DUALITY, EXTREME POINTS AND HULLS FOR NONCOMMUTATIVE PARTIAL CONVEXITY

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Abstract. This article studies generalizations of (matrix) convexity, including partial convexity and biconvexity, under the umbrella of Γ-convexity. Here Γ is a tuple of free symmetric polynomials determining the geometry of a Γ-convex set. The paper introduces the notions of Γ-operator systems and Γ-ucp maps and establishes a Webster-Winkler type categorical duality between Γ-operator systems and Γ-convex sets. Next, a notion of an extreme point for Γ-convex sets is defined, paralleling the concept of a free extreme point for a matrix convex set. To ensure the existence of such points, the matricial sets considered are extended to include an operator level. It is shown that the Γ-extreme points of an operator Γ-convex set $\boldsymbol{K}$ are in correspondence with the free extreme points of the operator convex hull of $\Gamma(\boldsymbol{K})$. From this result, a Krein-Milman theorem for Γ-convex sets follows. Finally, relying on the results of Helton and the first two authors, a construction of an approximation scheme for the Γ-convex hull of the matricial positivity domain (also known as a free semialgebraic set) $\mathcal{D}_p$ of a free symmetric polynomial p is given. The approximation consists of a decreasing family of Γ-analogs of free spectrahedra, whose projections, under mild assumptions, in the limit yield the Γ-convex hull of $\mathcal{D}_p$.

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1. Introduction

Convexity is a fundamental concept that plays a pivotal role across various branches of mathematics and its applications including optimization, economics, and geometric analysis. A set is **convex** if for any two points in the set the line segment joining them lies entirely within the set. This property not only simplifies mathematical models but also ensures the tractability of optimization problems by guaranteeing that local minima are global. Hence the elegant structure of convex sets and functions allows for the development of robust analytical tools. Understanding convexity is thus essential for advancing theory and practice across diverse scientific disciplines. Expanding upon the classical notion of convexity, noncommutative convexity [Arv08, DK15, DK+, DHM17, DM05, HM12], also known as matrix convexity [EW97, EH19, EHKM18, HKM16, WW99, Zal17], concerns convexity in matrix spaces. Noncommutative convexity is intimately connected with the study of operator systems and spaces [Pau02, HKM13, HKM17], and has applications in quantum physics [Eff09, AC21], and control theory and systems engineering [SIG17].

In this article we study generalizations of (noncommutative) convexity. An example is given by partially convex sets, i.e., the sets that are convex in some of the coordinates with the others held fixed [HHLM08]. The generalized notions of convexity we consider are summarized under the term Γ-convexity [JKMMP21, JKMMP22], where Γ is a tuple of free noncommutative polynomials.

The choice of the tuple Γ determines the geometry of a Γ-convex set. For example, by choosing $\Gamma = (x, y, xy + yx, i(xy - yx))$ one obtains the class of xy-convex sets. Such sets arise in the study of bilinear matrix inequalities (BMIs), which appear in control theory, optimization, and various applied mathematics fields [KSVS04]. These are finite sums of the form

$$A_0 + \sum_j A_j x_j + \sum_k B_k y_k + \sum_{p,q} C_{pq} x_p y_q \succeq 0 \tag{1.1}$$

for self-adjoint matrices A_j, B_k, C_{pq} [vAB00].

Another commonly studied type of Γ-convex sets is defined by the tuple $\Gamma = (x, y, y^2)$. In analogy with matrix convex sets and BMIs one considers sets of $(x, y) = (x_1, \dots, x_{\mathtt{g}}, y)$ describable in the form

$$A_0 + \sum A_j x_j + By + Cy^2 \succeq 0. \tag{1.2}$$

Such sets are convex in the variables x whenever y is held fixed.

An important area where matrix inequalities such as (1.1) and (1.2) arise is engineering systems problems governed by a signal flow diagram. Such problems naturally give rise to two classes of variables. The system variables depend on the choice of system parameters and produce polynomial inequalities in the state variables. The algebraic form of these inequalities involves noncommutative polynomials and depends only upon the flow diagram, and not the particular choice of system variables. See [dOH06, dOHMP09] and the citations therein. Convexity (or linearity, as in Linear Matrix Inequalities) in the state variables, for a given choice of system variables, is then an important optimization consideration.

Broadly speaking, our results are of three types. We establish a Γ-analog of the Webster-Winkler [WW99] duality between matrix convex sets and operator systems; introduce extreme points and establish a Krein-Milman theorem for operator Γ-convex sets; and construct a version of the Lasserre-Parrilo spectrahedral lift [Las09, Par06] of the Γ-convex hull of a semialgebraic set. The remainder of this introduction is organized as follows: Subsections 1.1 and 1.2 contain basic notation and terminology surrounding matrix and Γ-convex sets; The Hahn-Banach separation theorems in each case are reviewed in Subsection 1.3. With this background, we then preview the main results of the paper in Subsection 1.4.

1.1. **Free polynomials and their evaluations.** For $n, m \in \mathbb{N}$ and a vector space $\mathscr{V}$, let $M_{n,m}(\mathscr{V})$ denote the space of $n \times m$ matrices with entries from $\mathscr{V}$. In the case $\mathscr{V} = \mathbb{C}$, we use the abbreviation $M_n = M_{n,n}$, and denote by $I_n \in M_n$ the identity matrix. When $\mathscr{V} = \mathbb{C}^{\mathtt{g}}$ for some $\mathtt{g} \in \mathbb{N}$, the space $M_n(\mathbb{C}^{\mathtt{g}})$ is canonically identified with $M_n^{\mathtt{g}}$, the set of $\mathtt{g}$-tuples of $n \times n$ complex matrices. Denote by $\mathbb{S}_n \subseteq M_n$ the set of self-adjoint $n \times n$ complex matrices let $\mathbb{S}^{\mathtt{g}}$ denote the graded set $(\mathbb{S}_n^{\mathtt{g}})_n$. Likewise, let $\mathbb{M}(\mathscr{V}) = (M_n(\mathscr{V}))_n$ and $\mathbb{M}^{\mathtt{g}} = \mathbb{M}(\mathbb{C}^{\mathtt{g}})$.

