to bear forcefully and effectively on random matrices. The area is vast, so we do not dive in but refer the reader to an introduction [SV06, VDN92].

Nonlinear engineering systems. A classical technique in nonlinear systems theory developed by Fliess is based on manipulation of power series with noncommutative variables (the Chen series). The area has a new impetus coming from the problem of data compression, so now is a time when these correspondences are being worked out. A good entree to the subject is found at a 2011 conference web site

`http://www.th.physik.uni-bonn.de/people/fard/RPCCT2011/program.html`

1.2. **Further reading.** We pause here to offer some suggestions for further reading. For further engineering motivation we recommend the paper [SI95] or the longer version [SIG97] for related new directions. Descriptions of Positivstellensätze are in the surveys [HKM11, dOHMP09, HP07, Smü09] with the first three also briskly touring free convexity. The survey article [HMPV09] is aimed at engineers.

Noncommutative is a broad term, encompassing essentially all algebras. In between the extremes of commutative and free lie many important topics, such as Lie algebras, Hopf algebras, quantum groups, C^*-algebras, von Neumann algebras, etc. For instance, there are elegant noncommutative real algebraic geometry results for the Weyl Algebra [Smü05], cf. [Smü09].

1.3. **Guide to the tutorial.** The goal of this tutorial is to introduce the reader to the main results and techniques used to dissect free convexity. Fortunately, the subject is new and the techniques not too numerous so that one can quickly become an expert.

The basics of free, or nc, polynomials and their evaluations are developed in Section 2. The key notions are positivity and convexity for free polynomials. The principle fact is that the second directional derivatives (in direction h) of a free convex polynomial is a positive quadratic polynomials in h (just like in the commutative case). Free quadratic (in h) polynomials have a Gram type representation which thus figures prominently in studying convexity. The nuts and bolts of this Gram representation and some of its consequences, including Theorem 1.1, are the subjects of Sections 4 and 5 respectively.

The Gram representation techniques actually require only a small amount of convexity and thus there is a theory of geometry on free varieties having signed (e.g. positive) curvature. Some details are in Section 6.

A couple of free semialgebraic geometry results which have a heavy convexity component are described in the last section, Section 7 The first is an optimal free convex Positivstellensatz which generalizes Theorem 1.1. The second says that free convex semialgebraic sets are free spectrahedra, giving another example of the much more rigid structure in the free setting.

Section 3 introduces software which handles free noncommutative computations. You may find it useful in your free studies.

In what follows, mildly incorrectly, but in keeping with the usage in the literature, the terms noncommutative (abbreviated nc) and free are used synonymously.

2. Basics of nc Polynomials and their Convexity

This section treats the basics of polynomials in nc variables, nc differential calculus, and nc inequalities. There is also a brief introduction to nc rational functions and inequalities.

2.1. **Noncommutative polynomials.**

Before turning to the formalities, we give, by examples, an informal introduction to noncommutative (nc) polynomials.

A noncommutative polynomial p is a polynomial in a finite set $x = (x_1, \dots, x_g)$ of relation free variables. A canonical example, in the case of two variables $x = (x_1, x_2)$, is the *commutator*

$$c(x_1, x_2) = x_1x_2 - x_2x_1. \tag{2.1}$$

It is precisely the fact that x_1 and x_2 do not commute that makes c nonzero.

While a commutative polynomial $q \in \mathbb{R}[t_1, t_2]$ is naturally evaluated at points $t \in \mathbb{R}^2$, nc polynomials are naturally evaluated on tuples of square matrices. For instance, with

$$X_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and $X = (X_1, X_2)$, one finds

$$c(X) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Importantly, c can be evaluated on any pair (X, Y) of symmetric matrices of the same size. (Later in the section we will also consider evaluations involving not necessarily symmetric matrices.) Note that if X and Y are $n \times n$, then $c(X, Y)$ is itself an $n \times n$ matrix. In the case of $c(x, y) = xy - yx$, the matrix $c(X, Y) = 0$ if and only if X and Y commute. In particular, c is zero on $\mathbb{R}^2$ (2-tuples of 1×1 matrices).

For another example, if $d(x_1, x_2) = 1 + x_1x_2x_1$, then with X_1 and X_2 as above, we find

$$d(X) = I_2 + X_1X_2X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Note that although X is a tuple of symmetric matrices, it need not be the case that $p(X)$ is symmetric. Indeed, the matrix $c(X)$ above is not. In the present context, we say that p is *symmetric*, if $p(X)$ is symmetric whenever $X = (X_1, \dots, X_g)$ is a tuple of symmetric matrices. Another more algebraic definition of symmetric for nc polynomials appears in Section 2.2.

2.1.1. *Noncommutative convexity for polynomials.* Many standard notions for polynomials, and even functions, on $\mathbb{R}^g$ extend to the nc setting, though often with unexpected ramifications. For example, the commutative polynomial $q \in \mathbb{R}[t_1, t_2]$ is convex if, given $s, t \in \mathbb{R}^2$,

$$\frac{1}{2}\big(q(s) + q(t)\big) \geq q\Big(\frac{s+t}{2}\Big).$$

There is a natural ordering on symmetric $n \times n$ matrices defined by $X \succeq Y$ if the symmetric matrix $X - Y$ is positive semidefinite; i.e., if its eigenvalues are all nonnegative. Similarly, $X \succ Y$, if $X - Y$ is positive definite; i.e., all its eigenvalues are positive. This order yields a canonical notion of convex nc polynomial. Namely, a symmetric polynomial p is *convex* if for each n and each pair of g tuples of $n \times n$ symmetric matrices $X = (X_1, \dots, X_g)$ and $Y = (Y_1, \dots, Y_g)$, we have

$$\frac{1}{2}\big(p(X) + p(Y)\big) \succeq p\Big(\frac{X+Y}{2}\Big).$$

Equivalently,

$$\frac{p(X) + p(Y)}{2} - p\Big(\frac{X+Y}{2}\Big) \succeq 0. \tag{2.2}$$

Even in one variable, convexity for an nc polynomial is a serious constraint. For instance, consider the polynomial x^4. It is symmetric, but with

$$X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

it follows that

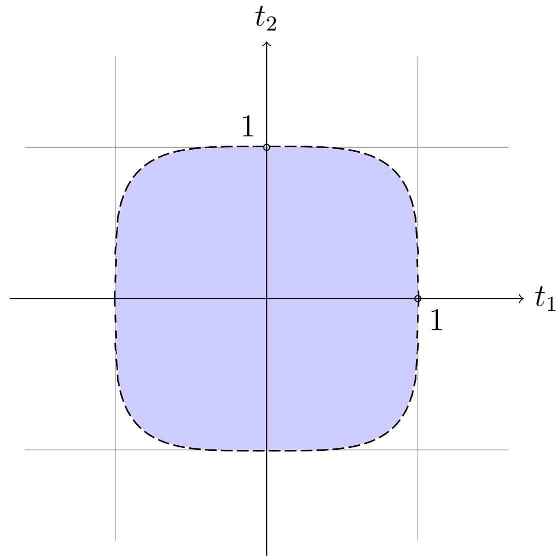
$$\frac{X^4 + Y^4}{2} - \big(\frac{1}{2}X + \frac{1}{2}Y\big)^4 = \begin{bmatrix} 164 & 120 \\ 120 & 84 \end{bmatrix}$$

is not positive semidefinite. Thus x^4 is not convex.

2.1.2. *Noncommutative polynomial inequalities and convexity.* The study of polynomial inequalities, real algebraic geometry or semialgebraic geometry, has a nc version. A *basic open semialgebraic set* is a subset of $\mathbb{R}^g$ defined by a list of polynomial inequalities; i.e., a set S is a basic open semialgebraic set if

$$S = \{t \in \mathbb{R}^g \colon p_1(t) > 0, \dots, p_k(t) > 0\}$$

for some polynomials $p_1, \dots, p_k \in \mathbb{R}[t_1, \dots, t_g]$.



$$\mathrm{ncTV}(1) = \{(t_1, t_2) \in \mathbb{R}^2 \colon 1 - t_1^4 - t_2^4 > 0\}.$$

Because noncommutative polynomials are evaluated on tuples of matrices, a nc (free) basic open semialgebraic set is a sequence. For positive integers n, let $(\mathbb{S}^{n\times n})^g$ denote the set of g-tuples of $n \times n$ symmetric matrices. Given symmetric nc polynomials $p_1, \dots, p_k$, let

$$\mathcal{S}(n) = \{X \in (\mathbb{S}^{n\times n})^g \colon p_1(X) \succ 0, \dots, p_k(X) \succ 0\}.$$

The sequence $\mathcal{S} = (\mathcal{S}(n))$ is then a *nc (free) basic open semialgebraic set*. The sequence

$$\mathrm{ncTV}(n) = \{X \in (\mathbb{S}^{n\times n})^2 \colon I_n - X_1^4 - X_2^4 \succ 0\}$$

is an entertaining example. When $n = 1$, $\mathrm{ncTV}(1)$ is a subset of $\mathbb{R}^2$ often called the *TV screen*. Numerically it can be verified, though it rather tricky to do so (see Exercise 2.7) that the set $\mathrm{ncTV}(2)$ is not a convex set. An analytic proof that $\mathrm{ncTV}(n)$ is not a convex set for some n can be found in [DHM07a]. It also follows by combining results in [HM+] and [HV07]. For properties of the classical commutative TV screen, see the Chapters **??** of Nie and **??** by Rostalski-Sturmfels in this book.

Example 2.1. Let $p_\epsilon := \epsilon^2 - \sum_{j=1}^{g} x_j^2$. An important example of a nc basic open semialgebraic set is the *ϵ-neighborhood of* 0,

$$\mathcal{N}_\epsilon := \bigcup_{n\in\mathbb{N}} \{X \in (\mathbb{S}^{n\times n})^g \colon p_\epsilon(X) \succ 0\}.$$

□

2.2. **Noncommutative polynomials, the formalities.** We now take up the formalities of nc polynomials, their evaluations, convexity, and positivity.

Let $x = \{x_1, \dots, x_g\}$ denote a g-tuple of free noncommuting variables and let $\mathbb{R}{<}x{>}$ denote the associative $\mathbb{R}$-algebra freely generated by x, i.e., the elements of $\mathbb{R}{<}x{>}$ are polynomials in the noncommuting variables x with coefficients in $\mathbb{R}$. Its elements are called *(nc) polynomials*. An element of the form aw where $0 \neq a \in \mathbb{R}$ and w is a *word* in the variables x is called a *monomial* and a its *coefficient*. Hence words are monomials whose coefficient is 1. Note that the empty word $\emptyset$ plays the role of the multiplicative identity for $\mathbb{R}{<}x{>}$.

There is a natural *involution* $^\intercal$ on $\mathbb{R}{<}x{>}$ that reverses words. For example, $(2 - 3x_1^2x_2x_3)^\intercal = 2 - 3x_3x_2x_1^2$. A polynomial p is a *symmetric polynomial* if $p^\intercal = p$. Later we will see that this notion of symmetric is equivalent to that in the previous subsection. For now we note that of

$$\begin{aligned} c(x) &= x_1x_2 - x_2x_1 \\ j(x) &= x_1x_2 + x_2x_1 \end{aligned}$$

j is symmetric, but c is not. Indeed, $c^\intercal = -c$. Because $x_j^\intercal = x_j$ we refer to the variables as *symmetric variables*. Occasionally we emphasize this point by writing $\mathbb{R}{<}x = x^\intercal{>}$ for $\mathbb{R}{<}x{>}$.

The *degree* of an nc polynomial p, denoted $\deg(p)$, is the length of the longest word appearing in p. For instance the polynomials c and j above both have degree two and the degree of

$$r(x) = 1 - 3x_1x_2 - 3x_2x_1 - 2x_1^2x_2^4x_1^2$$

is eight. Let $\mathbb{R}{<}x{>}_k$ denote the polynomials of degree at most k.

2.2.1. *Noncommutative matrix polynomials.* Given positive integers d, d', let $\mathbb{R}^{d\times d'}{<}x{>}$ denote the $d \times d'$ matrices with entries from $\mathbb{R}{<}x{>}$. Thus elements of $\mathbb{R}^{d\times d'}{<}x{>}$ are *matrix-valued nc polynomials*. The involution on $\mathbb{R}{<}x{>}$ naturally extends to a mapping $\intercal : \mathbb{R}^{d\times d'}{<}x{>} \to \mathbb{R}^{d'\times d}{<}x{>}$. In particular, if

$$P = \big[p_{i,j}\big]_{i,j=1}^{d,d'} \in \mathbb{R}^{d\times d'}{<}x{>},$$

then

$$P^\intercal = \big[p_{j,i}^\intercal\big]_{i,j=1}^{d,d'} \in \mathbb{R}^{d'\times d}{<}x{>}.$$

In the case that $d = d'$, such a P is symmetric if $P^\intercal = P$.

2.2.2. *Linear pencils.* Given a positive integer n, let $\mathbb{S}^{n\times n}$ denote the real symmetric $n\times n$ matrices. For $A_0, A_1, \dots, A_g \in \mathbb{S}^{d\times d}$, the expression

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}^{d\times d}\langle x\rangle \tag{2.3}$$

in the noncommuting variables x is a *symmetric affine linear pencil.* In other words, these are precisely the symmetric degree one matrix-valued nc polynomials. If $A_0 = I$, then L is *monic.* If $A_0 = 0$, then L is a *linear pencil.* The homogeneous linear part $\sum_{j=1}^{g} A_j x_j$ of a linear pencil L as in (2.3) will be denoted by $L^{(1)}$.

Example 2.2. Let

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The corresponding monic affine linear pencil is

$$I + \sum A_j x_j = \begin{bmatrix} 1 & x_1 & 0 & 0 \\ x_1 & 1 & x_2 & 0 \\ 0 & x_2 & 1 & x_3 \\ 0 & 0 & x_3 & 1 \end{bmatrix}$$

□

2.2.3. *Polynomial evaluations.* If $p \in \mathbb{R}^{d\times d'}\langle x\rangle$ is an nc polynomial and $X \in (\mathbb{S}^{n\times n})^g$, the *evaluation* $p(X) \in \mathbb{R}^{dn\times d'n}$ is defined by simply replacing x_i by X_i. Throughout we use lower case letters for variables and the corresponding capital letter for matrices substituted for that variable.

Example 2.3. Suppose $p(x) = Ax_1x_2$ where $A = \begin{bmatrix} -4 & 2 \\ 3 & 0 \end{bmatrix}$. That is,

$$p(x) = \begin{bmatrix} -4x_1x_2 & 2x_1x_2 \\ 3x_1x_2 & 0 \end{bmatrix}.$$

Thus $p \in \mathbb{R}^{2\times 2}\langle x\rangle$ and one example of an evaluation is

$$p\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = A \otimes \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = A \otimes \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 4 & 0 & -2 \\ -4 & 0 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, if p is a constant matrix-valued nc polynomial, $p(x) = A$, and $X \in (\mathbb{S}^{n\times n})^g$, then $p(X) = A \otimes I_n$. Here we have taken advantage of the usual *tensor* (or *Kronecker*) *product* of matrices. Given an $\ell \times \ell'$ matrix A and an $n \times n'$ matrix $B = (B_{i,j})$, by definition, $A \otimes B$ is the $n \times n'$ *block* matrix

$$A \otimes B = \left[AB_{i,j}\right],$$

with $\ell \times \ell'$ matrix entries. We have reserved the tensor product notation for the tensor product of matrices and have eschewed the strong temptation of using $A \otimes x_\ell$ in place of Ax_ℓ when x_ℓ is one of the variables. □

Proposition 2.4. *Suppose $p \in \mathbb{R}$<x>. In increasing levels of generality,*

1. *if $p(X) = 0$ for all n and all $X \in (\mathbb{S}^{n\times n})^g$, then $p = 0$;*
2. *if there is a nonempty nc basic open semialgebraic set $\mathcal{O}$ such that $p(X) = 0$ on $\mathcal{O}$ (meaning for every n and $X \in \mathcal{O}(n)$, $p(X) = 0$), then $p = 0$;*
3. *there is an N, depending only upon the degree of p, so that for any $n \geq N$ if there is an open subset $O \subseteq (\mathbb{S}^{n\times n})^g$ with $p(X) = 0$ for all $X \in O$, then $p = 0$.*

Proof. See Exercises 2.12, 2.15, and 2.18. ■

Exercise 2.1. Use Proposition 2.4 to prove the following statement:

Proposition 2.5. *Suppose $p \in \mathbb{R}$<x>. Show $p(X)$ is symmetric for every n and every $X \in (\mathbb{S}^{n\times n})^g$ if and only if $p^\intercal = p$.*

2.3. **Noncommutative convexity revisited and nc positivity.** Now we return with a bit more detail on our main theme, convexity. A symmetric polynomial p is *matrix convex*, if for each positive integer n, each pair of g-tuples $X = (X_1, \dots, X_g)$ and $Y = (Y_1, \dots, Y_g)$ in $(\mathbb{S}^{n\times n})^g$ and each $0 \leq t \leq 1$,

$$tp(X) + (1-t)p(Y) - p\big(tX + (1-t)Y\big) \succeq 0,$$

where, for an $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$, the notation $A \succeq 0$ means A is positive semidefinite. Synonyms for matrix convex include both *nc convex*, and simply *convex*.