Let $x = (x_1, \dots, x_{\mathtt{g}})$ denote a tuple of $\mathtt{g}$ (freely) noncommuting variables and let $\langle x \rangle$ denote the semigroup of words in the variables $x_1, \dots, x_{\mathtt{g}}$. We use 1 to denote the empty word $\emptyset$. Now denote by $\mathbb{C}\langle x \rangle$ the free algebra consisting of finite $\mathbb{C}$-linear combinations of the words in the variables $x_1, \dots, x_{\mathtt{g}}$. An element $p \in \mathbb{C}\langle x \rangle$ is called a **noncommutative (nc) polynomial**, or synonymously a **free polynomial**, and it is of the form

$$p = \sum_{w \in \langle x \rangle} p_w w, \tag{1.3}$$

where the sum is finite and $p_w \in \mathbb{C}$. A natural involution * on the semigroup $\langle x \rangle$ is defined by $x_j^* = x_j$ for $1 \le j \le \mathtt{g}$ and $(wu)^* = u^* w^*$ for $u, w \in \langle x \rangle$. The involution * naturally extends to $\mathbb{C}\langle x \rangle$ by

$$p^* = \sum_{w \in \langle x \rangle}^{\text{finite}} \overline{p_w} w^*,$$

where p is as in (1.3).

Free polynomials are **evaluated** at an $X \in \mathbb{S}^{\mathtt{g}}$. For a word

$$w = x_{j_1} x_{j_2} \cdots x_{j_N} \in \langle x \rangle,$$

and $X \in \mathbb{S}_n^{\mathtt{g}}$,

$$w(X) = X_{j_1} X_{j_2} \cdots X_{j_N} \in M_n;$$

and for p as in (1.3),

$$p(X) = \sum_{w\in\langle x\rangle}^{\text{finite}} p_w w(X) \in M_n.$$

Thus p determines a (graded) function $p : \mathbb{S}^{\mathtt{g}} \to \mathbb{M}$, where $\mathbb{M} = (M_n)_n$, and similarly, an $\mathtt{r}$-tuple $p = (p_1, \dots, p_{\mathtt{r}}) \in \mathbb{C}\langle x\rangle^{1\times\mathtt{r}} = M_{1,\mathtt{r}}(\mathbb{C}\langle x\rangle)$ determines a mapping $p : \mathbb{S}^{\mathtt{g}} \to \mathbb{M}^{\mathtt{r}}$.

A polynomial that is invariant under the involution * is called **symmetric**. It is well-known and easy to see that $p \in \mathbb{C}\langle x\rangle$ is symmetric if and only if it satisfies $p(X)^* = p(X)$ for all $X \in \mathbb{S}^{\mathtt{g}}$. In this case p determines a mapping $p : \mathbb{S}^{\mathtt{g}} \to \mathbb{S} = \mathbb{S}^1$. Since $x_j^* = x_j$, the variables x are referred to as **symmetric variables**.

More generally, a polynomial of the form (1.3) with the coefficients p_w lying in $\mathcal{B}(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$, is a $\mathcal{B}(\mathcal{H})$-valued noncommutative polynomial and $\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}\langle x\rangle$ is the space of $\mathcal{B}(\mathcal{H})$-valued noncommutative polynomials. In the case $\mathcal{H} = \mathbb{C}^\mu$ for some $\mu \in \mathbb{N}$, we obtain the space of all $\mu\times\mu$ matrix-valued noncommutative polynomials, denoted by $M_\mu(\mathbb{C}\langle x\rangle)$. Any polynomial $p \in \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}\langle x\rangle$ is **evaluated** at a tuple $X \in \mathbb{S}_n^{\mathtt{g}}$ using the (Kronecker) tensor product,

$$p(X) = \sum p_w \otimes w(X) \in \mathcal{B}(\mathcal{H}) \otimes M_n,$$

and p is **symmetric** if $p(X)^* = p(X)$ for all $X \in \mathbb{S}^{\mathtt{g}}$. Equivalently, a polynomial $p \in \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}\langle x\rangle$, or $p \in M_\mu(\mathbb{C}\langle x\rangle)$ in the case $\mathcal{H} = \mathbb{C}^\mu$, is symmetric if $p_{w^*} = p_w^*$ for all words w.

1.2. **Preliminaries on Γ-convexity.** Fix $\mathtt{g} \in \mathbb{N}$ and let $\Gamma = (\gamma_1, \dots, \gamma_{\mathtt{r}})$ denote a tuple of symmetric noncommutative polynomials with $\gamma_j = x_j$ for $1 \le j \le \mathtt{g} \le \mathtt{r}$. As in Subsection 1.1 we also use $\Gamma : \mathbb{S}^{\mathtt{g}} \to \mathbb{S}^{\mathtt{r}}$ to denote the mapping

$$\Gamma(X) = (\gamma_1(X), \dots, \gamma_{\mathtt{r}}(X)).$$

For instance, in the case $\mathtt{g} = 2$ and $\mathtt{r} = 4$ for $\Gamma(x, y) = (x, y, xy + yx, i(xy - yx))$ and $(X, Y) \in M_n^2$,

$$\Gamma(X, Y) = (X, Y, XY + YX, i(XY - YX)) \in M_n^4.$$

Definition 1.1. Let $\boldsymbol{K} = (K_n)_n \subseteq \mathbb{S}^{\mathtt{g}}$, where $K_n \subseteq \mathbb{S}_n^{\mathtt{g}}$ for each positive integer n, be a given graded set.

(1) The graded set $\boldsymbol{K}$ is **(uniformly) bounded** if there is an $R \in \mathbb{R}_{\ge 0}$ such that

$$\sum_{j=1}^{\mathtt{g}} X_j^2 \preceq R^2$$

for all n and $X \in K_n$. Equivalently, with the **norm** $\|X\|$ of $X \in K_n$ defined as the norm of the $n \times n\,\mathtt{g}$ matrix

$$\begin{pmatrix} X_1 & X_2 & \dots & X_{\mathtt{g}} \end{pmatrix},$$

the set $\boldsymbol{K}$ is bounded if there is an $R \ge 0$ such that $\|X\| \le R$ for all $X \in \boldsymbol{K}$.

(2) The set $\boldsymbol{K}$ is a **free set** if it is closed under direct sums and simultaneous unitary conjugations.[1]

(3) A pair (X, V), where $X \in \mathbb{S}_n^{\mathtt{g}}$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry, is a **Γ-pair** if it satisfies

$$V^*\Gamma(X)V = \Gamma(V^*XV).$$

Let $\mathcal{C}_\Gamma$ denote the collection of all Γ-pairs, parameterized over all choices of positive integers n, m.

(4) The set $\boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ is a **Γ-convex set** if it is free and if

$$X \in \boldsymbol{K} \text{ and } (X, V) \in \mathcal{C}_\Gamma \implies V^*XV \in \boldsymbol{K}.$$

(5) The **Γ-convex hull** of a free set $\boldsymbol{K}$ is the smallest Γ-convex set that contains $\boldsymbol{K}$. It is obtained as the intersection of all Γ-convex sets containing $\boldsymbol{K}$ and is denoted by $\Gamma\text{-}\operatorname{conv}(\boldsymbol{K})$. ■

Remark 1.2.

(a) Note that for every $n \times n$ unitary matrix U and $X \in \mathbb{S}_n^{\mathtt{g}}$, the tuple (X, U) is a Γ-pair.

(b) If an isometry V reduces a tuple X, then (X, V) is a Γ-pair. Indeed, writing the decomposition of X with respect to the range of V as $X = Y \oplus Z$,

$$V^*\Gamma(X)V = V^*(\Gamma(Y) \oplus \Gamma(Z))V = \Gamma(Y) = \Gamma(V^*(Y \oplus Z)V) = \Gamma(V^*XV).$$

(c) In the case when $\mathtt{r} = \mathtt{g}$, and thus $\Gamma(x) = x$, the notion of Γ-convexity reduces to **matrix convexity**. (See Subsection 2.2.) ■

Definition 1.3. Given a positive integer k, tuples $X^{(i)} \in K_{n_i}$ and $V_i \in M_{n_i,n}$ for $1 \leq i \leq k$ such that $\sum_{i=1}^k V_i^*V_i = I_n$, the tuple

$$\sum_{i=1}^k V_i^* X^{(i)} V_i \in M_n^{\mathtt{g}}$$

is a **matrix convex combination** often abbreviated by setting

$$X = \oplus_{i=1}^k X^{(i)} \quad \text{and} \quad V = \operatorname{col}(V_1, \dots, V_k) = \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix}, \tag{1.4}$$

so that

$$V^*XV = \sum_{i=1}^k V_i^* X^{(i)} V_i.$$

In particular, the condition $\sum_{i=1}^k V_i^*V_i = I_n$ is equivalent to V being an isometry.

[1]This means that if $X \in K_n$ and $Y \in K_m$, then $X \oplus Y \in K_{n+m}$; if $U \in M_n$ is unitary, then $U^*XU = (U^*X_1U, \dots, U^*X_{\mathtt{g}}U) \in K_n$.

A matrix convex combination of the form

$$\sum_{i=1}^{k} V_i^* X^{(i)} V_i, \tag{1.5}$$

is a **Γ-convex combination** if (X, V) of (1.4) is a Γ-pair. ■

The next two propositions [JKMMP21, Proposition 2.1, Proposition 2.2] give alternative descriptions of the Γ-convex hull of a free set. We include the easy proofs for self-containment.

Proposition 1.4. *A free set $\boldsymbol{K} \subseteq \mathbb{S}^g$ is a Γ-convex set if and only if it is closed under Γ-convex combinations. Moreover,*

$$\Gamma\text{-}\operatorname{conv}(\boldsymbol{K}) = \{V^* X V : X \in \boldsymbol{K},\ (X, V) \in \mathcal{C}_\Gamma\}.$$

Proof. The first statement is clear by the definition of a Γ-convex set. For the second claim it is easy to verify that $\mathscr{K} = \{V^* X V : X \in \boldsymbol{K},\ (X, V) \in \mathcal{C}_\Gamma\}$ is a Γ-convex set that contains $\boldsymbol{K}$. Since by definition, $\Gamma\text{-}\operatorname{conv}(\boldsymbol{K})$ must contain $\mathscr{K}$, we have that $\mathscr{K} = \Gamma\text{-}\operatorname{conv}(\boldsymbol{K})$. ■

In the case $r = \mathsf{g}$ so that $\Gamma = x$; that is, we are dealing with matrix convex sets, the Γ-convex hull is the **matrix convex hull** of $\boldsymbol{K}$ and we use the standard notation $\operatorname{matco}(\boldsymbol{K})$ instead of $\Gamma\text{-}\operatorname{conv}(\boldsymbol{K})$.

Proposition 1.5. *Suppose $\boldsymbol{K} \subseteq \mathbb{S}^{\mathsf{g}}$ is a free set and $X \in \mathbb{S}^{\mathsf{g}}$. The point X is in $\Gamma\text{-}\operatorname{conv}(\boldsymbol{K})$ if and only if $\Gamma(X)$ is in $\operatorname{matco}(\Gamma(\boldsymbol{K}))$. Equivalently,*

$$\Gamma^{-1}(\operatorname{matco}(\Gamma(\boldsymbol{K})) = \Gamma\text{-}\operatorname{conv}(\boldsymbol{K}).$$

Proof. If $X \in \Gamma\text{-}\operatorname{conv}(\boldsymbol{K})$, then by Proposition 1.4, there is a Y in $\boldsymbol{K}$ and an isometry V such that $V^*\Gamma(Y)V = \Gamma(V^*YV)$ and $X = V^*YV$. Thus, $\Gamma(X) = V^*\Gamma(Y)V$, which implies $\Gamma(X) \in \operatorname{matco}(\Gamma(\boldsymbol{K}))$.