Exercise 2.2. Show that the definition here of (matrix) convex is equivalent to that given in equation (2.2) in the informal introduction to nc polynomials.

As we have already seen in the informal introduction to nc polynomials, even in one-variable, convexity in the noncommutative setting differs from convexity in the commutative case because here Y need not commute with X. Thus, although the polynomial x^4 is a convex function of one real variable, it is not matrix convex. On the other hand, to verify that x^2 is a matrix convex polynomial, observe that

$$\begin{aligned} tX^2 + (1-t)Y^2 \quad &- \quad (tX + (1-t)Y)^2 \\ &= \quad t(1-t)(X^2 - XY - YX + Y^2) = t(1-t)(X-Y)^2 \succeq 0. \end{aligned}$$

A polynomial $p \in \mathbb{R}<x>$ is *matrix positive*, synonymously *nc positive* or simply *positive* if $p(X) \succeq 0$ for all tuples $X = (X_1, \dots, X_g) \in (\mathbb{S}^{n\times n})^g$. A polynomial p is a *sum of squares* if there exists $k \in \mathbb{N}$ and polynomials $h_1, \dots, h_k$ such that

$$p = \sum_{j=1}^{k} h_j^\intercal h_j.$$

Because, for a matrix A, the matrix $A^\intercal A$ is positive semidefinite, if p is a sum of squares, then p is positive. Though we will not discuss its proof in this chapter, we mention that, in contrast with the commutative case, the converse is true [Hel02, McC01].

Theorem 2.6. *If $p \in \mathbb{R}<x>$ is positive, then p is a sum of squares.*

As for convexity, note that $p(x)$ is convex if and only if the polynomial $q(x,y)$ in $2g$ nc variables given by

$$q(x,y) = \frac{1}{2}\big(p(x) + p(y)\big) - p\Big(\frac{x+y}{2}\Big)$$

is positive.

2.4. **Directional derivatives vs. nc convexity and positivity.**

Matrix convexity can be formulated in terms of positivity of the Hessian, just as in the case of a real variable. Thus we take a few moments to develop a very useful nc calculus.

Given a polynomial $p \in \mathbb{R}<x>$, the ℓ^{th} *directional derivative* of p in the "direction" h is

$$p^{(\ell)}(x)[h] := \left.\frac{d^\ell p(x+th)}{dt^\ell}\right|_{t=0}.$$

Thus $p^{(\ell)}(x)[h]$ is the polynomial that evaluates to

$$\left.\frac{d^\ell p(X+tH)}{dt^\ell}\right|_{t=0} \qquad \text{for every choice of} \quad X,\ H \in (\mathbb{S}^{n\times n})^g\,.$$

We let $p'(x)[h]$ denote the first derivative and the *Hessian*, denoted $p''(x)[h]$ of $p(x)$, is the second directional derivative of p in the direction h.

Equivalently, the Hessian of $p(x)$ can also be defined as the part of the polynomial

$$r(x)[h] := 2\big(p(x+h) - p(x)\big)$$

in

$$\mathbb{R}<x>[h] := \mathbb{R}<x_1,\dots,x_g,\ h_1,\dots,h_g>$$

that is homogeneous of degree two in h.

If $p'' \neq 0$, that is, if $p = p(x)$ is an nc polynomial of degree two or more, then the polynomial $p''(x)[h]$ in the $2g$ variables $x_1,\dots,x_g,h_1\dots,h_g$ is homogeneous of degree two in h and has degree equal to the degree of p.

Example 2.7.

(1) The Hessian of the polynomial $p = x_1^2 x_2$ is

$$p''(x)[h] = 2(h_1^2 x_2 + h_1 x_1 h_2 + x_1 h_1 h_2)\,.$$

(2) The Hessian of the polynomial $f(x) = x^4$ (just one variable) is

$$f''(x)[h] = 2(h^2x^2 + hxhx + hx^2h + xhxh + xh^2x + x^2h^2)$$

□

NC convexity is neatly described in terms of the Hessian.

Lemma 2.8. *$p \in \mathbb{R}<x>$ is nc convex if and only if $p''(x)[h]$ is nc positive.*

Proof. See Exercise 2.10. ■

Example 2.9. Various directional derivatives of p in (2.5) are

$$D_{x_1}p(x)[h_1] = h_1^\intercal x_1 + x_1^\intercal h_1 + \frac{3}{4}h_1x_2x_1^\intercal + \frac{3}{4}x_1x_2h_1^\intercal, \qquad D_{x_1}p(x)[h_2] = x_2 + \frac{3}{4}x_1h_2x_1^\intercal,$$

$$D_x p(x)[h] = h_1^\intercal x_1 + x_1^\intercal h_1 + h_2 + \frac{3}{4}h_1x_2x_1^\intercal + \frac{3}{4}x_1x_2h_1^\intercal + \frac{3}{4}x_1h_2x_1^\intercal,$$

□

2.5. **Symmetric, free, mixed, and classes of variables.** To this point, our variables x have been *symmetric* in the sense that, under the involution, $x_j^{\intercal} = x_j$. The corresponding polynomials, elements of $\mathbb{R}<x>$ are then the nc analog of polynomials in real variables, with evaluations at tuples $\mathbb{S}^{n\times n}$. In various applications and settings it is natural to consider nc polynomials in other types of variables.

2.5.1. *Free variables.* The nc analog of polynomials in complex variables are obtained by allowing evaluations on tuples X of not necessarily symmetric matrices. In this case, the involution must be interpreted differently and the variables are called *free*.

In this setting, given the nc variables $x = (x_1, \dots, x_g)$, let $x^{\intercal} = (x_1^{\intercal}, \dots, x_g^{\intercal})$ denote another collection of nc variables. On the ring $\mathbb{R}<x, x^{\intercal}>$ define the involution $^{\intercal}$ by the requiring $x_j \mapsto x_j^{\intercal}$; $x_j^{\intercal} \mapsto x_j$; $^{\intercal}$ reverses the order of words; and linearity. For instance, for

$$q(x) = 1 + x_1^{\intercal} x_2 - x_2^{\intercal} x_1 \in \mathbb{R}<x, x^{\intercal}>,$$

we have

$$q^{\intercal}(x) = 1 + x_2^{\intercal} x_1 - x_1^{\intercal} x_2.$$

Elements of $\mathbb{R}<x, x^{\intercal}>$ are polynomials in *free variables* and in this setting the variables themselves are *free*.

A polynomial $p \in \mathbb{R}<x, x^{\intercal}>$ is *symmetric* provided $p^{\intercal} = p$. In particular, q above is not symmetric, but

$$p = 1 + x_1^{\intercal} x_2 + x_2^{\intercal} x_1 \tag{2.4}$$

is.

A polynomial $p \in \mathbb{R}<x, x^{\intercal}>$ is *analytic* if there are no transposes; i.e., if p is a polynomial in x alone.

Elements of $\mathbb{R}<x, x^{\intercal}>$ are naturally evaluated on tuples $X = (X_1, \dots, X_g) \in (\mathbb{R}^{\ell\times\ell})^g$. For instance, if p is the polynomial in equation (2.4) and $X = (X_1, X_2) \in (\mathbb{R}^{2\times 2})^2$ where

$$X_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = X_2$$

then

$$p(X) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

The space $\mathbb{R}^{d\times d'}<x, x^{\intercal}>$ is defined by analogy with $\mathbb{R}^{d\times d'}<x>$ and evaluation of elements in $\mathbb{R}^{d\times d'}<x, x^{\intercal}>$ at a tuple $X \in (\mathbb{R}^{\ell\times\ell})^g$ is defined in the obvious way.

Exercise 2.3. State and prove analogs of Propositions 2.4 and 2.5 for $\mathbb{R}<x, x^{\intercal}>$ and evaluations from $(\mathbb{R}^{\ell\times\ell})^g$.

2.5.2. *Mixed variables.* At times it is desirable to mix free and symmetric variables. We won't introduce notation for this situation as it will generally be understood from the context. Here are some examples:

Example 2.10.

$$p(x) = x_1^{\intercal} x_1 + x_2 + \frac{3}{4} x_1 x_2 x_1^{\intercal}, \qquad x_2 = x_2^{\intercal}; \tag{2.5}$$

$$\operatorname{ric}(a_1, a_2, x) = a_1 x + x a_1^{\intercal} - x a_2 a_2^{\intercal} x, \qquad x = x^{\intercal},$$

In the first case x_1 is free, but x_2 is symmetric; and in the second a_1 and a_2 are free, but x is symmetric. Two additional remarks are in order about the second polynomial. First, it is a *Riccati polynomial* ubiquitous in control theory. Second, we have separated the variables into *two classes* of variables, the a variables and the x variable(s); thus $p \in \mathbb{R}{<}a, x = x^{\intercal}{>}$. In applications, the a variables can be chosen to represent known (system parameters), while the x variables are unknown(s). Of course, it could be that some of the a variables are symmetric and some free and ditto for the x variables. □

Continuing with the variable class warfare, consider the following matrix-valued example.

Example 2.11. Let

$$L(a_1, a_2, x) = \begin{bmatrix} a_1 x + x a_1^{\intercal} & a_2^{\intercal} x \\ x a_2 & 1 \end{bmatrix}.$$

We consider $L \in \mathbb{R}^{2\times 2}{<}a, x = x^{\intercal}{>}$; i.e., the a variables are free, and the x-variables symmetric. Note that L is linear in x if we consider a_1, a_2 fixed. Of course, if a_1, a_2 and x are all scalars, then using a Schur complement tells us there is a close relation between L in this example and the Riccati of the previous example. □

2.6. **Noncommutative rational functions.** While it is possible to define nc functions [Tay73, SV06, Voi04, Voi10, Pop06, Pop10, KVV-, HKM10a, HKM10b], in this section we content ourselves with a relatively informal discussion of nc rational functions [Coh95, Coh06, HMV06, KVV09].

2.6.1. *Rational functions, a gentle introduction.* Noncommutative *rational expressions* are obtained by allowing inverses of polynomials. An example is the discrete time algebraic Riccati equation (DARE)

$$r(a, x) = a_1^{\intercal} x a_1 - (a_1^{\intercal} x a_2) a_1 (a_3 + a_2^{\intercal} x a_2)^{-1} (a_2^{\intercal} x a_1) + a_4, \qquad x = x^{\intercal}.$$

It is a rational expression in the free variables a and the symmetric variable x, as is r^{-1}. An example, in free variables, which arises in operator theory is

$$s(x) = x^{\intercal}(1 - xx^{\intercal})^{-1}. \tag{2.6}$$

Thus, we define (scalar) *nc rational expressions* for free nc variables x by starting with nc polynomials and then applying successive arithmetic operations - addition, multiplication, and inversion. We emphasize that an expression includes the order in which it is composed and no two distinct expressions are identified, e.g., $(x_1)+(-x_1)$, $(-1) + (((x_1)^{-1})(x_1))$, and 0 are different nc rational expressions.

Evaluation on polynomials naturally extends to rational expressions. If r is a rational expression in free variables and $X \in (\mathbb{R}^{\ell\times\ell})^g$, then $r(X)$ is defined - in the obvious way - as long as any inverses appearing actually exist. Indeed, our main interest is in the evaluation of a rational expression. For instance, for the polynomial s above in one free variable, $s(X)$ is defined as long as $I - XX^{\intercal}$ is invertible and in this case,

$$s(X) = X^{\intercal}(I - XX^{\intercal})^{-1}.$$

Generally, a nc rational expression r can be evaluated on a g-tuple X of $n\times n$ matrices in its *domain of regularity*, $\operatorname{dom} r$, which is defined as the set of all g-tuples of square matrices of all sizes such that all the inverses involved in the calculation of $r(X)$ exist. For example, if $r = (x_1x_2 - x_2x_1)^{-1}$ then $\operatorname{dom} r = \{X = (X_1, X_2)\colon \det(X_1X_2 - X_2X_1) \neq 0\}$. We assume that $\operatorname{dom} r \neq \emptyset$. In other words, when forming nc rational expressions we never invert an expression that is nowhere invertible.

Two rational expressions r_1 and r_2 are *equivalent* if $r_1(X) = r_2(X)$ at any X where both are defined. For instance, for the rational expression t in one free variable,

$$t(x) = (1 - x^{\intercal}x)^{-1}x^{\intercal},$$

and s from equation (2.6), it is an exercise to check that $s(X)$ is defined if and only if $t(X)$ is and moreover in this case $s(X) = t(X)$. Thus s and t are equivalent rational expressions. We call an equivalence class of rational expressions a *rational function*. The set of all rational functions will be denoted by $\mathbb{R}\langle\!\langle x\rangle\!\rangle$.

Here is an interesting example of an nc rational function with nested inverses. It is taken from [Ber76, Theorem 6.3].

Example 2.12. Consider two free variables x, y. For any $r \in \mathbb{R}\langle\!\langle x, y\rangle\!\rangle$ let

$$W(r) := c\big(x, c(x,r)^2\big)\cdot c\big(x, c(x,r)^{-1}\big)^{-1} \in \mathbb{R}\langle\!\langle x, y\rangle\!\rangle. \tag{2.7}$$

Recall that c denotes the commutator (2.1). Bergman's nc rational function is given by:

$$b := W(y) \cdot W\big(c(x,y)\big) \cdot W\Big(c\big(x, c(x,y)\big)^{-1}\Big) \cdot W\Big(c\big(x, c(x, c(x,y))\big)^{-1}\Big) \in \mathbb{R}\langle\!\langle x, y\rangle\!\rangle. \tag{2.8}$$

□

Exercise 2.4. Consider the function W from (2.7). Let R, X be $n \times n$ matrices and assume $c\big(X, c(X,R)^{-1}\big)$ exists and is invertible. Prove:

(1) If $n = 2$, then $W(R) = 0$.
(2) If $n = 3$, then $W(R) = \det(c(X,R))$.

Exercise 2.5. Consider Bergman's rational function (2.8).

(1) Show that on a dense set of 2×2 matrices (X, Y), $b(X,Y) = 0$.
(2) Prove that on a dense set of 3×3 matrices (X, Y), $b(X,Y) = 1$.

The moral of Exercise 2.5 is that, unlike in the case of polynomial identities, a nc rational function that vanishes on (a dense set of) 3×3 matrices need not vanish on (a dense set of) 2×2 matrices.

2.6.2. *Matrices of Rational Functions; $LDL^\intercal$.* One of the main ways nc rational functions occur in systems engineering is in the manipulation of matrices of polynomials. Extremely important is the $LDL^\intercal$ *decomposition*. Consider the 2×2 matrix with nc entries

$$M = \begin{bmatrix} a & b^\intercal \\ b & c \end{bmatrix}$$

where $a = a^\intercal$. The entries themselves could be a nc polynomials, or even rational functions. If a is not zero, then M has the following decomposition

$$M = LDL^\intercal = \begin{bmatrix} I & 0 \\ ba^{-1} & I \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c - ba^{-1}b^\intercal \end{bmatrix} \begin{bmatrix} I & a^{-1}b^\intercal \\ 0 & I \end{bmatrix}.$$

Note that this formula holds in the case that c is itself a (square) matrix nc rational function and b (and thus $b^\intercal$) are vector-valued nc rational functions. On the hand, if both $a = c = 0$, then M is the block matrix,

$$M = \begin{bmatrix} 0 & b \\ b^\intercal & 0 \end{bmatrix}.$$

If we have $k \times k$ matrix M, iterating this procedure produces a decomposition of a permutation $\Pi M \Pi^\intercal$ of M of the form $\Pi M \Pi^\intercal = LDL^\intercal$ where D and L have the form

$$D = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & d_k & 0 & \cdots & 0 & 0 \\ 0 & \dots & 0 & D_{k+1} & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & D_\ell & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & E \end{bmatrix} \tag{2.9}$$

and L has the form,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & \ddots & 0 & 0 & 0 & 0 & \\ * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & I_2 & 0 & 0 & 0 \\ * & * & * & * & \ddots & 0 & 0 \\ * & * & * & * & * & I_2 & 0 \\ * & * & * & * & * & * & I_a \end{bmatrix}, \tag{2.10}$$

where d_j are symmetric rational functions, and the D_j are nonzero 2×2 matrices of the form

$$D_j = \begin{bmatrix} 0 & b_j \\ b_j^\intercal & 0 \end{bmatrix},$$

E is a square 0 matrix (possibly of size 0×0 - so absent), and I_2 is the 2×2 identity and the $*$'s represent possibly nonzero rational expressions (in some cases matrices of rational), some of the 0s are zero matrices (of the appropriate sizes), and a is the dimension of the space that E acts upon. The permutation Π is necessary in cases where the procedure hits a 0 on the diagonal, necessitating a permutation to bring a nonzero diagonal entry into the "pivot" position.

Theorem 2.13. *Suppose $M(x) \in \mathbb{R}\langle x\rangle^{\ell\times\ell}$ is symmetric, and $\Pi M \Pi^\intercal = LDL^\intercal$ where L, D are $\ell \times \ell$ matrices with nc rational entries as in equations (2.10) and (2.9) and L respectively. If n is a positive integer and $X \in (\mathbb{S}^{n\times n})^g$ is in the domains of both L and D, then $M(X)$ is positive semidefinite if and only if $D(X)$ is positive semidefinite.*

Proof. The proof is an easy exercise based on the fact that a square block lower triangular matrix whose diagonal blocks are invertible is itself invertible. In this case, $L(X)$ is block lower triangular with the $n \times n$ identity I_n as each diagonal entry. Thus $M(X)$ and $D(X)$ are congruent, so have the same number of negative eigenvalues. ∎

Remark 2.14. Note that if D has any 2×2 blocks D_j, the $D(X) \succeq 0$ if and only if each $D_j(X) = 0$. Thus, if D has any 2×2 blocks, generically $D(X)$, and hence $M(X)$, is not positive semidefinite (recall we assume, without loss of generality that D_j are not zero).

2.6.3. *More on rational functions.* The matrix positivity and convexity properties of nc rational functions go just like those for polynomials. One only tests r on a matrices X in the domain of regularity. The definition of directional derivatives goes as before and it is easy to compute them formally. There are issues of equivalences which we avoid here, instead referring the reader to [Coh95, KVV09] or our treatment in [HMV06].

We emphasize that proving the assertions above takes considerable effort, because of dealing with the equivalence relation. In practice one works with rational expressions, and calculations with nc rational expressions themselves are straightforward. For instance, computing the derivative of a symmetric nc rational function r leads to an expression of the form

$$Dr(x)[h] = \text{symmetrize}\left[\sum_{\ell=1}^{k} a_\ell(x) h b_\ell(x)\right],$$

where a_ℓ, b_ℓ are nc rational functions of x, and the symmetrization of a (not necessarily symmetric) rational expression s is $\frac{s+s^\intercal}{2}$.

2.7. **Exercises.** Section 3 gives a very brief chapter on nc computer algebra and some might enjoy playing with computer algebra in working some of these exercises.

Define for use in later exercises the nc polynomials

$$\begin{aligned} p &= x_1^2 x_2^2 - x_1x_2x_1x_2 - x_2x_1x_2x_1 - x_2^2x_1^2 \\ q &= x_1x_2x_3 + x_2x_3x_1 + x_3x_1x_2 - x_1x_3x_2 - x_2x_1x_3 - x_3x_2x_1 \\ s &= x_1x_3x_2 - x_2x_3x_1. \end{aligned}$$

Exercise 2.6.

(a) What is the derivative with respect to x_1 in direction h_1 of q, s and u.
(b) Concerning the formal derivative with respect to x_1 in direction h_1.
   (i) Show the derivative of $r(x_1) = {x_1}^{-1}$ is $-x_1^{-1}h_1x_1^{-1}$.
   (ii) What is the derivative of $u(x_1, x_2) = x_2(1 + 2x_1)^{-1}$?

Exercise 2.7. Consider the polynomials p, q, s and rational functions r, u from above.

(a) Evaluate the polynomials p, q, s on some matrices of size 1×1, 2×2 and 3×3.

(b) Redo part (a) for the rational functions r, u.

Try to use Mathematica or Matlab.

Exercise 2.8. Show $c = x_1x_2 - x_2x_1$ is not symmetric, by finding n and $X = (X_1, X_2)$ such that $c(X)$ is not a symmetric matrix.

Exercise 2.9. Consider the following polynomials in two and three variables, respectively:

$$\begin{aligned} h_1 &= c^2 = (x_1x_2)^2 - x_1x_2^2x_1 - x_2x_1^2x_2 + (x_2x_1)^2, \\ h_2 &= h_1x_3 - x_3h_1. \end{aligned}$$

(a) Compute $h_1(X_1, X_2)$ and $h_2(X_1, X_2, X_3)$ for several choices of 2×2 matrices X_j. What do you find? Can you formulate and prove a statement?
(b) What happens if you plug in 3×3 matrices into h_1 and h_2?

Exercise 2.10. Prove that a symmetric nc polynomial p is matrix convex if and only if the Hessian $p''(x)[h]$ is matrix positive, by completing the following exercise.

Fix n, suppose ℓ is a positive linear functional on $\mathbb{S}^{n \times n}$, and consider

$$f = \ell \circ p : (\mathbb{S}^{n \times n})^g \to \mathbb{R}.$$

(a) Show f is convex if and only if $\frac{d^2 f(X+tH)}{dt^2} \geq 0$ at $t = 0$ for all $X, H \in (\mathbb{S}^{n \times n})^g$.

Given $v \in \mathbb{R}^n$, consider the linear functional $\ell(M) := v^\intercal M v$ and let $f_v = \ell \circ p$.

(b) *Geometric*: Fix n. Show, each f_v satisfies the convexity inequality if and only if p satisfies the convexity inequality on $(\mathbb{S}^{n \times n})^g$; and
(b) *Analytic*: show, for each $v \in \mathbb{R}^n$, $f_v''(X)[H] \geq 0$ for every $X, H \in (\mathbb{S}^{n \times n})^g$ if and only if $p''(X)[H] \succeq 0$ for every $X, H \in (\mathbb{S}^{n \times n})^g$.

Exercise 2.11. For $n \in \mathbb{N}$ let

$$s_n = \sum_{\tau \in \mathrm{Sym}_n} \mathrm{sign}(\tau) x_{\tau(1)} \cdots x_{\tau(n)}$$

be a polynomial of degree n in n variables. Here Sym_n denotes the symmetric group on n elements.

(a) Prove that s_4 is a polynomial identity for 2×2 matrices. That is, for any choice of 2×2 matrices $X_1, \dots, X_4$, we have

$$s_4(X_1, \dots, X_4) = 0.$$

(b) Fix $d \in \mathbb{N}$. Prove that there exists a nonzero polynomial p vanishing on all tuples of $d \times d$ matrices.

Several of the next exercises use a version of the shift operators on Fock space. With g fixed, the corresponding *Fock space*, $\mathcal{F} = \mathcal{F}_g$, is the Hilbert space obtained from $\mathbb{R}<x>$ by declaring the words to be an orthonormal basis; i.e., if v, w are words, then

$$\langle v, w\rangle = \delta_{v,w},$$

where $\delta_{v,w} = 1$ if $v = w$ and is 0 otherwise. Thus $\mathcal{F}_g$ is the closure of $\mathbb{R}<x>$ in this inner product. For each j, the operator S_j on $\mathcal{F}_g$ densely defined by $S_j p = x_j p$, for $p \in \mathbb{R}<x>$ is an isometry (preserves the inner product) and hence extends to an isometry on all of $\mathcal{F}_g$. Of course, S_j acts on an infinite dimensional Hilbert space and thus is not a matrix.

Exercise 2.12. Given a natural number k, note that $\mathbb{R}<x>_k$ is a finite dimensional (and hence closed) subspace of $\mathcal{F} = \mathcal{F}_g$. The dimension of $\mathbb{R}<x>_k$ is

$$\sigma(k) = \sum_{j=0}^{k} g^j. \tag{2.11}$$

Let $V : \mathbb{R}<x>_k \to \mathcal{F}$ denote the inclusion and

$$T_j = V^\intercal S_j V_k.$$

Thus T_j does act on a finite dimensional space, and $T = (T_1, \dots, T_g) \in (\mathbb{R}^{n\times n})^g$, for $n = \sigma(k)$.

(a) Show, if v is a word of length at most $k-1$, then

$$T_j v = x_j v;$$

and $T_j v = 0$ if the length of v is k.

(b) Determine $T_j^\intercal$;

(c) Show, if p is a nonzero polynomial of degree at most k and $Y_j = T_j + T_j^\intercal$, then $p(Y)\emptyset \neq 0$;

(d) Conclude, if, for every n and $X \in (\mathbb{S}^{n\times n})^g$, $p(X) = 0$, then p is 0.

Exercise 2.12 shows there are no nc polynomials vanishing on all tuples of (symmetric) matrices of all sizes. The next exercise will lead the reader through an alternative proof inspired by standard methods of polynomial identities.

Exercise 2.13. Let $p \in \mathbb{R}<x>_n$ be an analytic polynomial that vanishes on $(\mathbb{R}^{n\times n})^g$ (same fixed n). Write $p = p_0 + p_1 + \cdots + p_n$, where p_j is the homogeneous part of p of degree j.

(a) Show that p_j also vanishes on $(\mathbb{R}^{n\times n})^g$.

(b) A polynomial q is called multilinear if it is homogeneous of degree one with respect to all of its variables. Equivalently, each of its monomials contains all variables exactly once, i.e.,

$$q = \sum_{\pi \in S_n} \alpha_\pi X_{\pi(1)} \cdots X_{\pi(n)}.$$

Using the staircase matrices $E_{11}, E_{12}, E_{22}, E_{23}, \dots, E_{n-1\,n}, E_{nn}$ show that a nonzero multilinear polynomial q of degree n cannot vanish on all $n \times n$ matrices.

(c) By (a) we may assume p is homogeneous. By induction on the biggest degree a variable in p can have, prove that $p = 0$. *Hint:* What are the degrees of the variables appearing in

$$p(x_1 + \hat{x}_1, x_2, \dots, x_g) - p(x_1, x_2, \dots, x_g) - p(\hat{x}_1, x_2, \dots, x_g)?$$

Exercise 2.14. Redo Exercise 2.13 for a polynomial

(a) $p \in \mathbb{R}\langle x, x^\intercal \rangle$, not necessarily analytic, vanishing on all tuples of matrices;
(b) $p \in \mathbb{R}\langle x \rangle$ vanishing on all tuples of symmetric matrices.

Exercise 2.15. Show, if $p \in \mathbb{R}\langle x \rangle$ vanishes on a nonempty basic open semialgebraic set, then $p = 0$.

Exercise 2.16. Suppose $p \in \mathbb{R}\langle x \rangle$, n is a positive integer and $O \subseteq (\mathbb{S}^{n\times n})^g$ is an open set. Show, if $p(X) = 0$ for each $X \in O$, then $P(X) = 0$ for each $X \in (\mathbb{S}^{n\times n})^g$. *Hint:* given $X_0 \in O$ and $X \in (\mathbb{S}^{n\times n})^g$, consider the matrix valued polynomial,

$$q(t) = p(X_0 + tX).$$

Exercise 2.17. Suppose $r \in \mathbb{R}\langle\!\langle x \rangle\!\rangle$ is a rational function and there is a nonempty nc basic open semialgebraic set $\mathcal{O} \subseteq \operatorname{dom}(r)$ with $r|_{\mathcal{O}} = 0$. Show that $r = 0$.

Exercise 2.18. Prove item (3) of Proposition 2.4. You may wish to use Exercises 2.16 and 2.12.

Exercise 2.19. Prove the following proposition:

Proposition 2.15. *If $\pi : \mathbb{R}\langle x \rangle \to \mathbb{R}^{n\times n}$ is an involution preserving homomorphism, then there is an $X \in (\mathbb{S}^{n\times n})^g$ such that $\pi(p) = p(X)$; i.e., all finite dimensional representations of $\mathbb{R}\langle x \rangle$ are evaluations.*

Exercise 2.20. Do the algebra to show

$$x^\intercal (1 - x x^\intercal)^{-1} = (1 - x^\intercal x)^{-1} x^\intercal.$$

(This is a key fact used in the model theory for contractions [NFBK10].)

Exercise 2.21. Give an example of symmetric 2×2 matrices X, Y such that $X \succeq Y \succeq 0$, but $X^2 \not\succeq Y^2$.

This failure of a basic order property of $\mathbb{R}$ for $\mathbb{S}^{n\times n}$ is closely related to the rigid nature of positivity and convexity in the nc setting.

Exercise 2.22. Antiderivatives.

(a) Is $q(x)[h] = xh + hx$ the derivative of any nc polynomial p? If so what is p?
(b) Is $q(x)[h] = hhx + hxh + xhh$ the second derivative of any nc polynomial p? If so what is p?
(c) Describe in general which polynomials $q(x)[h]$ are the derivative of some nc polynomial $p(x)$.
(d) Check you answer against the theory in [GHV+].

Exercise 2.23. (Requires background in algebra) Show that $\mathbb{R}\langle\!\langle x\rangle\!\rangle$ is a division ring; i.e., the nc rational functions form a ring in which every nonzero element is invertible.

Exercise 2.24. In this exercise we will establish that it is possible to embed the free algebra $\mathbb{R}<x_1,\dots,x_g>$ into $\mathbb{R}<x,y>$ for any $g \in \mathbb{N}$.

(a) Show that the subalgebra of $\mathbb{R}<x,y>$ generated by xy^n, $n \in \mathbb{N}_0$, is free.
(b) Ditto for the subalgebra generated by

$$x_1 = x, \quad x_2 = c(x_1, y), \quad x_3 = c(x_2, y), \quad \dots, \quad x_n = c(x_{n-1}, y), \dots.$$

Here, as before, c is the commutator, $c(a,b) = ab - ba$.

A comprehensive study of free algebras and nc rational functions from an algebraic viewpoint is developed in [Coh95, Coh06].

Exercise 2.25. As a hard exercise, numerically verify that the set

$$\mathrm{ncTV}(2) = \{X \in (\mathbb{S}^{2\times 2})^2 : 1 - X_1^4 - X_2^4 \succ 0\}$$

is not convex. That is, find $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ where X_1, X_2, Y_1, Y_2 are 2×2 symmetric matrices such that both

$$1 - X_1^4 - X_2^4 \succ 0 \quad \text{and} \quad 1 - Y_1^4 - Y_2^4 \succ 0,$$

but

$$1 - \Big(\frac{X_1 + Y_1}{2}\Big)^4 - \Big(\frac{X_2 + Y_2}{2}\Big)^4 \not\succ 0.$$

You may wish to write a numerical search routine.

3. Computer algebra support

There are several computer algebra packages available to ease the first contact with free convexity and positivity. In this section we briefly describe two of them:

(1) `NCAlgebra` running under Mathematica;
(2) `NCSOStools` running under Matlab.

The former is more universal in that it implements manipulation with noncommutative variables, including nc rationals, and several algorithms pertaining to convexity. The latter is focused on nc positivity and numerics.

3.1. **NCAlgebra.** NCAlgebra [HOMS+] runs under Mathematica and gives it the capability of manipulating noncommuting algebraic expressions. An important part of the package (which we shall not go into here) is NCGB, which computes noncommutative Groebner Bases and has extensive sorting and display features as well as algorithms for automatically discarding "redundant" polynomials.

We recommend the user to have a look at the Mathematica notebook `NCBasicCommandsDemo` available from the NCAlgebra website

`http://math.ucsd.edu/~ncalg/`

for the basic commands and their usage in NCAlgebra. Here is a sample.

The basic ingredients are (symbolic) variables, which can be either noncommutative or commutative. At present, single-letter lower case variables are noncommutative by default and all others are commutative by default. To change this one can employ

NCAlgebra Command: `SetNonCommutative[listOfVariables]` to make all the variables appearing in listOfVariables noncommutative. The converse is given by

NCAlgebra Command: `SetCommutative`.

Example 3.1. Here is a sample session in Mathematica running NCAlgebra.

```
In[1]:= a ** b - b ** a
Out[1]= a ** b - b ** a

In[2]:= A ** B - B ** A
Out[2]= 0

In[3]:= A ** b - b ** a
Out[3]= A b - b ** a
```

```
In[4]:= CommuteEverything[a ** b - b ** a]
Out[4]= 0

In[5]:= SetNonCommutative[A, B]
Out[5]= {False, False}

In[6]:= A ** B - B ** A
Out[6]= A ** B - B ** A

In[7]:= SetNonCommutative[A];SetCommutative[B]
Out[7]= {True}

In[8]:= A ** B - B ** A
Out[8]= 0
```

□

Slightly more advanced is the NCAlgebra command to generate the directional derivative of a polynomial $p(x, y)$ with respect to x, which is denoted by $D_x p(x, y)[h]$:

NCAlgebra Command: `DirectionalD[Function` p, x, h`]`, and is abbreviated

NCAlgebra Command: `DirD`.

Example 3.2. Consider

```
a = x ** x ** y - y ** x ** y
```

Then

```
DirD[a, x, h] = (h ** x + x ** h) ** y - y ** h ** y
```

or in expanded form,

```
NCExpand[DirD[a, x, h]] = h ** x ** y + x ** h ** y - y ** h ** y
```

Note that we have used

NCAlgebra Command: `NCExpand[Function` p`]` to expand a noncommutative expressions. The command comes with a convenient abbreviation

NCAlgebra Command: `NCE`. □

NCAlgebra is capable of much more. For instance, is a given noncommutative function "convex"? You type in a function of noncommutative variables; the command

NCAlgebra Command: `NCConvexityRegion[Func, ListOfVariables]` tells you where the (symbolic) Function is convex in the Variables. The algorithm comes from the paper of Camino, Helton, Skelton, Ye [CHSY03].

NCAlgebra Command: $\{L, D, U, P\}$:=`NCLDUDecomposition[Matrix]`. Computes the LDU Decomposition of Matrix and returns the result as a 4 tuple. The last entry is a Permutation matrix which reveals which pivots were used. If Matrix is symmetric then $U = L^\intercal$.

The NCAlgebra website comes with extensive documentation. A more advanced notebook with a hands on demonstration of applied capabilities of the package is `DemoBRL.nb`; it derives the Bounded Real Lemma for a linear system.

Exercise 3.1. For the polynomials and rational functions defined at the beginning of Section 2.7, use NCAlgebra to calculate

(a) `p**q` and `NCExpand[p**q]`
(b) `NCCollect[p**q, x1]`
(c) `D[p,x1,h1]` and `D[u,x1,h1]`

3.1.1. *Warning.* The Mathematica substitute commands `\.`, `\>` and `\:>` are not reliable in NCAlgebra, so a user should use NCAlgebra's Substitute command.