For the converse assume $\Gamma(X) \in \operatorname{matco}(\Gamma(\boldsymbol{K}))$. Thus there is a $Y \in \boldsymbol{K}$ and an isometry V such that $\Gamma(X) = V^*\Gamma(Y)V$. A comparison of the first g coordinates gives $X = V^*YV$, from which we deduce that (Y, V) is Γ-pair. Since $Y \in \boldsymbol{K}$ and (Y, V) is a Γ-pair, Proposition 1.4 implies $X \in \Gamma\text{-}\operatorname{conv}(\boldsymbol{K})$. ■

Example 1.6. In the case of two variables $(x, y) = (x_1, x_2)$ and $\Gamma = \{x, y, y^2\}$ we write **y^2-convex** to mean $\{x, y, y^2\}$-convex. It was shown in [JKMMP21, §4], in this case, $((X, Y), V)$ is a Γ-pair if and only if the isometry V reduces Y and that a free set $\boldsymbol{K}$ is y^2-convex if and only if it is convex in x for any fixed y. That is, $(X_1, Y), (X_2, Y) \in K_n$ implies $(\frac{X_1+X_2}{2}, Y) \in K_n$. An example of a y^2-convex set is

$$\boldsymbol{K} = \{(X, Y) \mid -(Y^2 + I) \preceq X \preceq Y^2 + I,\ Y^2 \preceq I\},$$

which we consider later in Example 3.30.

In the case $\Gamma(x,y) = (x, y, xy + yx, i(xy - yx))$, a free set that is Γ-convex is known as an xy**-convex set**. An xy-convex set is convex in both x and y separately. See [HHLM08, Mag16, DHM17, JKMMP21, BHM23] for more results on this topic. ■

1.3. **A Hahn-Banach separation theorem for Γ-convex sets.** This section presents a Hahn-Banach separation theorem of an outlier from a closed Γ-convex set, where the separation is given by a Γ-analog of a linear pencil. We continue to let $\Gamma(x) = (\gamma_1, \dots, \gamma_{\mathtt{r}})$, where the γ_i are symmetric and $\gamma_i = x_i$ for $i = 1, \dots, \mathtt{g}$.

For a (complex) Hilbert space $\mathcal{H}$, let $\mathbb{S}_{\mathcal{H}}$ denote the self-adjoint (hermitian) operators on $\mathcal{H}$ and let $\mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ denote the set of $\mathtt{g}$-tuples of elements of $\mathbb{S}_{\mathcal{H}}$.

Definition 1.7. A **Γ-pencil** with coefficients $A = (A_0, A_1, \dots A_{\mathtt{r}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{r}+1}$ is the $\mathcal{B}(\mathcal{H})$-valued affine linear polynomial

$$L^{\Gamma}(x) = L_A^{\Gamma}(x) = A_0 + \sum_{i=1}^{\mathtt{g}} A_i\, x_i + \sum_{i=\mathtt{g}+1}^{\mathtt{r}} A_i \gamma_i(x). \tag{1.6}$$

For a tuple $X \in M_n^{\mathtt{g}}$,

$$L^{\Gamma}(X) = A_0 \otimes I_n + \sum_{j=1}^{\mathtt{g}} A_i \otimes X_i + \sum_{j=\mathtt{g}+1}^{\mathtt{r}} A_j \otimes \gamma_j(X) \in \mathcal{B}(\mathcal{H}) \otimes M_n \cong \mathcal{B}(H \otimes \mathbb{C}^n) \cong \mathcal{B}(H^n).$$

The matricial **positivity domain** of L^{Γ}, denoted by $\mathcal{D}_A^{\Gamma}$, is the graded set

$$\mathcal{D}_A^{\Gamma} = \left(\mathcal{D}_A^{\Gamma}(n)\right)_n = (\{X \in \mathbb{S}_n^{\mathtt{g}} \mid L^{\Gamma}(X) \succeq 0\})_n$$

and is known a **Γ-spectrahedron** when $\mathcal{H}$ is finite-dimensional and a **Hilbertian Γ-spectrahedron** in case $\mathcal{H}$ is potentially infinite-dimensional. A Γ-pencil is called **monic** if $A_0 = I$. ■

In the case that $\mathtt{g} = \mathtt{r}$ (and $\Gamma(x) = x$) the expression,

$$L(x) = L_A(x) = A_0 + \sum_{j=1}^{\mathtt{g}} A_j\, x_j$$

is a **linear pencil** and the positivity domain $\mathcal{D}_A^{\Gamma} = \mathcal{D}_A$ is a **Hilbertian free spectrahedron**. In particular, a Γ-pencil L^{Γ} has the form $L^{\Gamma} = L_A \circ \Gamma$, for a tuple $A \in \mathbb{S}_{\mathcal{H}}^{\mathtt{r}+1}$.

As examples, a monic Γ-pencil with $\Gamma = (x, y, xy + yx, i(xy - yx))$ is of the form

$$L^{\Gamma}(x,y) = I + A_x x + A_y y + Bxy + B^* yx,$$

where A_x, A_y are self-adjoint; and a monic Γ-pencil with $\Gamma = (x, y, y^2)$ is of the form

$$L^{\Gamma}(x,y) = I + A_x x + A_y y + By^2,$$

where A_x, A_y, and B are self-adjoint.

We now recall a weaker form of the finite-dimensional version of the Effros-Winkler [EW97] Hahn-Banach separation theorem for matrix convex sets given in [HM12, Proposition 6.4] (or see, e.g., [Kri19, Lemma 2.1(a)]). Typically, an adjective describing a free set is a **level-wise description**. For instance, a matrix convex set $K \subseteq \mathbb{S}^{\mathrm{g}}$ is **closed** if each K_n is closed (in any locally convex topology, equivalently norm topology) on M_n^{g}. An exception to this convention is boundedness, since, as it turns out, a matrix convex set $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$ is level-wise bounded if and only if it is bounded. See Lemma 2.1. If the origin $0 = (0, \dots, 0) \in \mathbb{C}^{\mathrm{g}}$ is contained in K_1, then $0 \in M_n^{\mathrm{g}}$, the tuple of zero matrices, is in K_n for each n. A tuple $Y \in \mathbb{S}^g$ has **size** ℓ means $Y \in \mathbb{S}_\ell^{\mathrm{g}}$.