Example 3.3. Here is an example of unsatisfactory behavior of the built-in Mathematica function.

```
In[1]:= (x ** a ** b) /. {a ** b -> c}
Out[1]= x ** a ** b
```

On the other hand, NCAlgebra performs as desired:

```
In[2]:= Substitute[x ** a ** b, a ** b -> c]
Out[2]= x ** c
```
□

3.2. **NCSOStools.** A reader mainly interested in positivity of noncommutative polynomials might be better served by NCSOStools [CKP11]. NCSOStools is an open source Matlab toolbox for

(a) basic symbolic computation with polynomials in noncommuting variables;

(b) constructing and solving sum of hermitian squares (with commutators) programs for polynomials in noncommuting variables.

It is normally used in combination with standard semidefinite programming software to solve these constructed LMIs.

The NCSOStools website

`http://ncsostools.fis.unm.si`

contains documentation and a demo notebook `NCSOStoolsdemo` to give the user a gentle introduction into its features.

Example 3.4. Despite some ability to manipulate symbolic expressions, Matlab cannot handle noncommuting variables. They are implemented in NCSOStools.

NCSOStools Command: `NCvars` x introduces a noncommuting variable x into the workspace. □

NCSOStools is well equipped to work with commutators and sums of (hermitian) squares. Recall: a *commutator* is an expression of the form $fg - gf$.

Exercise 3.2. Use NCSOStools to check whether the polynomial $x^2yx + yx^3 - 2xyx^2$ is a sum of commutators. (*Hint*: Try the `NCisCycEq` command.) If so, can you find such an expression?

Let us demonstrate an example with sums of squares.

Example 3.5. Consider

```
f = 5 + x^2 - 2*x^3 + x^4 + 2*x*y + x*y*x*y - x*y^2 + x*y^2*x
    -2*y + 2*y*x + y*x^2*y - 2*y*x*y + y*x*y*x - 3*y^2 - y^2*x + y^4
```

Is f matrix positive? By Theorem 2.6 it suffices to check whether f is a sum of squares. This is easily done using

NCSOStools Command: `NCsos`(f), which checks the polynomial f is a sum of squares. Running NCsos(f) tells us that f is indeed a sum of squares. What NCSOStools does, is transform this question into a semidefinite program (SDP) and then calls a solver. NCsos comes with several options. Its full command line is

```
[IsSohs,X,base,sohs,g,SDP_data,L] = NCsos(f,params)
```

The meaning of the output is as follows:

- `IsSohs` equals 1 if the polynomial f is a sum of hermitian squares and 0 otherwise;
- `X` is the Gram matrix solution of the corresponding SDP returned by the solver;

- `base` is a list of words which appear in the SOHS decomposition;
- `sohs` is the SOHS decomposition of f;
- `g` is the NCpoly representing $\sum_i m_i^\intercal m_i$;
- `SDP_data` is a structure holding all the data used in SDP solver;
- `L` is the operator representing the dual optimization problem (i.e., the dual feasible SDP matrix). □

Exercise 3.3. Use NCSOStools to compute the smallest eigenvalue $f(X,Y)$ can attain for a pair of symmetric matrices (X,Y). Can you also find a minimizer pair (X,Y)?

Exercise 3.4. Let $f = y^2 + (xy-1)^\intercal(xy-1)$. Show that

(a) $f(X,Y)$ is always positive semidefinite.
(b) For each $\epsilon > 0$ there is a pair of symmetric matrices (X,Y) so that the smallest eigenvalue of $f(X,Y)$ is ϵ.
(c) Can $f(X,Y)$ be singular?

The moral of Example 3.4 is that even if an nc polynomial is bounded from below, it need not attain its minimum.

Exercise 3.5. Redo the Exercise 3.4 for $f(x) = x^\intercal x + (xx^\intercal - 1)^\intercal(xx^\intercal - 1)$.

4. A Gram like representation

The next two sections are devoted to a powerful representation of quadratic functions q in nc variables which takes a strong form when q is matrix positive; we call it a *QuadratischePositivstellensatz*. Ultimately we shall apply this to $q(x)[h] = p''(x)[h]$ and show that if p is matrix convex (i.e., q is matrix positive), then p has degree two. We begin by illustrating our grand scheme with examples.

4.1. Illustrating the ideas.

Example 4.1. The (symmetric) polynomial $p(x) = x_1x_2x_1 + x_2x_1x_2$ (in symmetric variables) has Hessian $q(x)[h] = p''(x)[h]$ which is homogeneous quadratic in h and is

$$q(x)[h] = 2h_1h_2x_1 + 2h_1x_2h_1 + 2h_2h_1x_2 + 2h_2x_1h_2 + 2x_1h_2h_1 + 2x_2h_1h_2.$$

We can write q in the form

$$q(x)[h] = \begin{bmatrix} h_1 & h_2 & x_2h_1 & x_1h_2 \end{bmatrix} \begin{bmatrix} 2x_2 & 0 & 0 & 2 \\ 0 & 2x_1 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_1x_2 \\ h_2x_1 \end{bmatrix}.$$

□

The representation of q displayed above is of the form

$$q(x)[h] = V(x)[h]^\intercal Z(x)V(x)[h]$$

where Z is called the *middle matrix* (MM) and V the *border vector* (BV). The MM does not contain h. The BV is linear in h with h always on the left. In Section 4.2 we define this border vector-middle matrix (BV-MM) representation generally for nc polynomials $q(x)[h]$ which are homogeneous of degree two in the h variables. Note the entries of the BV are distinct monomials.

Example 4.2. Let $p = x_2x_1x_2x_1 + x_1x_2x_1x_2$. Then

$$\begin{aligned} q = p'' = {} & 2h_1h_2x_1x_2 + 2h_1x_2h_1x_2 + 2h_1x_2x_1h_2 + 2h_2h_1x_2x_1 + 2h_2x_1h_2x_1 + 2h_2x_1x_2h_1 \\ & + 2x_1h_2h_1x_2 + 2x_1h_2x_1h_2 + 2x_1x_2h_1h_2 + 2x_2h_1h_2x_1 + 2x_2h_1x_2h_1 + 2x_2x_1h_2h_1. \end{aligned}$$

The BV-MM representation for q is

$$q = \begin{bmatrix} h_1 & h_2 & x_2h_1 & x_1h_2 & x_1x_2h_1 & x_2x_1 \end{bmatrix} \begin{bmatrix} 0 & 2x_2x_1 & 2x_2 & 0 & 0 & 2 \\ 2x_1x_2 & 0 & 0 & 2x_1 & 2 & 0 \\ 2x_1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2x_2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_1x_2 \\ h_2x_1 \\ h_1x_2x_1 \\ h_2x_1x_2 \end{bmatrix}$$

□

Example 4.3. In one variable with $h_1 = h_1^\intercal$ we abbreviate it to h. Fix some nc variables not necessarily symmetric $w := (a, b, d, e)$ and consider

$$q(w)[h] := hah + e^\intercal hbh + hb^\intercal he + e^\intercal hdhe. \tag{4.1}$$

which is a quadratic function of h. It can be written in the BV-MM form

$$q(w)[h] = \begin{bmatrix} h & e^\intercal h \end{bmatrix} \begin{bmatrix} a & b^\intercal \\ b & d \end{bmatrix} \begin{bmatrix} h \\ he \end{bmatrix}. \tag{4.2}$$

The representation is unique.

Observe (4.2) contrasts strongly with the commutative case wherein (4.1) takes the form

$$q(w)[h] = h(a + e^\intercal b + b^\intercal g + e^\intercal de)h.$$

□

Example 4.4. The Hessian of $p(x) = x^4$ is

$$\begin{aligned} q(x)[h] := p''(x)[h] &= 2(x^2h^2 + xh^2x + h^2x^2) \\ &\quad + 2(xhxh + hxhx) \\ &\quad + hx^2h \end{aligned} \tag{4.3}$$

is a polynomial that is homogeneous of degree two in x and homogeneous of degree two in h that can be expressed as

$$q(x)[h] = 2\begin{bmatrix} h & xh & x^2h \end{bmatrix} \begin{bmatrix} x^2 & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h \\ hx \\ hx^2 \end{bmatrix}.$$

□

Notice that the contribution of the main antidiagonal of the MM for q in Example 4.4 (all $1's$) corresponds to the right hand side of first line of (4.3). Indeed, each antidiagonal corresponds to a line of (4.3).

Exercise 4.1. In Example 4.4, for which symmetric matrices X is $Z(X)$ positive semidefinite?

Exercise 4.2. What is the MM for $p(x) = x^3$? For which symmetric matrices is X is $Z(X)$, the MM, positive semidefinite?

Exercise 4.3. Compute middle matrix representations using `NCAlgebra`. The command is

$\{lt, mq, rt\}$ =`NCMatrixOfQuadratic[`q`, `$\{h, k\}$`]`

In the output mq is the MM and rt is the BV and lt is $(rt)^\intercal$. For examples, see `NCConvexityRegionDemo.nb` In the `NC/DEMOS directory`.

4.1.1. *The positivity of q vs. positivity of the MM.* In this section we let $q(x)[h]$ denote a polynomial which is homogeneous of degree two in h, but which is not necessarily the Hessian of a nc polynomial. While we have focused on Hessians, such a q will still have a BV-MM representation. *So what good is this representation?* After all one expects that q could have wonderful properties, such as positivity, which are not shared by its middle matrix. No, the striking thing is that positivity of q implies positivity of the MM. Roughly we shall prove what we call the *QuadratischePositivstellensatz*, which is essentially Theorem 3.1 of [CHSY03].

Theorem 4.5. *If the polynomial*[2] $q(x)[h]$ *is homogeneous quadratic in* h*, then* q *is matrix positive if and only if its middle matrix* Z *is matrix positive.*

More generally, suppose $\mathcal{O}$ *is a nonempty nc basic open semialgebraic set. If* $q(X)[H]$ *is positive semidefinite for all* $n \in \mathbb{N}$*,* $X \in \mathcal{O}(n)$ *and* $H \in (\mathbb{S}^{n\times n})^g$*, then* $Z(X) \succeq 0$ *for all* $X \in \mathcal{O}$*.*

We emphasize that, in the theorem, the convention that the terms of the border vector are distinct is in force.

To foreshadow Section 5 and to give an idea of the proof and we illustrate it on an example in one variable. This time we use a *free* rather than symmetric variable since proofs are a bit easier.

Proof of Theorem 4.5 for an example. Consider the noncommutative quadratic function q given by

$$q(w)[h] := h^\intercal bh + e^\intercal h^\intercal ch + h^\intercal c^\intercal he + e^\intercal h^\intercal ahe \tag{4.4}$$

where $w = (a, b, c, e)$. The border vector $V(w)[h]$ and the coefficient matrix $Z(w)$ with noncommutative entries are

$$V(w)[h] = \begin{bmatrix} h \\ he \end{bmatrix} \qquad \text{and} \qquad Z(w) = \begin{bmatrix} b & c^\intercal \\ c & a \end{bmatrix},$$

that is, q has the form

$$q(w)[h] = V(w)[h]^\intercal Z(w)V(w)[h] = \begin{bmatrix} h^\intercal & e^\intercal h^\intercal \end{bmatrix} \begin{bmatrix} b & c^\intercal \\ c & a \end{bmatrix} \begin{bmatrix} h \\ he \end{bmatrix}.$$

Now, if in equation (4.4) the elements a, b, c, e, h are replaced by matrices in $\mathbb{R}^{n\times n}$, then the noncommutative quadratic function $q(w)[h]$ becomes a matrix valued function $q(W)[H]$. The matrix valued function $q[H]$ is matrix positive if and only if $v^\intercal q(W)[H]v \geq 0$ for all vectors $v \in \mathbb{R}^n$ and all $H \in \mathbb{R}^{n\times n}$. Or equivalently, the following inequality must hold

$$\begin{bmatrix} v^\intercal H^\intercal & v^\intercal E^\intercal H^\intercal \end{bmatrix} Z \begin{bmatrix} Hv \\ HEv \end{bmatrix} \geq 0. \tag{4.5}$$

Let

$$y^\intercal := \begin{bmatrix} v^\intercal H^\intercal & v^\intercal E^\intercal H^\intercal \end{bmatrix}.$$

Then (4.5) is equivalent to $y^\intercal Z\, y \geq 0$. Now it suffices to prove that all vectors of the form y sweep $\mathbb{R}^{2n}$. This will be completely analyzed in full generality in Section 5.1 but next we give the proof for our simple situation.

[2]This theorem is true (but not proved here) for q which are nc rational in x.

Suppose for a given v, with $n \geq 2$, the vectors v and Ev are linearly independent. Let $y = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be any vector in $\mathbb{R}^{2n}$, then we can choose $H \in \mathbb{R}^{n \times n}$ with the property that $v_1 = Hv$ and $v_2 = HEv$. It is clear that

$$\mathcal{R}^v := \left\{ \begin{bmatrix} Hv \\ HEv \end{bmatrix} : H \in \mathbb{R}^{n \times n} \right\} \tag{4.6}$$

is all $\mathbb{R}^{2n}$ as required.

Thus we are finished unless for all v the vectors v and Ev are linearly dependent. That is for all v, $\lambda_1(v)v + \lambda_2(v)Ev = 0$ for nonzero $\lambda_1(v)$ and $\lambda_2(v)$. Note $\lambda_2(v) \neq 0$, unless $v = 0$. Set $\tau(v) := \frac{\lambda_1(v)}{\lambda_2(v)}$, then the linear dependence becomes $\tau(v)v + Ev = 0$ for all v. It turns out that this does not happen unless $E = \tau I$ for some $\tau \in \mathbb{R}$. This is a baby case of Theorem 5.11 which comes later and is a subject unto itself.

To finish the proof pick a v which makes $\mathcal{R}^v$ equal all of $\mathbb{R}^{2n}$. Then $v^\intercal q(W)[H]v \geq 0$ implies that $Z \succeq 0$, by (4.5). ■

4.2. **Details of the Middle Matrix representation.** The following representation for symmetric nc polynomials $q(x)[h]$ that are of degree ℓ in x and homogeneous of degree two in h is exploited extensively in this subject:

$$q(x)[h] = \begin{bmatrix} V_0^\intercal & V_1^\intercal & \cdots & V_{\ell-1}^\intercal & V_\ell^\intercal \end{bmatrix} \begin{bmatrix} Z_{00} & Z_{01} & \cdots & Z_{0,\ell-1} & Z_{0\ell} \\ Z_{10} & Z_{11} & \cdots & Z_{1,\ell-1} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ Z_{\ell-1,0} & Z_{\ell-2,1} & \cdots & 0 & 0 \\ Z_{\ell 0} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{\ell-1} \\ V_\ell \end{bmatrix}, \tag{4.7}$$

where:

(1) The degree d of $q(x)[h]$ is $d = \ell + 2$.
(2) $V_j = V_j(x)[h]$, $j = 0, \ldots, \ell$, is a vector of height g^{j+1} whose entries are monomials of degree j in the x variables and degree one in the h variables. The h always appears to the left. In particular, $V(x)[h]$ is a vector of height $g\sigma(\ell)$, where as in (2.11),

$$\sigma(\ell) = 1 + g + \cdots + g^\ell.$$

(3) $Z_{ij} = Z_{ij}(x)$, is a matrix of size $g^{i+1} \times g^{j+1}$ whose entries are polynomials in the noncommuting variables $x_1, \ldots, x_g$ of degree $\leq \ell - (i + j)$. In particular, $Z_{i,\ell-i} = Z_{i,\ell-i}(x)$ is a constant matrix for $i = 0, \ldots, \ell$.

(4) $Z_{ij}^{\intercal} = Z_{ji}$.

Usually the entries of the vectors V_j are ordered lexicographically.

We note that the vector of monomials, $V(x)[h]$, might contain monomials that are not required in the representation of the nc quadratic q. Therefore, we can omit all monomials from the border vector that are not required. This gives us a *minimal length* border vector and prevents extraneous zeros from occurring in the middle matrix. The matrix Z in the representation (4.7) will be referred to as the *middle matrix* (*MM*) *of the polynomial* $q(x)[h]$ and the vectors $V_j = V_j(x)[h]$ with monomials as entries will be referred to as *border vectors* (*BV*). It is easy to check that a minimal length border vector contains distinct monomials and once the ordering of entries of V is set the MM for a given q is unique, see Lemma 4.7 below.

Example 4.6. Returning to Example 4.2, we have for the MM representation of q that

$$V_0 = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \qquad V_1 = \begin{bmatrix} h_2x_1 \\ h_1x_2 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} h_1x_2x_1 \\ h_2x_1x_2 \end{bmatrix}$$

and, for instance,

$$Z_{00} = \begin{bmatrix} 0 & 2x_2x_1 \\ 2x_1x_2 & 0 \end{bmatrix}, \qquad Z_{01} = \begin{bmatrix} 2x^2 & 0 \\ 0 & 2x_1 \end{bmatrix}, \qquad Z_{02} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Note that generically for a polynomial q in two variables the V_j have additional terms. For instance, usually V_1 is the column

$$\begin{bmatrix} h_1x_1 \\ h_1x_2 \\ h_2x_1 \\ h_2x_2 \end{bmatrix}.$$

Likewise generically V_2 has eight terms. As for the Z_{ij}, for instance Z_{01} is generically 2×4. □

Lemma 4.7. *The entries in the middle matrix $Z(x)$ are uniquely determined by the polynomial $q(x)[h]$ and the border vector $V(x)[h]$.*

Proof. Note every monomial in $q(x)[h]$ has the form

$$m_L h_i m_M h_j m_R.$$

Define

$$\mathcal{R}_j := \{h_j m : \ m_L h_i m_M h_j m \text{ is a term in } q(x)[h]\}.$$

Given the representation $V^{\intercal} Z V$ for q, let E_V denote the monomials in V. Then it is clear that each monomial in E_V must occur in some term of q, so it appears in

$\mathcal{R}_j$ for some j. Conversely, each term $h_j m$ in $\mathcal{R}_j$ corresponds to at least one term $m_L h_i m_M h_j m$ of q, so it must be in E_V. ■

Exercise 4.4. Prove the degree bound on the Z_{ij} in (3).
Hint: Read Example 4.8 first.

Example 4.8. If $p(x)$ is a symmetric polynomial of degree $d = 4$ in g noncommuting variables, then the middle matrix $Z(x)$ in the representation of the Hessian $p''(x)[h]$ is

$$Z(x) = \begin{bmatrix} Z_{00}(x) & Z_{01}(x) & Z_{02}(x) \\ Z_{10}(x) & Z_{11}(x) & 0 \\ Z_{20}(x) & 0 & 0 \end{bmatrix},$$

where the block entries $Z_{ij} = Z_{ij}(x)$ have the following structure:

Z_{00} is a $g \times g$ matrix with nc polynomial entries of degree ≤ 2,
Z_{01} is a $g \times g^2$ matrix with with nc polynomial entries of degree ≤ 1,
Z_{02} is a $g \times g^3$ matrix with constant entries.

All of these are proved merely by keeping track of the degrees. For example, the contribution of Z_{02} to p'' is $V_0^\intercal Z_{02} V_2$ whose degree is

$$\deg(V_0^\intercal) + \deg(Z_{02}) + \deg(V_2) = 1 + \deg(Z_{02}) + 3 \leq 4,$$

so $\deg(Z_{02}) = 0$. □

4.3. **The Middle Matrix of p''.** The middle matrix $Z(x)$ of the Hessian $p''(x)[h]$ of an nc symmetric polynomial $p(x)$ plays a key role. These middle matrices have a very rigid structure similar to that in Example 4.4. We illustrate with an example and then with exercises.

Example 4.9. As a warm up we first illustrate that $Z_{02}(X) = 0$ if and only if $Z_{11}(X) = 0$ for Example 4.2. To this end, observe that the contribution of the MM's extreme outer diagonal element Z_{02} to q is as follows

$$\frac{1}{2} V_0(x)[h]^\intercal Z_{02}(x) V_2(x)[h] = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}^\intercal \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} h_1 x_2 x_1 \\ h_2 x_1 x_2 \end{bmatrix} = 2h_1 h_2 x_1 x_2 + 2h_2 h_1 x_2 x_1.$$

Substitute $h_j \rightsquigarrow x_j$ and get $2x_1x_2x_1x_2 + 2x_2x_1x_2x_1$ which is $2p(x)$. That is,

$$p(x) = \frac{1}{2} V_0(x)[x]^\intercal Z_{02}(x) V_2(x)[x],$$

where $V_k(x)[h]$ is the homogeneous, in x, of degree k part of the border vector V. Obviously, $Z_{02} = 0$ implies $p = 0$. □

Exercise 4.5. Show $p(x)$ can also be obtained from Z_{11} in a similar fashion; i.e.,

$$p(x) = \frac{1}{2} V_1(x)[x]^\intercal Z_{11}(x) V_1(x)[x].$$

Exercise 4.6. Suppose p is homogeneous of degree d and its Hessian q has the border vector middle matrix representation $q(x)[h] = V(x)[h]^\intercal Z(x) V(x)[h]$.

(a) Show,

$$p = \frac{1}{2} V_0(x)[x]^\intercal Z_{0\ell} V_\ell(x)[x]$$

with $\ell = d - 2$. Prove this formula for $d = 2$, $d = 4$.

(b) Show that likewise,

$$p = \frac{1}{2} V_1(x)[x]^\intercal Z_{1,\ell-1}(x) V_{\ell-1}(x)[x]$$

Do not cheat and look this up in [DGHM09], but do compare with Exercise 4.4.

Exercise 4.7. Let Z denote the middle matrix for the Hessian of a nc polynomial p. Show, if $i + j = i' + j'$, then $Z_{ij} = 0$ if and only if $Z_{i'j'} = 0$.

4.4. **Positivity of the Middle Matrix and the demise of nc convexity.** This section focuses on positivity of the middle matrix of a Hessian.

Why should we focus on the case where $Z(x)$ is positive semidefinite? In [HMe98] it was shown that a polynomial $p \in \mathbb{R}<x>$ is matrix convex if and only if its Hessian $p''(x)[h]$ is positive (see Exercise 2.10). Moreover, if $Z(x)$ is positive, then the degree of $p(x)$ is at most two [HM04a]. The proof of this degree constraint given in Proposition 4.10 below using the more manageable bookkeeping scheme in this paper, begins with the following exercise.

Exercise 4.8. Show that

$$\begin{bmatrix} A & B \\ B^\intercal & 0 \end{bmatrix},$$

is positive semidefinite if and only if $A \succeq 0$ and $B = 0$. More refined versions of this fact appear as exercises later, see Exercise 4.11.

As we shall see we need not require our favorite functions be positive everywhere. It is possible to work locally, namely on an open set.

Proposition 4.10. *Let $p = p(x)$ be a symmetric polynomial of degree d in g nc variables and let $Z(x)$ denote the middle matrix (MM) in the BV-MM representation of the Hessian $p''(x)[h]$. If $Z(X) \succeq 0$ for all X in some nonempty nc basic open semialgebraic set $\mathcal{O}$, then d is at most two.*

Proof. Arguing by contradiction, suppose $d \geq 3$, then $p''(x)[h]$ is of degree $\ell = d-2 \geq 1$ in x and its middle matrix is of the form

$$Z = \begin{bmatrix} Z_{00} & \cdots & Z_{0\ell} \\ \vdots & \iddots & \vdots \\ Z_{\ell 0} & \cdots & 0 \end{bmatrix}.$$

Therefore, $Z(X)$ is of the form

$$Z(X) = \begin{bmatrix} A & B \\ B^\intercal & 0 \end{bmatrix},$$

where $A = A^\intercal$ and $B^\intercal = \begin{bmatrix} Z_{0\ell}(X) & 0 & \cdots & 0 \end{bmatrix}$. From Exercise 4.6, p_d, the homogeneous of degree d part of p, can be reconstructed from $Z_{0\ell}$. Now there is an $X \in \mathcal{O}$ such that $p_d(X)$ is nonzero, as otherwise p_d vanishes on a basic open semialgebraic set and is equal to 0. It follows that there is an $X \in \mathcal{O}$ such that $Z_{0\ell}(X)$ is not zero. Hence $B(X)$ is not zero which implies, by Exercise 4.8, the contradiction that $Z(X)$ is not positive semidefinite. ■

We have now reached our goal of showing that convex polynomials have degree ≤ 2.

Theorem 4.11. *If $p \in \mathbb{R}<x>$ is a symmetric polynomial which is convex on a nonempty nc basic open semialgebraic set $\mathcal{O}$, then it has degree at most two.*

There is a version of the theorem for free variables; i.e., with $p \in \mathbb{R}<x, x^\intercal>$.

Proof. The convexity of p on $\mathcal{O}$ is equivalent to $p''(X)[H]$ being positive semidefinite for all X in $\mathcal{O}$, see Exercise 2.10. By the QuadratischePositivstellensatz the middle matrix $Z(x)$ for $p''(x)[h]$ is positive on $\mathcal{O}$; that is, $Z(X) \succeq 0$ for all $X \in \mathcal{O}$. Proposition 4.10 implies degree p is at most 2. ■

4.5. **The signature of the middle matrix.** This section introduces the notion of the *signature* $\mu_\pm(Z(x))$ of $Z(x)$, the middle matrix of a Hessian, or more generally a polynomial $q(x)[h]$ which is homogeneous of degree two in h.

The *signature of a symmetric matrix* M is a triple of integers:

$$\big(\mu_-(M),\ \mu_0(M),\ \mu_+(M)\big),$$

where $\mu_-(M)$ is the number of negative eigenvalues (counted with multiplicity); $\mu_+(M)$ is the number of positive eigenvalues; and $\mu_0(M)$ is the dimension of the null space of M.

Lemma 4.12. *A nc symmetric polynomial $q(x)[h]$ homogeneous of degree two in h has middle matrix Z of the form in (4.7) and Z being positive semidefinite implies Z is of the form*

$$\begin{bmatrix} Z_{00} & Z_{01} & \cdots & Z_{0,\lfloor\frac{\ell}{2}\rfloor} & 0 & \cdots \\ Z_{10} & Z_{11} & \cdots & Z_{1,\lfloor\frac{\ell}{2}\rfloor} & 0 & \cdots \\ \vdots & \vdots & \iddots & \vdots & \vdots & \iddots \\ Z_{\lfloor\frac{\ell}{2}\rfloor,0} & Z_{\lfloor\frac{\ell}{2}\rfloor,1} & \cdots & Z_{\lfloor\frac{\ell}{2}\rfloor,\lfloor\frac{\ell}{2}\rfloor} & 0 & \iddots \\ 0 & 0 & \cdots & 0 & 0 & \iddots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \end{bmatrix}.$$

This lemma follows immediately from a much more general lemma.

Lemma 4.13. *If*

$$E = \begin{bmatrix} A & B & C \\ B^\intercal & D & 0 \\ C^\intercal & 0 & 0 \end{bmatrix}$$

is a real symmetric matrix, then

$$\mu_\pm(E) \geq \mu_\pm(D) + \operatorname{rank} C.$$

This can be proved using the $LDL^\intercal$ decomposition which we shall not do here but suggest the reader apply the $LDL^\intercal$ hammer to the following simpler exercise.

4.6. **Exercises.**

Exercise 4.9. True of False? If p_d is homogeneous of degree d and we let Z denote the middle matrix of the Hessian $p''(x)[h]$, then for each $k \leq d-2$ the degree of $Z_{i,k-i}$ is independent of i.

Exercise 4.10. Redo Exercise 2.10 for convexity on a nc basic open semialgebraic set.

Exercise 4.11. If $F = \begin{bmatrix} A & C \\ C^\intercal & 0 \end{bmatrix}$, then $\mu_\pm(F) \geq \operatorname{rank} C$. (If you cannot do the general case, assume A is invertible.)

Exercise 4.12. If $p(x)$ is a symmetric polynomial of degree $d = 2$ in g noncommuting variables, then the middle matrix $Z(x)$ in the representation of the Hessian $p''(x)[h]$ is equal to the $g \times g$ constant matrix Z_{00}. Substituting $X \in (\mathbb{S}^{n\times n})^g$ for x gives

$$\mu_\pm(Z(X)) \geq \mu_\pm(Z_{00})$$

Exercise 4.13. Let $f \in \mathbb{R}<x>_{2d}$ and let $V \in <x>_d^{\sigma(d)}$ be a vector consisting of all words in x of degree $\leq d$. Prove:

(a) there is a matrix $G \in \mathbb{R}^{\sigma(d)\times\sigma(d)}$ with $f = V^\intercal GV$ (any such G is called a *Gram matrix* for f);
(b) if f is symmetric, then there is a symmetric Gram matrix for f.

Exercise 4.14. Find all Gram matrices for

(a) $f = x_1^4 + x_1^2x_2 - x_1x_2^2 + x_2x_1^2 - x_2^2x_1 + x_1^2 - x_2^2 + 2x_1 - x_2 + 4$;
(b) $f = c(x_1, x_2)^2$.

Exercise 4.15. Show: if $f \in \mathbb{R}<x>$ is homogeneous of degree $2d$, then it has a unique Gram matrix $G \in \mathbb{R}^{\sigma(d)\times\sigma(d)}$.

5. Der QuadratischePositivstellensatz

In this section we present the proof of the QuadratischePositivstellensatz, (Theorem 4.5) which is based on the fact that local linear dependence of nc rationals (or nc polynomials) implies global linear dependence, a fact itself based on the forthcoming CHSY Lemma [CHSY03].

5.1. The Camino, Helton, Skelton, Ye (CHSY) Lemma.

At the root of the CHSY Lemma [CHSY03] is the following linear algebra fact:

Lemma 5.1. *Fix $n > d$. If $\{z_1, \dots, z_d\}$ is a linearly independent set in $\mathbb{R}^n$, then the codimension of*

$$\left\{ \begin{bmatrix} Hz_1 \\ Hz_2 \\ \vdots \\ Hz_d \end{bmatrix} : H \in \mathbb{S}^{n\times n} \right\} \subseteq \mathbb{R}^{nd}$$

is $\frac{d(d-1)}{2}$. Especially important is, this codimension is independent of n.

The following exercise is a variant of the Lemma 5.1 which is easier to prove. Thus we suggest attempting it before launching into the proof of the lemma.

Exercise 5.1. Prove if $\{z_1, \dots, z_d\}$ is a linearly independent set in $\mathbb{R}^n$, then

$$\left\{ \begin{bmatrix} Hz_1 \\ Hz_2 \\ \vdots \\ Hz_d \end{bmatrix} : H \in \mathbb{R}^{n\times n} \right\} = \mathbb{R}^{nd}$$

Hint: it goes like the proof of (4.6).

Proof of Lemma 5.1. Consider the mapping $\Phi : \mathbb{S}^{n\times n} \to \mathbb{R}^{nd}$ given by

$$H \mapsto \begin{bmatrix} Hz_1 \\ Hz_2 \\ \vdots \\ Hz_d \end{bmatrix}.$$

Since the span of $\{z_1, \ldots, z_d\}$ has dimension d, it follows that the kernel of Φ has dimension $\kappa = \frac{(n-d)(n-d+1)}{2}$ and hence the range has dimension $\frac{n(n+1)}{2} - \kappa$. To see this assertion, it suffices to assume that the span of $\{z_1, \ldots, z_d\}$ is the span of $\{e_1, \ldots, e_d\} \subseteq \mathbb{R}^n$ (the first d standard basis vectors in $\mathbb{R}^n$). In this case (since H is symmetric) $Hz_j = 0$ for all j if and only if

$$H = \begin{bmatrix} 0 & 0 \\ 0 & H' \end{bmatrix},$$

where H' is a symmetric matrix of size $(n-d)\times(n-d)$; in other words, this is the kernel of Φ.

From this we conclude that the codimension of the range of Φ is

$$nd - \Big(\frac{n(n+1)}{2} - \kappa\Big) = \frac{d(d-1)}{2}.$$ ∎

Next is a straightforward extension of Lemma 5.1.

Lemma 5.2 ([CHSY03])**.** *If $n > d$ and $\{z_1, \ldots, z_d\}$ is a linearly independent subset of $\mathbb{R}^n$, then the codimension of*

$$\Big\{ \oplus_{j=1}^{g} \begin{bmatrix} H_jz_1 \\ H_jz_2 \\ \vdots \\ H_jz_d \end{bmatrix} : H = (H_1, \ldots, H_g) \in (\mathbb{S}^{n\times n})^g \Big\} \subseteq \mathbb{R}^{gnd}$$

is $g\frac{d(d-1)}{2}$ and is independent of n.

Proof. See Exercise 5.2. ∎

Finally, the form in which we generally apply the lemma is the following.

Lemma 5.3. *Let $v \in \mathbb{R}^n$, $X \in (\mathbb{S}^{n\times n})^g$. If the set $\{m(X)v \colon m \in <x>_d\}$ is linearly independent, then the codimension of*

$$\{V(X)[H]v \colon H \in (\mathbb{S}^{n\times n})^g\}$$

is $g^{\frac{\kappa(\kappa-1)}{2}}$, *where* $\kappa = \sigma(d) = \sum_{j=0}^{d} g^j$ *and where*

$$V = \bigoplus_{i=1}^{g} \bigoplus_{m \in <x>_d} H_i m$$

is the border vector associated to $<x>_d$. *Again, this codimension is independent of* n *as it only depends upon the number of variables* g *and the degree* d *of the polynomial.*

Proof. Let $z_m = m(X)v$ for $m \in <x>_d$. There are at most κ of these. Now apply the previous lemma. ∎

5.2. **Linear Dependence of Symbolic Functions.** The main result in this section, Theorem 5.11 says roughly that if each evaluation of a set $G_1, \dots G_\ell$ of rational functions produces linearly dependent matrices, then they satisfy a universal linear dependence relation. We begin with a clean and easily stated consequence of Theorem 5.11.