Theorem 1.8. *Let $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$ be a closed matrix convex set containing the origin. If $\ell \in \mathbb{N}$ and $Y \in \mathbb{S}_\ell^{\mathrm{g}} \setminus K_\ell$, then there is a monic linear pencil L of size ℓ such that $L(X) \succeq 0$ for all X in $\boldsymbol{K}$, but $L(Y) \not\succeq 0$.*

Theorem 1.8 begets the following Γ-analog of the Effros-Winkler matricial Hahn-Banach separation theorem.

Theorem 1.9 ([JKMMP21, Theorem 1.4])**.** *Suppose $\Gamma(0) = 0$ and $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$ is a Γ-convex set containing 0. If the matrix convex hull of $\Gamma(\boldsymbol{K}) \subseteq \mathbb{S}^{\mathrm{r}}$ is closed and if $Y \in \mathbb{S}_\ell^{\mathrm{g}} \setminus K_\ell$, then there is a monic Γ-pencil L^Γ of size ℓ such that $L^\Gamma(X)$ is positive semidefinite for all X in $\boldsymbol{K}$, but $L^\Gamma(Y)$ is not positive semidefinite.*

Proof. By Proposition 1.5, we have $\Gamma(Y) \notin \operatorname{matco}(\Gamma(\boldsymbol{K}))_\ell$. Since $\operatorname{matco}(\Gamma(\boldsymbol{K}))$ is, by assumption, closed, it is a a closed matrix convex subset of $\mathbb{S}^{\mathrm{r}}$ containing 0. Thus, by Theorem 1.8, there is a monic linear pencil

$$L(z) = I_\ell + \sum_{j=1}^{\mathrm{r}} A_j z_j$$

of size ℓ such that $L(Z) \succeq 0$ for all $Z \in \operatorname{matco}(\Gamma(\boldsymbol{K}))$, but $L(\Gamma(Y)) \not\succeq 0$. Now $L^\Gamma = L \circ \Gamma$ is a monic Γ-pencil of size ℓ that is positive semidefinite on $\boldsymbol{K}$, but is not positive semidefinite at Y. ■

Remark 1.10. Theorem 1.9 remains valid if $\overline{\operatorname{matco}}(\Gamma(\boldsymbol{K}))$ is replaced by any closed matrix convex set $\boldsymbol{J}$ containing $\Gamma(\boldsymbol{K})$ such that

$$\Gamma\text{-}\operatorname{conv}(\boldsymbol{K}) = \Gamma^{-1}(\boldsymbol{J} \cap \operatorname{range}(\Gamma)).$$ ■

1.4. **Main results and guide to the paper.** This paper is structured around three main themes. First, it introduces a categorical duality in the Γ-convex setting, relating Γ-convex sets to Γ-operator systems via the concept of Γ-ucp maps. Second, it explores the concept of extreme points in Γ-convex sets $\boldsymbol{K}$, including a Krein-Milman-type theorem showing that under natural assumptions, the Γ-extreme points of $\boldsymbol{K}$ span $\boldsymbol{K}$. Finally, it presents a free analog of the Lasserre-Parrilo lifts for Γ-convex sets, constructing a sequence of free

Γ-spectrahedra whose projections offer increasingly better approximations and clamp down on the Γ-convex hull of the free semialgebraic set $\mathcal{D}_p = \{X \mid p(X) \succeq 0\}$.

1.4.1. *Duality in the* Γ*-convex setting.* Section 2 begins by revisiting the categorical duality between matrix convex sets and operator systems [WW99, Proposition 3.5]. Subsection 2.3 then introduces concrete Γ-operator systems as well as Γ-ucp maps, the Γ analogs of unital completely positive (ucp) maps.

For any tuple of self-adjoint operators $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$, let $\mathcal{R} = \mathcal{R}_A^\Gamma$ denote the operator system

$$\operatorname{span}\{I_{\mathcal{H}}, \gamma_j(A) \mid j = 1, \dots, \mathtt{r}\} \subseteq \mathcal{B}(\mathcal{H}). \tag{1.7}$$

In particular, the dimension of $\mathcal{R}$, as a vector subspace of $\mathcal{B}(\mathcal{H})$, is at most $\mathtt{r} + 1$. In Proposition 2.8 we show that if the Hilbertian spectrahedron $\mathcal{D}_{\Gamma(A)} \subseteq \mathbb{S}^{\mathtt{r}}$ is bounded, then the operator system $\mathcal{R}$ in (1.7) has dimension $\mathtt{r} + 1$.

Definition 2.14. Suppose $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$.

(a) The operator system $\mathcal{R}_A^\Gamma = \operatorname{span}\{I_{\mathcal{H}}, \gamma_j(A) \mid j = 1, \dots, \mathtt{r}\}$ is called a **Γ-operator system**.

(b) Let $\mathcal{R} = \mathcal{R}_A^\Gamma$ be a Γ-operator system and suppose $n \in \mathbb{N}$. A linear map $\varphi : \mathcal{R} \to M_n$ is a **Γ-ucp map** if it is ucp, and, for all i,

$$\varphi(\gamma_i(A)) = \gamma_i(\varphi(A_1), \dots, \varphi(A_{\mathtt{g}})). \qquad \blacksquare$$

A morphism of Γ-operator systems is a Γ-ucp map. An **isomorphism of Γ-operator systems** is, by definition, a bijective Γ-ucp map whose inverse is also Γ-ucp. To mimic the Webster-Winkler duality, we associate to any Γ-operator system $\mathcal{R}$ as in (1.7) the graded set $\check{\mathcal{R}} = W^\Gamma(A) = (W_n^\Gamma(A))_n$ with

$$W_n^\Gamma(A) = \{(\varphi(A_1), \dots, \varphi(A_{\mathtt{g}})) \in \mathbb{S}_n^{\mathtt{g}} \mid \varphi : \mathcal{R} \to M_n \text{ is } \Gamma\text{-ucp}\}.$$

Proposition 2.16 establishes that $\check{\mathcal{R}}$ is a compact Γ-convex set. Conversely, if $\boldsymbol{K}$ is a compact Γ-convex set, we set

$$\widehat{Y} = \bigoplus_{Y \in \boldsymbol{K}} Y \quad \text{and} \quad I = \bigoplus_{Y \in \boldsymbol{K}} I_{\operatorname{size}(Y)} \tag{1.8}$$

and associate to $\boldsymbol{K}$ the Γ-operator system $\widehat{\boldsymbol{K}} = \operatorname{span}\{I, \gamma_1(\widehat{Y}), \dots, \gamma_{\mathtt{r}}(\widehat{Y}))\}$, leading to the following duality between operations $\widehat{\ }$ and $\check{\ }$ in the Γ-convex setting. A tuple $A \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ is **semi-finite** if it is an at most countable direct sum of matrix tuples; that is, $A = \oplus_{j \in J} A^{(j)}$, where $J \subseteq \mathbb{N}$, and for each $j \in J$ there is a positive integer n_j such that $A^{(j)} \in \mathbb{S}_{n_j}^{\mathtt{g}}$.

Theorem 2.21 (Duality for Γ-convex sets). *The above operations* $\widehat{\ }$ *and* $\check{\ }$ *are dual to one another:*

(a) *Suppose* $A = (A_1, \ldots, A_{\mathtt{g}}) \in \mathbb{S}^{\mathtt{g}}_{\mathcal{H}}$ *is semi-finite and let* $\mathcal{R} = \operatorname{span}\{I_{\mathcal{H}}, \gamma_1(A), \ldots, \gamma_{\mathtt{r}}(A)\}$. *If* $\mathcal{D}_{\Gamma(A)}$ *is bounded, then* $\mathcal{R}$ *and* $\widehat{\widehat{\mathcal{R}}}$ *are isomorphic* Γ*-operator systems.*

(b) *Suppose* $\Gamma(0) = 0$ *and let* $\boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ *denote a closed and bounded* Γ*-convex set with* $0 \in K_1$. *If* $\boldsymbol{K} = \Gamma^{-1}(\overline{\operatorname{matco}}(\Gamma(\boldsymbol{K})))$, *then* $\boldsymbol{K} = \widecheck{\widecheck{\boldsymbol{K}}} = W^{\Gamma}(\widehat{Y})$ *for* $\widehat{Y}$ *defined as in equation* (1.8).

We note that while the tuple $\widehat{Y}$ is not semi-finite, it is isomorphic, as a Γ-operator system to a Γ-operator system defined by a semi-finite tuple. That is, there exists a semi-finite tuple $\widehat{E}$ acting on a separable Hilbert space $\mathscr{E}$ such that $\operatorname{span}\{I, \widehat{Y}\}$ and $\operatorname{span}\{I_{\mathscr{E}}, \widehat{E}\}$ are isomorphic as operator systems via the unital map that sends $\widehat{Y}_j$ to $\widehat{E}_j$. (See Remark 2.5.)

1.4.2. *Extreme points of* Γ*-convex sets.* Section 3 extends the concept of free extreme points of matrix convex sets to the Γ-convex setting. Ensuring the existence of such extreme points requires modifying the definition of a matrix (or Γ-)convex set to an *operator* (Γ-)*convex set* $\boldsymbol{K} = (K_n)_{n \in \mathbb{N} \cup \{\infty\}}$ to include an infinite-dimensional level $K_\infty \subseteq \mathcal{B}(\mathcal{H})^{\mathtt{g}}_{\mathrm{sa}}$, where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space. An operator (Γ-)convex set is thus closed under generalized matrix convex combinations with operator coefficients. The following definition is an adaptation of the concept of an *nc extreme point* from [DK+], a notion closely related to that of a free extreme point, tailored to the study of operator convex sets explored here.

Definition 3.23. Let $\boldsymbol{K} = (K_n)_{n \in \mathbb{N} \cup \{\infty\}}$ be an operator Γ-convex set (that is not assumed closed). A tuple $X \in K_n$ for $n \in \mathbb{N} \cup \{\infty\}$ is a **Γ-extreme point** if any expression of X as an operator Γ-convex combination

$$X = \sum_{i=1}^{k} V_i^* X^{(i)} V_i,$$

where $X^{(i)} \in K_{n_i}$, the $V_i \in M_{n_i,n}$ are all nonzero with $\sum_{i=1}^k V_i^* V_i = I_n$, and

$$(X, V) = (\oplus_i X^{(i)}, \operatorname{col}(V_1, \ldots, V_k))$$

is a Γ-pair, implies that for each i, the matrix V_i is a scalar multiple of an isometry $W_i \in M_{n_i,n}$ such that $(X^{(i)}, W_i)$ is a Γ-pair and, with respect to the range of V_i,

$$X^{(i)} = Y^{(i)} \oplus Z^{(i)}$$

for some $Y^{(i)}, Z^{(i)} \in \boldsymbol{K}$ with $Y^{(i)}$ unitarily equivalent to X. Denote the set of Γ-extreme points of $\boldsymbol{K}$ by $\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})$. ■

Proposition 3.25 is key to establishing the existence of Γ-extreme points. It asserts that under mild assumptions, if $\boldsymbol{K}$ is an operator Γ-convex set and $\Gamma(X)$ is a free extreme point of $\operatorname{opco}(\Gamma(\boldsymbol{K}))$, the operator convex hull of $\Gamma(\boldsymbol{K})$, then X is a Γ-extreme point of $\boldsymbol{K}$. As in the case of matrix convexity, the (closed) operator convex hull of a graded set $\boldsymbol{S} = (S_n)_n \subseteq \mathbb{S}^{\mathtt{g}}$ is, by definition, the smallest (closed) operator convex set that contains $\boldsymbol{S}$. Language such as the τ-*closed* operator convex hull indicates that the closure is taken with respect to the topology τ, typically either the strong operator topology (SOT) or the weak*-topology (w*). We use the notation $\overline{\mathrm{opco}}^{\tau}(\boldsymbol{S})$. The definition of a τ-closed operator Γ-convex hull, denoted $\Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{S})$[2] is similar. Finally, we prove a Krein-Milman type theorem for Γ-convex sets, whose proof relies critically on [DK+, Theorem 6.4.2].