In Section 2.1.2 we defined nc basic open semialgebraic sets. Here we define a nc basic semialgebraic set. Given matrix-valued symmetric nc polynomials ρ and $\tilde{\rho}$, let

$$\mathcal{D}_+^{\rho}(n) = \{X \in (\mathbb{S}^{n\times n})^g \colon \rho(X) \succ 0\},$$

and

$$\mathcal{D}^{\tilde{\rho}}(n) = \{X \in (\mathbb{S}^{n\times n})^g \colon \tilde{\rho}(X) \succeq 0\}.$$

Then $\mathcal{D}$ is a *nc basic semialgebraic set* if there exists $\rho_1, \dots, \rho_k$ and $\tilde{\rho}_1, \dots, \tilde{\rho}_{\tilde{k}}$ such that $\mathcal{D} = (\mathcal{D}(n))_{n\in\mathbb{N}}$ where

$$\mathcal{D}(n) = \big(\bigcap_j \mathcal{D}_0^{\rho_j}(n)\big) \cap \big(\bigcap_j \mathcal{D}^{\tilde{\rho}_{\tilde{j}}}(n)\big).$$

Theorem 5.4. *Suppose* $G_1, \dots, G_\ell$ *are rational expressions and* $\mathcal{D}$ *is a nonempty nc basic semialgebraic set on which each* G_j *is defined. If, for each* $X \in \mathcal{D}(n)$ *and vector* $v \in \mathbb{R}^n$ *the set* $\{G_j(X)v \colon j = 1, 2, \dots, \ell\}$ *is linearly dependent, then the set* $\{G_j(X) \colon j = 1, 2, \dots, \ell\}$ *is linearly dependent on* $\mathcal{D}$, *i.e. there exists a nonzero* $\lambda \in \mathbb{R}^\ell$ *such that*

$$0 = \sum_{j=1}^{\ell} \lambda_j G_j(X) \qquad \text{for all } X \in \mathcal{D}.$$

If, in addition, $\mathcal{D}$ *contains an* ϵ*-neighborhood of* 0 *for some* $\epsilon > 0$, *then there exists a nonzero* $\lambda \in \mathbb{R}^\ell$ *such that*

$$0 = \sum_{j=1}^{\ell} \lambda_j G_j.$$

Corollary 5.5. *Suppose $G_1, \ldots, G_\ell$ are rational expressions. If, for each $n \in \mathbb{N}$, $X \in (\mathbb{S}^{n\times n})^g$, and vector $v \in \mathbb{R}^n$ the set $\{G_j(X)v \colon j = 1, 2, \ldots, \ell\}$ is linearly dependent, then the set $\{G_j \colon j = 1, 2, \ldots, \ell\}$ is linearly dependent, i.e., there exists a nonzero $\lambda \in \mathbb{R}^\ell$ such that*

$$\sum_{j=1}^{\ell} \lambda_j G_j = 0.$$

Corollary 5.6. *Suppose $G_1, \ldots, G_\ell$ are rational expressions. If, for each $n \in \mathbb{N}$ and $X \in (\mathbb{S}^{n\times n})^g$, the set $\{G_j(X) \colon j = 1, 2, \ldots, \ell\}$ is linearly dependent, then the set $\{G_j \colon j = 1, 2, \ldots, \ell\}$ is linearly dependent.*

The point is that the λ_j are independent of X. Before proving Theorem 5.4 we shall introduce some terminology pursuant to our more general result.

5.2.1. *Direct Sums.* We present some definitions about direct sum and sets which respect direct sums, since they are important tools.

Definition 5.7. Our definition of the *direct sum* is the usual one. Given pairs (X_1, v_1) and (X_2, v_2) where X_j are $n_j \times n_j$ matrices and $v_j \in \mathbb{R}^{n_j}$,

$$(X_1, v_1) \oplus (X_2, v_2) = (X_1 \oplus X_2, v_1 \oplus v_2)$$

where

$$X_1 \oplus X_2 := \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \qquad v_1 \oplus v_2 := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

We extend this definition to μ terms, $(X_1, v_1), \ldots, (X_\mu, v_\mu)$ in the expected way.

In the definition below, we consider a set $\mathcal{B}$ which is the sequence

$$\mathcal{B} := (\mathcal{B}(n))_n,$$

where each $\mathcal{B}(n)$ is a set whose members are pairs (X, v) where X is in $(\mathbb{S}^{n\times n})^g$ and $v \in \mathbb{R}^n$.

Definition 5.8. The *set $\mathcal{B}$ is said to respect direct sums* if (X^j, v^j) with $X^j \in (\mathbb{S}^{n_j\times n_j})^g$ and $v^j \in \mathbb{R}^{n_j}$ for $j = 1, \ldots, \mu$ is contained in the set $\mathcal{B}$ $(\mathcal{B}(n_j))$ implies that the direct sum

$$(X^1 \oplus \ldots \oplus X^\mu, v^1 \oplus \ldots \oplus v^\mu) = (\oplus_{j=1}^\mu X^j, \oplus_{j=1}^\mu v^j)$$

is also contained in $\mathcal{B}$ $(\mathcal{B}(\sum n_j))$.

Definition 5.9. By a *natural map* G on $\mathcal{B}$, we mean a sequence of functions $G(n) : \mathcal{B}(n) \to \mathbb{R}^n$, which *respects direct sums* in the sense that, if $(X^j, v^j) \in \mathcal{B}(n_j)$ for $j = 1, 2, \dots, \mu$, then

$$G(\sum_1^{\mu} n_j)(\oplus X^j, \oplus v^j) = \oplus_1^{\mu} G(n_j)(X^j, v^j).$$

Typically we omit the argument n, writing $G(X)$ instead of $G(n)(X)$.

Examples of sets which respect direct sums and of natural maps are provided by the following example.

Example 5.10. Let ρ be a rational expression.

(1) The set $\mathcal{B}^\rho = \{(X, v) : X \in \mathcal{D}^\rho \cap (\mathbb{S}^{n\times n})^g,\ v \in \mathbb{R}^n,\ n \in \mathbb{N}\}$ respects direct sums.
(2) If G is a matrix-valued nc rational expression whose domain contains $\mathcal{D}^\rho$, then G determines a natural map on $\mathcal{B}(\rho)$ by $G(n)(X, v) = G(X)v$. In particular, every nc polynomial determines a natural map on every nc basic semialgebraic set $\mathcal{B}$.

□

5.2.2. *Main Result on Linear Dependence.*

Theorem 5.11. *Suppose $\mathcal{B}$ is a set which respects direct sums and $G_1, \dots, G_\ell$ are natural maps on $\mathcal{B}$. If for each $(X, v) \in \mathcal{B}$ the set $\{G_1(X, v), \dots, G_\ell(X, v)\}$ is linearly dependent, then there exists a nonzero $\lambda \in \mathbb{R}^\ell$ so that*

$$0 = \sum_{j=1}^{\ell} \lambda_j G_j(X, v)$$

for every $(X, v) \in \mathcal{B}$. We emphasize that λ is independent of (X, v).

Before proving 5.11, we use it to prove an important earlier theorem.

Proof of Theorem 5.4. Let $\mathcal{B}$ be given by

$$\mathcal{B}(n) = \{(X, v) \colon X \in \mathcal{D}^\rho \cap (\mathbb{S}^{n\times n})^g \text{ and } v \in \mathbb{R}^n\}.$$

Let G_j denote the natural maps, $G_j(X, v) = G_j(X)v$. Then $\mathcal{B}$ and $G_1, \dots, G_\ell$ satisfy the hypothesis of Theorem 5.11 and so the first conclusion of Theorem 5.4 follows.

The last conclusion follows because an nc rational function r vanishing on an nc basic open semialgebraic set is 0 on all $\operatorname{dom}(r)$ and hence is zero, cf. Exercise 2.17. ■

5.2.3. *Proof of Theorem* 5.11. We start with a finitary version of Theorem 5.11:

Lemma 5.12. *Let $\mathcal{B}$ and G_i be as in Theorem 5.11. If $\mathcal{S}$ is a finite subset of $\mathcal{B}$, then there exists a nonzero $\lambda(\mathcal{S}) \in \mathbb{R}^\ell$ such that*

$$\sum_{j=1}^{\ell} \lambda(\mathcal{S})_j G_j(X)v = 0,$$

for every $(X, v) \in \mathcal{S}$.

Proof. The proof relies on taking direct sums of matrices. Write the set $\mathcal{S}$ as

$$\mathcal{S} = \big\{(X^1, v^1), \dots, (X^\mu, v^\mu)\big\},$$

where each $(X^i, v^i) \in \mathcal{B}$. Since $\mathcal{B}$ respects direct sums,

$$(X, v) = (\oplus_{\nu=1}^{\mu} X^\nu, \oplus_{\nu=1}^{\mu} v^\nu) \in \mathcal{B}.$$

Hence, there exists a nonzero $\lambda(\mathcal{S}) \in \mathbb{R}^\ell$ such that

$$0 = \sum_{j=1}^{\ell} \lambda(\mathcal{S})_j G_j(X, v).$$

Since each G_j respects direct sums, the desired conclusion follows. ■

Proof of Theorem 5.11. The proof is essentially a compactness argument, based on Lemma 5.12. Let $\mathbb{B}$ denote the unit sphere in $\mathbb{R}^\ell$.

To $(X, v) \in \mathcal{B}$ associate the set

$$\Omega_{(X,v)} = \big\{\lambda \in \mathbb{B} \colon \lambda \cdot G(X)v = \sum_j \lambda_j G_j(X, v) = 0\big\}.$$

Since $(X, v) \in \mathcal{B}$, the hypothesis on $\mathcal{B}$ says $\Omega_{(X,v)}$ is nonempty. It is evident that $\Omega_{(X,v)}$ is a closed subset of $\mathbb{B}$ and is thus compact.

Let $\boldsymbol{\Omega} := \{\Omega_{(X,v)} \colon (X, v) \in \mathcal{B}\}$. Any finite sub-collection from $\boldsymbol{\Omega}$ has the form $\{\Omega_{(X,v)} \colon (X, v) \in \mathcal{S}\}$ for some finite subset $\mathcal{S}$ of $\mathcal{B}$, and so by Lemma 5.12 has a nonempty intersection. In other words, $\boldsymbol{\Omega}$ has the finite intersection property. The compactness of $\mathbb{B}$ implies that there is a $\lambda \in \mathbb{B}$ which is in every $\Omega_{(X,v)}$. This is the desired conclusion of the theorem. ■

5.3. **Proof of the QuadratischePositivstellensatz.** We are now ready to give the proof of Theorem 4.5. Accordingly, let $\mathcal{O}$ be a given basic open semialgebraic set. Suppose

$$q(x)[h] = V(x)[h]^\intercal\, Z(x)\, V(x)[h], \tag{5.1}$$

where V is the border vector and Z is the middle matrix; cf. (4.7). Clearly, if Z is matrix-positive on $\mathcal{O}$, then $q(X)[H]$ is positive semidefinite for each n, each $X \in \mathcal{O}(n)$ and $H \in (\mathbb{S}^{n\times n})^g$.

The converse is less trivial and requires the CHSY Lemma plus our main result on linear dependence of nc rational functions. Let ℓ denote the degree of $q(x)[h]$ in the variable x. In particular, the border vector in the representation of $q(x)[h]$ itself has degree ℓ in x. Recall σ_ℓ from Exercise 2.12.

Suppose for some s and g-tuple of symmetric matrices $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_g) \in \mathcal{O}(s)$, the matrix $Z(\tilde{X})$ is not positive semidefinite. By Lemma 5.3 and Theorem 5.4, there is an t, a $Y \in \mathcal{O}(t)$, and a vector η so that $\{m(Y)\eta\colon m \in {<}x{>}_\ell\}$ is linearly independent. Let $X = \tilde{X} \oplus Y$ and $\gamma = 0 \oplus \eta \in \mathbb{R}^{s+t}$. Then $Z(X)$ is not positive semidefinite and $\{m(X)\gamma\colon m \in {<}x{>}_\ell\}$ is linearly independent.

Let $N = g^{\frac{\kappa(\kappa-1)}{2}} + 1$, where κ is given in Lemma 5.3 and let $n = (s+t)N$. Consider $W = X \otimes I_N = (X_1 \otimes I_N, \dots, X_g \otimes I_N)$ and vector $\omega = \gamma \otimes e$, for any nonzero vector $e \in \mathbb{R}^{N+1}$. The set $\{m(W)\omega\colon m \in {<}x{>}_\ell\}$ is linearly independent and thus by Lemma 5.3, the codimension of $\mathcal{M} = \{V(W)[H]\omega\colon H \in (\mathbb{S}^{n\times n})^g\}$ is at most $N-1$. On the other hand, because $Z(X)$ has a negative eigenvalue, the matrix $Z(W)$ has an eigenspace $\mathcal{E}$, corresponding to a negative eigenvalue, of dimension at least N. It follows that $\mathcal{E} \cap \mathcal{M}$ is nonempty; i.e., there is an $H \in (\mathbb{S}^{n\times n})^g$ such that $V(W)[H]\omega \in \mathcal{E}$. In particular, this together with (5.1) implies

$$\langle q(W)[H]\omega, \omega\rangle = \langle Z(W)V(W)[H]\omega, V(W)\omega\rangle < 0$$

and thus, $q(W)[H]$ is not positive semidefinite. ∎

5.4. **Exercises.**

Exercise 5.2. Prove Lemma 5.2.

Exercise 5.3. Let $A \in \mathbb{R}^{n\times n}$ be given. Show, if the rank of A is r, then the matrices $A, A^2, \dots, A^{r+1}$ are linearly dependent.

In the next exercise employ the Fock space (see Section 2.7) to prove a strengthening of Corollary 5.5 for nc polynomials.

Exercise 5.4. Suppose $p_1, \ldots, p_\ell \in \mathbb{R}\langle x\rangle_k$ are nc polynomials. Show, if the set of vectors

$$\{p_1(X)v, \ldots, p_\ell(X)v\} \tag{5.2}$$

is linearly dependent for every $(X, v) \in (\mathbb{S}^{\sigma\times\sigma})^g \times \mathbb{R}^\sigma$, where $\sigma = \sigma(k) = \dim \mathbb{R}\langle x\rangle_k$, then $\{p_1, \ldots, p_\ell\}$ is linearly dependent.

Exercise 5.5. Redo Exercise 5.4 under the assumption that the vectors (5.2) are linearly dependent for all $(X, v) \in O \times \mathbb{R}^\sigma$, where $O \subseteq (\mathbb{S}^{\sigma\times\sigma})^g$ is a nonempty open set.

For a more algebraic view of the linear dependence of nc polynomials we refer to [BK+].

Exercise 5.6. Prove that $f \in \mathbb{R}\langle x\rangle$ is a sum of squares if and only if it has a positive semidefinite Gram matrix. Are then all of f's Gram matrices positive semidefinite?

6. NC varieties with positive curvature have degree two

This section looks at noncommutative varieties and their geometric properties. We see a very strong rigidity when they have positive curvature which generalizes what we have already seen about convex polynomials (their graph is a positively curved variety) having degree two.

In the classical setting of a surface defined by the zero set

$$\nu(p) = \{x \in \mathbb{R}^g \colon p(x) = 0\}$$

of a polynomial $p = p(x_1, \ldots, x_g)$ in g commuting variables, the second fundamental form at a smooth point x_0 of $\nu(p)$ is the quadratic form,

$$h \mapsto -\langle (\operatorname{Hess} p)(x_0)h, h\rangle, \tag{6.1}$$

where $\operatorname{Hess} p$ is the Hessian of p, and $h \in \mathbb{R}^g$ is in the tangent space to the surface $\nu(p)$ at x_0; i.e., $\nabla p(x_0) \cdot h = 0$.[3]

We shall show that in the noncommutative setting that the zero set $\mathcal{V}(p)$ of a noncommutative polynomial p (subject to appropriate irreducibility constraints)

[3]The choice of the minus sign in (6.1) is somewhat arbitrary. Classically the sign of the second fundamental form is associated with the choice of a smoothly varying vector that is normal to $\nu(p)$. The zero set $\nu(p)$ has positive curvature at x_0 if the second fundamental form is either positive semidefinite or negative semidefinite at x_0. For example, if we define $\nu(p)$ using a concave function p, then the second fundamental form is negative semidefinite, while for the same set $\nu(-p)$ the second fundamental form is positive semidefinite.

having positive curvature (even in a small neighborhood) implies that p is convex - and thus, p has degree at most two - and $\mathcal{V}(p)$ has positive curvature everywhere; see Theorem 6.4 for the precise statements.

In fact there is a natural notion of the signature $C_\pm(\mathcal{V}(p))$ of a variety $\mathcal{V}(p)$ and the bound

$$\deg(p) \le 2C_\pm(\mathcal{V}(p)) + 2$$

on the degree of p in terms of the signature $C_\pm(\mathcal{V}(p))$ was obtained in [DHM07b]. The convention that $C_+(\mathcal{V}(p)) = 0$ corresponds to positive curvature, since in our examples, defining functions p are typically concave or quasiconcave. One could consider characterizing p for which $C_\pm(\mathcal{V}(p))$ satisfies less restrictive hypothesis than equal zero and this has been done to some extent in [DGHM09]; however, this higher level of generality is beyond our focus here. Since our goal is to present the basic ideas, we stick to positive curvature.

6.1. **NC varieties and their curvature.**

We next define a number of basic geometric objects associated to the nc variety determined by an nc polynomial p.