Theorem 3.27 (**Krein-Milman theorem for Γ-convex sets**). *Suppose* $\Gamma(0) = 0$. *If* $\boldsymbol{K}$ *is an SOT-closed and bounded operator* Γ*-convex set that contains* 0, *then* $\Gamma\text{-}\operatorname{ext}(\boldsymbol{K}) \neq \emptyset$ *and*

$$\boldsymbol{K} = \Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})).$$

1.4.3. *Lasserre-Parrilo lifts for* Γ*-convex sets.* Section 4 provides a Γ-analog of the results in [HKM16], namely a free version of the Lasserre-Parrilo construction in the Γ-convex setting. Fixing a symmetric matrix-valued noncommutative polynomial $p \in M_\mu(\mathbb{C}\langle x, y\rangle)$,

$$p(x) = \sum_{|\gamma| \le \delta} p_\gamma \gamma,$$

of degree $\le \delta$ in $\mathtt{g}$ variables x, where $p_\alpha \in M_\mu$, we give a construction of a sequence $\mathcal{D}^{\Gamma}_{A(d)}$ of free Γ-spectrahedra in increasingly larger spaces whose projections give better and better approximations to the Γ-convex hull of the operator positivity domain $\mathcal{D}_p^\infty$ of p,

$$\mathcal{D}_p^\infty = (\{X \in \mathbb{S}_n^{\mathtt{g}} \mid p(X) \succeq 0\})_{n \in \mathbb{N} \cup \{\infty\}}.$$

Firstly, by leveraging the theory of moments, in Theorem 4.7 we employ free Hankel matrices to explicitly present the operator Γ-convex hull of a bounded[3] $\mathcal{D}_p^\infty$ as the projection of an Hilbertian Γ-spectrahedron. Second, truncating this construction yields the clamping down Theorem 4.10: the operator Γ-convex hull of $\mathcal{D}_p^\infty$ is the decreasing intersection of an explicit countable family of projections of free Γ-spectrahedra.

Finally, Subsection 4.6 gives two criteria for when this intersection stabilizes in finitely many steps (in which case the operator Γ-convex hull of $\mathcal{D}_p^\infty$ is the projection of a free Γ-spectrahedron, a so-called Γ-spectrahedrop).

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2. Categorical duality

The categorical duality between compact matrix convex sets and operator systems was established in [WW99]. We refer to [Pau02, §13] for a treatment of the abstract characterization of operator systems via the Choi-Effros axioms. An **abstract operator system** $\mathcal{R}$ is

[2]See Remark 3.19 for the rationale for sot as a subscript, and not as a superscript
[3]More precisely, for Archimedean p, see (4.14)

a matrix ordered $*$-vector space with an Archimedean matrix order unit. This means firstly, that $\mathcal{R}$ is a $*$-ordered vector space; that is, $\mathcal{R}$ is a vector space equipped with an involution $*$ giving rise to a real subspace $\mathcal{R}_{\mathrm{sa}} \subseteq \mathcal{R}$ of self-adjoint elements. The space $\mathcal{R}$ also has a cone of positive elements and an Archimedean order unit. The key feature of an operator system is that, for each n, the matrix space $M_n(\mathcal{R})$ also has such a structure and the structures on different matrix levels are compatible. For instance, $\mathcal{R}$ has an Archimedean order unit e that induces an Archimedean matrix order unit. Namely, for each n, we have $I_n \otimes e$ is an Archimedean order unit for $M_n(\mathcal{R})$. Thus, when dealing with $\mathcal{R}$ we always also consider its associated family of matrix spaces $(M_n(\mathcal{R}))_n = \mathbb{M}(\mathcal{R})$.

There is also the notion of a **concrete operator system**, namely a closed self-adjoint unital subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. By a result due to Choi-Effros (see [Pau02, Theorem 13.1]), every abstract operator system $\mathcal{R}$ has a concrete realization[4]. On the other hand, it is easy to check that every concrete operator system satisfies the Choi-Effros axioms of an abstract operator system, since $\mathcal{B}(\mathcal{H})$ is an operator system by way of the identification of $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\oplus^n\mathcal{H})$.

There is a duality between operator systems and matrix convex sets that we describe in Subsection 2.1. In the case of matrix convex sets over the vector space $\mathbb{C}^{\mathrm{g}}$ and finite-dimensional operator systems $\mathcal{R}$, certain topological considerations in the general setting disappear and there are rather transparent proofs of the Webster-Winkler duality between convex sets and operator systems that we present in Subsection 2.2. The duality for Γ-convex sets and Γ-operator systems appears in Subsection 2.3.

2.1. **Duality: matrix affine and ucp maps.** Given a matrix convex set $\boldsymbol{K}$, the set of matrix affine maps $\mathrm{A}(\boldsymbol{K})$ is an operator system; and given an operator system $\mathcal{R}$, the set $\mathrm{UCP}(\mathcal{R})$ of unital completely positive (ucp) maps from $\mathcal{R}$ into matrix algebras is a matrix convex set. The operations $\boldsymbol{K} \mapsto \mathrm{A}(\boldsymbol{K})$ and $\mathcal{R} \mapsto \mathrm{UCP}(\mathcal{R})$ [WW99] are dual to one another as we now describe.