6.1.1. *Varieties, tangent planes, and the second fundamental form.*

The *variety* (zero set) of a $p \in \mathbb{R}\langle x\rangle$ is

$$\mathcal{V}(p) := \bigcup_{n\ge 1} \mathcal{V}_n(p),$$

where

$$\mathcal{V}_n(p) := \left\{(X,v) \in (\mathbb{S}^{n\times n})^g \times \mathbb{R}^n \colon p(X)v = 0\right\}.$$

The *clamped tangent plane* to $\mathcal{V}(p)$ at $(X,v) \in \mathcal{V}_n(p)$ is

$$\mathcal{T}_p(X,v) := \{H \in (\mathbb{S}^{n\times n})^g \colon p'(X)[H]v = 0\}.$$

The *clamped second fundamental form* for $\mathcal{V}(p)$ at $(X,v) \in \mathcal{V}_n(p)$ is the quadratic form

$$\mathcal{T}_p(X,v) \to \mathbb{R}, \quad H \mapsto -\langle p''(X)[H]v, v\rangle.$$

Note that

$$\{X \in (\mathbb{S}^{n\times n})^g \colon (X,v) \in \mathcal{V}(p) \text{ for some } v \ne 0\} = \{X \in (\mathbb{S}^{n\times n})^g \colon \det(p(X)) = 0\}$$

is a variety in $(\mathbb{S}^{n\times n})^g$ and typically has a *true* (commutative) tangent plane at many points X, which of course has codimension one, whereas the clamped tangent plane at a typical point $(X,v) \in \mathcal{V}_n(p)$ has codimension on the order of n and is contained inside the true tangent plane.

6.1.2. *Full rank points.* The point $(X, v) \in \mathcal{V}(p)$ is a *full rank point* of p if the mapping

$$(\mathbb{S}^{n\times n})^g \to \mathbb{R}^n, \quad H \mapsto p'(X)[H]v$$

is onto. The full rank condition is a nonsingularity condition which amounts to a smoothness hypothesis. Such conditions play a major role in real algebraic geometry, see [BCR98, §3.3].

As an example, consider the classical real algebraic geometry case of $n = 1$ (and thus $X \in \mathbb{R}^g$) with the commutative polynomial $\check{p}$ (which can be taken to be the *commutative collapse* of the polynomial p). In this case, a full rank point $(X, 1) \in \mathbb{R}^g \times \mathbb{R}$ is a point at which the gradient of $\check{p}$ does not vanish. Thus, X is a nonsingular point for the zero variety of $\check{p}$.

Some perspective for $n > 1$ is obtained by counting dimensions. If $(X, v) \in (\mathbb{S}^{n\times n})^g \times \mathbb{R}^n$, then $H \mapsto p'(X)[H]v$ is a linear map from the $g(n^2+n)/2$ dimensional space $(\mathbb{S}^{n\times n})^g$ into the n dimensional space $\mathbb{R}^n$. Therefore, the codimension of the kernel of this map is no bigger than n. This codimension is n if and only if (X, v) is a full rank point and in this case the clamped tangent plane has codimension n.

6.1.3. *Positive curvature.* As noted earlier, a notion of positive (really nonnegative) curvature can be defined in terms of the clamped second fundamental form.

The variety $\mathcal{V}(p)$ has *positive curvature* at $(X, v) \in \mathcal{V}(p)$ if the clamped second fundamental form is nonnegative at (X, v); i.e., if

$$-\langle p''(X)[H]v, v\rangle \geq 0 \quad \text{for every} \quad H \in \mathcal{T}_p(X, v)\,.$$

6.1.4. *Irreducibility: The minimum degree defining polynomial condition.* While there is no tradition of what is an effective notion of irreducibility for nc polynomials, there is a notion of minimal degree nc polynomial which is appropriate for the present context. In the commutative case the polynomial $\check{p}$ on $\mathbb{R}^g$ is a *minimal degree defining polynomial* for $\nu(\check{p})$ if there does not exist a polynomial q of lower degree such that $\nu(\check{p}) = \nu(q)$. This is a key feature of irreducible polynomials.

Definition 6.1. A symmetric nc polynomial p is a *minimum degree defining polynomial* for a nonempty set $\mathcal{S} \subseteq \mathcal{V}(p)$ if whenever $q \neq 0$ is another (*not necessarily symmetric*) nc polynomial such that $q(X)v = 0$ for each $(X, v) \in \mathcal{S}$, then

$$\deg(q) \geq \deg(p).$$

Note this contrasts with [DHM07a], where minimal degree meant a slightly weaker inequality holds.

The reader who is so inclined can simply choose $\mathcal{S} = \mathcal{V}(p)$ or $\mathcal{S}$ equal to the full rank points of $\mathcal{V}(p)$.

Now we give an example to illustrate these ideas.

6.2. **A very simple example.** In the following example, the null space

$$\mathcal{T} = \mathcal{T}_p(X, v) = \{H \in (\mathbb{S}^{n\times n})^g \colon p'(X)[H]v = 0\}$$

is computed for certain choices of p, X, and v. Recall that if $p(X)v = 0$, then the subspace $\mathcal{T}$ is the *clamped tangent plane* introduced in Subsection 6.1.1.

Example 6.2. Let $X \in \mathbb{S}^{n\times n}$, $v \in \mathbb{R}^n$, $v \neq 0$, let $p(x) = x^k$ for some integer $k \geq 1$. Suppose that $(X, v) \in \mathcal{V}(p)$, that is, $X^k v = 0$. Then, since

$$X^k v = 0 \iff Xv = 0 \quad \text{when } X \in \mathbb{S}^{n\times n},$$

it follows that p is a minimum degree defining polynomial for $\mathcal{V}(p)$ if and only if $k = 1$.

It is readily checked that

$$(X, v) \in \mathcal{V}(p) \Longrightarrow p'(X)[H]v = X^{k-1}Hv,$$

and hence that X is a full rank point for p if and only if X is invertible.

Now suppose $k \geq 2$. Then,

$$\langle p''(X)[H]v, v\rangle = 2\langle HX^{k-2}Hv, v\rangle.$$

Therefore, if $k > 2$

$$(X, v) \in \mathcal{V}(p) \quad \text{and} \quad p'(X)[H]v = 0 \quad \Longrightarrow \quad XHv = 0, \text{ and so}$$
$$\langle p''(X)[H]v, v\rangle = 0.$$

To count the dimension of $\mathcal{T}$ we can suppose without loss of generality that

$$X = \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^\intercal,$$

where $Y \in \mathbb{S}^{(n-1)\times(n-1)}$ is invertible. Then, for the simple case under consideration,

$$\mathcal{T} = \{H \in \mathbb{S}^{n\times n} \colon h_{21}, \ldots, h_{n1} = 0\},$$

where h_{ij} denotes the ij entry of H. Thus,

$$\dim \mathcal{T} = \frac{n^2 + n}{2} - (n - 1), \qquad \text{i.e.,} \qquad \operatorname{codim} \mathcal{T} = n - 1.$$

□

Remark 6.3. We remark that

$$X^k v = 0 \quad \text{and} \quad \langle p''(X)[H]v, v\rangle = 0 \Longrightarrow p'(X)[H]v = 0 \quad \text{if } k = 2t \geq 4,$$

as follows easily from the formula

$$\langle p''(X)[H]v, v\rangle = 2\langle X^{t-1}Hv, X^{t-1}Hv\rangle.$$

Exercise 6.1. Let $A \in \mathbb{S}^{n\times n}$ and let $\mathcal{U}$ be a maximal strictly negative subspace of $\mathbb{R}^n$ with respect to the quadratic form $\langle Au, u\rangle$. Prove: there exists a complementary subspace $\mathcal{V}$ of $\mathcal{U}$ in $\mathbb{R}^n$ such that $\langle Av, v\rangle \geq 0$ for every $v \in \mathcal{V}$.

6.3. Main Result: Positive curvature and the degree of p.

Theorem 6.4. *Let p be a symmetric nc polynomial in g symmetric variables, let $\mathcal{O}$ be a nc basic open semialgebraic set and let $\mathcal{S}$ denote the full rank points of p in $\mathcal{V}(p) \cap \mathcal{O}$. If*

(1) *$\mathcal{S}$ is nonempty;*
(2) *$\mathcal{V}(p)$ has positive curvature at each point of $\mathcal{S}$; and*
(3) *p is a minimum degree defining polynomial for $\mathcal{S}$,*

then $\deg(p)$ *is at most two and p is concave.*

6.4. Ideas and proofs.

Our aim is to give the idea behind the proof of Theorem 6.4 under much stronger hypotheses. We saw earlier the positivity of a quadratic on a nc basic open set $\mathcal{O}$ imparts positivity to its MM there. The following shows this happens for thin sets (nc varieties) too. Thus, the following theorem generalizes the QuadratischePositivstellensatz, Theorem 4.5.

Theorem 6.5. *Let $p, \mathcal{O}, \mathcal{S}$ be as in Theorem 6.4. Let $q(x)[h]$ be a polynomial which is quadratic in h having MM representation $q = V^\intercal Z V$ for which* $\deg(V) \leq \deg(p)$. *If*

$$v^\intercal q(X)[H]v \geq 0 \quad \textit{for all} \quad (X, v) \in \mathcal{S} \textit{ and all } H, \tag{6.2}$$

then $Z(X)$ is positive semidefinite for all X with $(X, v) \in \mathcal{S}$.

Proof. The proof of this theorem follows the proof of the QuadratischePositivstellensatz, modified to take into account the set $\mathcal{S}$.

Suppose for each $(X, v) \in \mathcal{S}$ there is a linear combination $G_{(X,v)}(x)$ of the words $\{m(x)\colon \deg(m) < \deg(p)\}$ with $G_{(X,v)}(X)v = 0$ for all $(X, v) \in \mathcal{S}$. Then by Theorem 5.11 (note that $\mathcal{S}$ is closed under direct sums), there is a linear combination $G \in \mathbb{R}\langle x\rangle_{\deg(p)-1}$ with $G(X)v = 0$. However, this is absurd by the minimality of p. Hence there is an $(Y, v) \in \mathcal{S}$ such that $\{m(Y)v\colon \deg(m) < \deg(p)\}$ is linearly independent.

Assume for some g-tuple of symmetric matrices $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_g)$, there is a vector $\tilde{v}$ such that $(\tilde{X}, \tilde{v}) \in \mathcal{S}$, and the matrix $Z(\tilde{X})$ is not positive semidefinite. Let $X = \tilde{X} \oplus Y$ and $\gamma = \tilde{v} \oplus v$. Then $(X, \gamma) \in \mathcal{S}(\ell)$ for some ℓ; the matrix $Z(X)$ is not positive semidefinite; and $\{m(X)\gamma \colon \deg(m) < \deg(p)\}$ is linearly independent.

Let $N = g^{\frac{\kappa(\kappa-1)}{2}} + 1$, where κ is given in Lemma 5.3 and let $n = \ell N$. Consider $W = X \otimes I_N = (X_1 \otimes I_N, \dots, X_g \otimes I_N)$ and vector $\omega = \gamma \otimes e$, where $e \in \mathbb{R}^N$ is the vector with each entry equal to 1. Then, $(W, \omega) \in \mathcal{S}(n)$, and the set $\{m(W)\omega \colon m \in <x>_\ell\}$ is linearly independent and thus by Lemma 5.3, the codimension of $\mathcal{M} = \{V(W)[H]\omega \colon H \in (\mathbb{S}^{n\times n})^g\}$ is at most $N-1$. On the other hand, because $Z(X)$ has a negative eigenvalue, the matrix $Z(W)$ has an eigenspace $\mathcal{E}$, corresponding to a negative eigenvalue, of dimension at least N. It follows that $\mathcal{E} \cap \mathcal{M}$ is nonempty; i.e., there is an $H \in (\mathbb{S}^{n\times n})^g$ such that $V(W)[H]\omega \in \mathcal{E}$. In particular,

$$\langle q(W)[H]\omega, \omega\rangle = \langle Z(W)V(W)[H]\omega, V(W)\omega\rangle < 0$$

and thus, $q(W)[H]$ is not positive semidefinite. ∎

6.4.1. *The modified Hessian.* Our main tool for analyzing the curvature of noncommutative varieties is a variant of the Hessian for symmetric nc polynomials p. The curvature of $\mathcal{V}(p)$ is defined in terms of Hess (p) compressed to tangent planes, for each dimension n. This compression of the Hessian is awkward to work with directly, and so we associate to it a quadratic polynomial $q(x)[h]$ carrying all of the information of p'' compressed to the tangent plane, but having the key property (6.2). We shall call this q we construct the relaxed Hessian. The first step in constructing the relaxed Hessian is to consider the simpler *modified Hessian*

$$p''_{\lambda,0}(x)[h] := p''(x)[h] + \lambda\, p'(x)[h]^{\intercal} p'(x)[h].$$

which captures the conceptual idea. Suppose $X \in (\mathbb{S}^{n\times n})^g$ and $v \in \mathbb{R}^n$. We say that the *modified Hessian is negative* at (X, v) if there is a $\lambda_0 < 0$, so that for all $\lambda \le \lambda_0$,

$$0 \le -\langle p''_{\lambda,0}(X)[H]v, v\rangle$$

for all $H \in (\mathbb{S}^{n\times n})^g$. Given a subset $\mathcal{S} = (\mathcal{S}(n))_{n=1}^{\infty}$, with $\mathcal{S}(n) \subseteq (\mathbb{S}^{n\times n})^g \times \mathbb{R}^n$, we say that the *modified Hessian is negative on* $\mathcal{S}$ if it is negative at each $(X, v) \in S$.

Now we turn to motivation.

Example 6.6. *The classical* $n = 1$ *case.* Suppose that p is strictly smoothly quasi-concave, meaning that all superlevel sets of p are strictly convex with strictly positively curved smooth boundary. Suppose that the gradient ∇p (written as a row vector) never vanishes on $\mathbb{R}^g$. Then $G = \nabla p (\nabla p)^{\intercal}$ is strictly positive, at each point

X in $\mathbb{R}^g$. Fix such an X; the modified Hessian can be decomposed as a block matrix subordinate to the tangent plane to the level set at X, denoted T_X, and to its orthogonal complement (the gradient direction):

$$T_X \oplus \{\lambda \nabla p \colon \lambda \in \mathbb{R}\}.$$

In this decomposition the modified Hessian has the form

$$R = \begin{bmatrix} A & B \\ B^\intercal & D + \lambda G \end{bmatrix}.$$

Here, in the case of $\lambda = 0$, R is the Hessian and the second fundamental form is A or $-A$, depending on convention and the rather arbitrary choice of inward or outward normal to ν. If we select our normal direction to be ∇p, then $-A$ is the classical second fundamental form as is consistent with the choice of sign in our definition in Subsection 6.1.3. (All this concern with the sign is unimportant to the content of this chapter and can be ignored by the reader.)

Next, in view of the presumed strict positive curvature of each level set ν, the matrix A at each point of ν is negative definite but the Hessian could have a negative eigenvalue. However, by standard Schur complement arguments, R will be negative definite if

$$D + \lambda G - B^\intercal A^{-1} B \prec 0$$

on this region. Thus, strict convexity assumptions on the sublevel sets of p make the modified Hessian negative definite for negative enough λ. One can make this negative definiteness uniform in X in various neighborhoods under modest assumptions. $\square$

Very unfortunately in the noncommutative case, Remark 6.8 [DHM+] implies that if n is large enough, then the second fundamental form will have a nonzero null space, thus strict negative definiteness of the A part of the modified Hessian is impossible.

Our trick, to deal with the likely reality that A is only positive semidefinite, and obtain a negative definite R, is to add another negative term, say δI, with arbitrarily small $\delta < 0$. After adding such δ, the argument based on choosing $-\lambda$ large succeeds as before. This δ term plus the λ term produces the "relaxed Hessian", to be introduced next, and proper selection of these terms make it negative definite.

6.4.2. *The relaxed Hessian.* Recall Let $V_k(x)[h]$ denotes the vector of polynomials with entries $h_j w(x)$, where $w \in {<}x{>}$ runs through the set of g^k words of length k, $j = 1, \ldots, g$. Although the order of the entries is fixed in some of our earlier applications (see e.g. [DHM07b, (2.3)]) it is irrelevant for the moment. Thus, $V_k = V_k(x)[h]$ is a vector of height g^{k+1}, and the vectors

$$V(x)[h] = \mathrm{col}(V_0, \dots, V_{d-2}) \quad \text{and} \quad \widetilde{V}(x)[h] = \mathrm{col}(V_0, \dots, V_{d-1})$$

are vectors of height $g\sigma(d-2)$ and $g\sigma(d-1)$ respectively. Note that

$$\widetilde{V}(x)[h]^{\intercal}\widetilde{V}(x)[h] = \sum_{j=1}^{g} \sum_{\deg(w)\le d-1} w(x)^{\intercal} h_j^2 w(x).$$

The *relaxed Hessian* of the symmetric nc polynomial p of degree d is defined to be

$$p''_{\lambda,\delta}(x)[h] := p''_{\lambda,0}(x)[h] + \delta\, \widetilde{V}(x)[h]^{\intercal}\widetilde{V}(x)[h] \in \mathbb{R}\langle x\rangle[h].$$

Suppose $X \in (\mathbb{S}^{n\times n})^g$ and $v \in \mathbb{R}^n$. We say that the *relaxed Hessian is negative* at (X, v) if for each $\delta < 0$ there is a $\lambda_\delta < 0$, so that for all $\lambda \le \lambda_\delta$,

$$0 \le -\langle p''_{\lambda,\delta}(X)[H]v, v\rangle$$

for all $H \in (\mathbb{S}^{n\times n})^g$. Given a $\mathcal{S} = (\mathcal{S}(n))_{n=1}^{\infty}$, with $\mathcal{S}(n) \subseteq (\mathbb{S}^{n\times n})^g \times \mathbb{R}^n$, we say that the *relaxed Hessian is positive* (*resp., negative*) *on* $\mathcal{S}$ if it is positive (resp., negative) at each $(X, v) \in S$.

The following theorem provides a link between the signature of the clamped second fundamental form with that of the relaxed Hessian.

Theorem 6.7. *Suppose p is a symmetric nc polynomial of degree d in g symmetric variables and $(X, v) \in (\mathbb{S}^{n\times n})^g \times \mathbb{R}^n$. If $\mathcal{V}(p)$ has positive curvature at $(X, v) \in \mathcal{V}_n(p)$, i.e., if*

$$\langle p''(X)[H]v, v\rangle \le 0 \quad \text{for every } H \in \mathcal{T}_p(X, v),$$

then for every $\delta < 0$ there exists a $\lambda_\delta < 0$ such that for all $\lambda \le \lambda_\delta$,

$$\langle p''_{\lambda,\delta}(X)[H]v, v\rangle \le 0 \quad \text{for every } H \in (\mathbb{S}^{n\times n})^g;$$

i.e., the relaxed Hessian of p is negative at (X, v).

We leave the proof of Theorem 6.7 to the reader.

The basic idea of the proof of Theorem 6.4, is to obtain a negative relaxed Hessian q from Theorem 6.7 and then apply Theorem 6.5. We begin with the following lemma.

Lemma 6.8. *Suppose R and T are operators on a finite dimensional Hilbert space $H = K \oplus L$. Suppose further that, with respect to this decomposition of H, the operator $R = CC^{\intercal}$ for*

$$C = \begin{bmatrix} r \\ c \end{bmatrix} : L \to K \oplus L \quad \text{and} \quad T = \begin{bmatrix} T_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If c is invertible and if for every $\delta > 0$ there is a $\eta > 0$ such that for all $\lambda > \eta$,

$$T + \delta I + \lambda R \succeq 0,$$

then $T \succeq 0$.

Proof. Write

$$T + \delta I + \lambda R = \begin{bmatrix} T_0 + \delta I + \lambda r r^\intercal & \lambda r c^\intercal \\ \lambda c r^\intercal & \delta + \lambda c c^\intercal \end{bmatrix}.$$

From Schur complements it follows that

$$T_0 + \delta I + r(\lambda - \lambda^2 c^\intercal (\delta + \lambda c c^\intercal)^{-1} c) r^\intercal \succeq 0.$$

Now

$$\begin{aligned} r(\lambda - \lambda^2 c^\intercal (\delta + \lambda c c^\intercal)^{-1} c) r^\intercal &= \lambda r c^\intercal ((c c^\intercal)^{-1} - \lambda(\delta + \lambda c c^\intercal)^{-1}) c r^\intercal \\ &= \lambda r c^\intercal \delta (c c^\intercal)^{-1} (\delta + \lambda (c c^\intercal))^{-1} c r^\intercal \\ &\preceq \delta r (c c^\intercal)^{-1} r^\intercal. \end{aligned}$$

Hence,

$$T_0 + \delta I + \delta r (c c^\intercal)^{-1} r^\intercal \succeq 0.$$

Since the above inequality holds for all $\delta > 0$, it follows that $T_0 \succeq 0$. ■

We now have enough machinery developed to prove Theorem 6.4.

Proof of Theorem 6.4. Fix $\lambda, \delta > 0$ and consider $q(x)[h] = -p''_{\lambda,\delta}(x)[h]$. We are led to investigate the middle matrix $Z^{\lambda,\delta}$ of $q(x)[h]$, whose border vector $V(x)[h]$ includes all monomials of the form $h_j m$, where m is a word in x only of length at most $d-1$; here d is the degree of p. Indeed,

$$Z^{\lambda,\delta} = Z + \delta I + \lambda W,$$

where Z is the middle matrix for $-p''(x)[h]$, and W is the middle matrix for $p'(x)[h]^\intercal p'(x)[h]$. With an appropriate choice of ordering for the border vector V, we have, $W = CC^\intercal$, where

$$C(x) = \begin{bmatrix} w(x) \\ c \end{bmatrix},$$

for a nonzero vector c; and at the same time,

$$Z(x) = \begin{bmatrix} Z^{0,0}(x) & 0 \\ 0 & 0 \end{bmatrix}.$$

By the curvature hypothesis at a given X with $(X, v) \in \mathcal{S}$, Theorem 6.7 implies for every $\delta > 0$ there is an $\eta > 0$ such that if $\lambda > \eta$

$$\langle q(X)[H]v, v\rangle \geq 0 \qquad \text{for all } (X, v) \in \mathcal{S} \text{ and all } H.$$

Hence, by Theorem 6.5, the middle matrix, $Z^{\lambda,\delta}(X)$ for $q(x)[h]$ is positive semidefinite. We are in the setting of Lemma 6.8 from which we obtain $Z^{0,0}(X) \succeq 0$. If this held for X in a nc basic open semi-algebraic set, then Theorem 4.11 forces p to have degree no greater than 2. The proof of that theorem applies easily here to finish this proof. ∎

6.5. **Exercises.**

Exercise 6.2. Compute the BV-MM representation for the relaxed Hessian of x^3 and x^4.

7. Convex semialgebraic nc sets

In this section we will give a brief overview of convex semialgebraic nc sets and positivity of nc polynomials on them. We shall see that their structure is much more rigid than that of their commutative counterparts. For example, roughly speaking, each convex semialgebraic nc set is a spectrahedron; i.e., a solution set of a linear matrix inequality (cf. Section 7.1 below). Similarly, every nc polynomial nonnegative on a spectrahedron admits a sum of squares representation with weights and optimal degree bounds (see Section 7.2 for details and precise statements).

7.1. **nc Spectrahedra.**

Let L be an affine linear pencil. Then the solution set of the linear matrix inequality (LMI) $L(x) \succ 0$ is

$$\mathcal{D}_L = \bigcup_{n\in\mathbb{N}} \left\{X \in (\mathbb{S}^{n\times n})^g \colon L(X) \succ 0\right\},$$

and is called a *nc spectrahedron*. The set $\mathcal{D}_L$ is convex in the sense that each

$$\mathcal{D}_L(n) := \left\{X \in (\mathbb{S}^{n\times n})^g \colon L(X) \succ 0\right\}$$

is convex. It is also a noncommutative basic open semialgebraic set as defined in Section 2.1.2 above. The main theorem of this section is the converse, a result which has implications for both semidefinite programming and systems engineering.

Most of the time we will focus on monic linear pencils. An affine linear pencil L is called *monic* if $L(0) = I$, i.e., $L(x) = I + A_1x_1 + \cdots + A_gx_g$. Since we are mostly interested in the set $\mathcal{D}_L$, there is no harm in reducing to this case whenever $\mathcal{D}_L \neq \emptyset$; see Exercise 7.1.

Let $p \in \mathbb{R}^{\delta\times\delta}<x>$ be a given symmetric noncommutative $\delta \times \delta$-valued matrix polynomial. Assuming that $p(0) \succ 0$, the positivity set $\mathcal{D}_p(n)$ of a noncommutative symmetric polynomial p in dimension n is the component of 0 of the set

$$\{X \in (\mathbb{S}^{n\times n})^g \ : \ p(X) \succ 0\}.$$

The *positivity set*, $\mathcal{D}_p$, is the sequence of sets $(\mathcal{D}_p(n))_{n\in\mathbb{N}}$. The noncommutative set $\mathcal{D}_p$ is called *convex* if, for each n, $\mathcal{D}_p(n)$ is convex.

Theorem 7.1 (Helton-McCullough [HM+]). *Fix p a $\delta \times \delta$ symmetric matrix of polynomials in noncommuting variables. Assume*

(1) $p(0)$ *is positive definite;*
(2) $\mathcal{D}_p$ *is bounded; and*
(3) $\mathcal{D}_p$ *is convex.*

Then there is a monic linear pencil L such that

$$\mathcal{D}_L = \mathcal{D}_p.$$

Here we shall confine ourselves to a few words about the techniques involved in the proof, and refer the reader to [HM+] for the full proof. Since we are dealing with matrix convex sets, it is not surprising that the starting point for our analysis is the matricial version of the Hahn-Banach Separation theorem of Effros and Winkler [EW97] which (itself a part of the theory of operator spaces and completely positive maps [BL04, Pa02, Pi03]) says that given a point x not inside a matrix convex set there is a (finite) linear matrix inequality which separates x from the set. For a general matrix convex set $\mathcal{C}$, the conclusion is then that there is a collection, likely infinite, of finite LMIs which cut out $\mathcal{C}$.

In the case $\mathcal{C}$ is matrix convex and also semialgebraic, the challenge is to prove that there is actually a *finite* collection of (finite) LMIs which define $\mathcal{C}$. The techniques used to meet this challenge have little relation to the methods of noncommutative calculus and positivity in the previous sections. Indeed a basic tool (of independent interest) is a degree bounded type of free Zariski closure of a single point $(X, v) \in (\mathbb{S}^{n\times n})^g \times \mathbb{R}^n$,

$$Z_d(X,v) := \bigcup_m \{(Y,w) \in (\mathbb{S}^{m\times m})^g \times \mathbb{R}^m : q(Y)w = 0 \text{ if } q(X)v = 0,\ q \in \mathbb{R}<x>_d\}.$$

Chief among a pleasant list of natural properties is the fact that there is an (X, v) with $X \in \partial\mathcal{D}_p$ and $p(X)v = 0$ for which $Z_d(X, v)$ contains all pairs (Y, w) such that $Y \in \partial\mathcal{D}_p$ and $p(Y)w = 0$. Combining this with the Effros-Winkler Theorem and battling degeneracies is a bit tricky, but voila separation prevails in the end. See [HM+] for the details.

An unexpected consequence of Theorem 7.1 is that projections of noncommutative semialgebraic sets may not be semialgebraic, see Exercise 7.2. For perspective, in the commutative case of a basic open semialgebraic subset $\mathcal{C}$ of $\mathbb{R}^g$, there is a stringent condition, called the "*line test*", which, in addition to convexity, is necessary for $\mathcal{C}$ to be a spectrahedron. In two dimensions the line test is necessary and sufficient [HV07], a result used by Lewis-Parrilo-Ramana [LPR05] to settle a 1958 conjecture of Peter Lax on hyperbolic polynomials.

In summary, if a (commutative) bounded basic open semialgebraic convex set is a spectrahedron, then it must pass the highly restrictive line test; whereas a nc basic open semialgebraic set is a spectrahedron if and only if it is convex.

7.2. **Noncommutative Positivstellensätze under convexity assumptions.** An algebraic certificate for positivity of a polynomial p on a semialgebraic set S is a Positivstellensatz. The familiar fact that a polynomial p in one-variable which is positive on $\mathbb{R}$ is a sum of squares is an example.

The theory of Positivstellensätze - a pillar of the field of real algebraic geometry - underlies the main approach currently used for global optimization of polynomials. See [Las10] or Chapter **??** of Parrilo for a beautiful treatment of this, and other, applications of commutative real algebraic geometry. Further, because convexity of a polynomial p on a set S is equivalent to positivity of the Hessian of p on S, this theory also provides a link between convexity and semialgebraic geometry. Indeed, this link in the noncommutative setting ultimately lead to the conclusion the a matrix convex noncommutative polynomial has degree at most two, cf. Section 4.4.

In this section we give a result of opposite type. We present a noncommutative Positivstellensatz for a polynomial to be nonnegative on a convex semialgebraic nc set (i.e., on a spectrahedron). Again, this result is cleaner and more rigid than the commutative counterparts (cf. Theorem 2.6).

Theorem 7.2 ([HKM++]). *Suppose L is a monic linear pencil. Then a noncommutative polynomial p is* positive semidefinite *on $\mathcal{D}_L$ if and only if it has a weighted sum of squares representation with optimal degree bounds. Namely,*

$$p = s^{\intercal}s + \sum_{j}^{\text{finite}} f_j^{\intercal} L f_j, \tag{7.1}$$

where s, f_j are vectors of noncommutative polynomials of degree no greater than $\frac{\deg(p)}{2}$.

The main ingredient of the proof is an analysis of rank preserving extensions of truncated noncommutative Hankel matrices; see [HKM++] for details. We point out that with $L = 1$, Theorem 7.2 recovers Theorem 2.6.

Theorem 7.2 contrasts sharply with the commutative setting, where the degrees of s, f_j are vastly greater than $\deg(p)$ and assuming only p nonnegative yields a clean Positivstellensatz so seldom that the cases are noteworthy.

7.3. **Exercises.**

Exercise 7.1. Suppose L is an affine linear pencil such that $0 \in \mathcal{D}_L(1)$. Show that there is a monic linear pencil $\check{L}$ with $\mathcal{D}_L = \mathcal{D}_{\check{L}}$.

Exercise 7.2. Chapters **??** and **??** discuss sets $D \subseteq \mathbb{R}^g$ which have a semidefinite representation as a strict generalization of a spectrahedron. For instance, consider the TV screen (cf. Section 2.1.2)

$$\mathrm{ncTV}(1) = \{X \in \mathbb{R}^2 \colon 1 - X_1^4 - X_2^4 > 0\} \subseteq \mathbb{R}^2$$

and the monic pencil

$$L(x, y) = \begin{bmatrix} 1 & x_1 \\ x_1 & y_1 \end{bmatrix} \oplus \begin{bmatrix} 1 & x_2 \\ x_2 & y_2 \end{bmatrix} \oplus \begin{bmatrix} 1 + y_1 & y_2 \\ y_2 & 1 - y_1 \end{bmatrix}.$$

It is readily verified that ncTV(1) is the projection, onto the first two (the x) coordinates of the set $\mathcal{D}_L(1)$; i.e.,

$$\mathrm{ncTV}(1) = \{X \in \mathbb{R}^2 \colon \exists Y \in \mathbb{R}^2 \ L(X, Y) \succ 0\}.$$

(1) Show that ncTV(1) is not a spectrahedron. (Hint: How often is $L_{\mathrm{TV}}(tX, tY)$ for $t \in \mathbb{R}$ singular?)
(2) Show that ncTV is not the projection of the nc spectrahedron $\mathcal{D}_L$.
(3) Show that ncTV is not the projection of *any* nc spectrahedron.
(4) Is ncTV(2) a projection of a spectrahedron? (Feel free to use the results about ncTV and LMI representable sets (spectrahedra), stated without proofs, from Section 2.1.2 and Section 7.1.)

Exercise 7.3. Suppose $f \in \mathbb{R}[x]$ is a real univariate polynomial nonnegative on $\mathbb{R}$. Prove that there are $g, h \in \mathbb{R}[x]$ with $f = g^2 + h^2$.

Exercise 7.4. If q is a symmetric concave matrix-valued polynomial with $q(0) = I$, then there exists a linear pencil L and a matrix-valued linear polynomial Λ such that

$$q = I - L - \Lambda^{\intercal}\Lambda.$$

Exercise 7.5. Consider the monic linear pencil

$$M(x) = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}.$$

(1) Determine $\mathcal{D}_M$.
(2) Show that $1 + x$ is positive semidefinite on $\mathcal{D}_M$.
(3) Construct a representation for $1 + x$ of the form (7.1).

Exercise 7.6. Consider the univariate affine linear pencil

$$L(x) = \begin{bmatrix} 1 & x \\ x & 0 \end{bmatrix}.$$

(1) Determine $\mathcal{D}_L$.
(2) Show that x is positive semidefinite on $\mathcal{D}_L$.
(3) Does x admit a representation of the form (7.1)?

Exercise 7.7. Let L be an affine linear pencil. Prove that:

(1) $\mathcal{D}_L$ is bounded if and only if $\mathcal{D}_L(1)$ is bounded;
(2) $\mathcal{D}_L = \emptyset$ if and only if $\mathcal{D}_L(1) = \emptyset$.

Exercise 7.8. Let $L = I + A_1x_1 + \cdots + A_gx_g$ be a monic linear pencil and assume that $\mathcal{D}_L(1)$ is bounded. Show that $I, A_1, \ldots, A_g$ are linearly independent.

Exercise 7.9. Let

$$\Delta(x_1, x_2) = I + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix}$$

and

$$\Gamma(x_1, x_2) = I + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{bmatrix}$$

be affine linear pencils. Show:

(1) $\mathcal{D}_\Delta(1) = \mathcal{D}_\Gamma(1)$.
(2) $\mathcal{D}_\Gamma(2) \subsetneq \mathcal{D}_\Delta(2)$.
(3) Is $\mathcal{D}_\Delta \subseteq \mathcal{D}_\Gamma$? What about $\mathcal{D}_\Gamma \subseteq \mathcal{D}_\Delta$?

Exercise 7.10. Let $L = A_1x_1 + \cdots + A_gx_g \in \mathbb{S}^{d\times d}<x>$ be a (homogeneous) linear pencil. Then the following are equivalent:

(i) $\mathcal{D}_L(1) \neq \emptyset$;
(ii) If $u_1, \ldots, u_m \in \mathbb{R}^d$ with $\sum_{i=1}^m u_i^\intercal L(x) u_i = 0$, then $u_1 = \cdots = u_m = 0$.

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