Given (locally convex Hausdorff) vector spaces $\mathscr{V}$ and $\mathscr{V}'$ and a matrix convex set $\boldsymbol{K} \subseteq \mathbb{M}(\mathscr{V})$, a **matrix affine map** $\theta : \boldsymbol{K} \to \mathbb{M}(\mathscr{V}')$ is a sequence of maps $\theta_n : K_n \to M_n(\mathscr{V}')$ satisfying,

$$\theta_n(\sum_{i=1}^m V_i^* A^{(i)} V_i) = \sum_{i=1}^m V_i^* \theta_{n_i}\left(A^{(i)}\right) V_i,$$

whenever $A^{(i)} \in K_{n_i}$ and $V_i \in M_{n_i,n}$ satisfy $\sum_{j=1}^m V_j^* V_j = I_n$. Matrix convex sets $\boldsymbol{K} \subseteq \mathbb{M}(\mathscr{V})$ and $\boldsymbol{K}' \subseteq \mathbb{M}(\mathscr{V}')$ are **isomorphic as matrix convex sets** if there exists a matrix affine homeomorphism $F : \boldsymbol{K} \to \boldsymbol{K}'$. Thus F is a bijective matrix affine map and each $F_n : K_n \to$

[4]There is a concrete operator system $\mathcal{S}$ such that $\mathcal{R}$ and $\mathcal{S}$ are isomorphic as operator systems as described below.

K_n' is a homeomorphism; that is, $F = (F_n)_n$ is a sequence of homeomorphisms satisfying

$$F_n(\sum_{i=1}^{m} V_i^* A^{(i)} V_i) = \sum_{i=1}^{m} V_i^* F_{n_i}\left(A^{(i)}\right) V_i,$$

whenever $A^{(i)} \in K_{n_i}$ and $V_i \in M_{n_i,n}$ satisfy $\sum_{i=1}^m V_i^* V_i = I_n$.

Assuming $\boldsymbol{K}$ is compact, meaning for each n the set $K_n \subseteq M_n(\mathscr{V})$ is compact in the product topology, let $\mathrm{A}(\boldsymbol{K}, M_r)$ denote the continuous matrix affine mappings $\theta : \boldsymbol{K} \to \mathbb{M}(M_r)$. In the case $r = 1$, so that $M_r = M_1 = \mathbb{C}$, let $\mathrm{A}(\boldsymbol{K}) = \mathrm{A}(\boldsymbol{K}, \mathbb{C})$.

The space

$$\mathrm{A}(\boldsymbol{K}) = \{\theta = (\theta_n : K_n \to M_n)_{n\in\mathbb{N}} \mid \theta \text{ continuous matrix affine}\}$$

is an abstract operator system. Indeed, the involution * on $\mathrm{A}(\boldsymbol{K})$ is pointwise conjugation,

$$\theta_n^*(X) = \theta_n(X)^* \tag{2.1}$$

for all n and $X \in K_n$, and the positive cone C_1 consists of all $\theta \in \mathrm{A}(\boldsymbol{K})$ such that

$$\theta(X) \succeq 0$$

for all $X \in \boldsymbol{K}$. It remains to describe the cones C_r for $M_r(\mathrm{A}(\boldsymbol{K}))$ and an Archimedean matrix order unit for $\mathbb{M}(\mathrm{A}(\boldsymbol{K}))$. After identifying

$$M_r(\mathrm{A}(\boldsymbol{K})) \cong \mathrm{A}(\boldsymbol{K}, M_r),$$

the involution * on $M_r(\mathrm{A}(\boldsymbol{K}))$ is defined as pointwise conjugation as in (2.1) and the positive cone C_r is defined to consist of all the maps $\theta \in \mathrm{A}(\boldsymbol{K}, M_r)_{\mathrm{sa}}$ that are pointwise positive semidefinite,

$$\theta_m(X) \succeq 0$$

for all m and $X \in K_m$. It is easy to check that the cones C_n are pointed; that is, $C_r \cap (-C_r) = (0)$ holds for all r, and they satisfy the compatibility condition $\alpha^* C_r\, \alpha \subseteq C_s$ for all r, s and $r \times s$ complex matrices α. Further, letting $e = (e_n)$ denote the continuous matrix affine map given by the sequence of constant functions $e_n : K_n \to M_n$ defined by $e_n(v) = I_n$, the sequence of continuous affine maps $\mathbb{I} = (e \otimes 1_r)_r$ from $(M_r(\mathrm{A}(\boldsymbol{K})))_r \cong (\mathrm{A}(\boldsymbol{K}, M_r))_r$ is an Archimedean matrix order unit. If $\boldsymbol{K}$ and $\boldsymbol{K}'$ are isomorphic as matrix convex sets via an affine homeomorphism F, then $\mathrm{A}(\boldsymbol{K}') \ni f \mapsto f \circ F \in \mathrm{A}(\boldsymbol{K})$ determines a complete order isomorphism. Thus $\mathrm{A}(\boldsymbol{K})$ and $\mathrm{A}(\boldsymbol{K}')$ are isomorphic as operator systems. The first part of [WW99, Proposition 3.5] states that every abstract operator system is of the form $\mathrm{A}(\boldsymbol{K})$ for a compact matrix convex set $\boldsymbol{K}$.

We now turn to the space $\mathrm{UCP}(\mathcal{R})$ of ucp matrix-valued maps on an operator system $\mathcal{R}$. A linear map $\Phi : \mathcal{R} \to M_n$ is k-positive if its k-th ampliation $\Phi_k = \Phi \otimes I_k : M_k(\mathcal{R}) \to M_k(M_n)$ is positive. Here for any $B = (B_{i,j}) \in M_k(\mathcal{R})$,

$$\Phi_k(B) = \big(\Phi(B_{i,j})\big)$$

and Φ_k is positive if $\Phi_k(B)$ is positive semidefinite whenever B is. The map Φ is **completely positive** if it is k-positive for all k. The **matrix state space** of $\mathcal{R}$ is the sequence $\mathrm{UCP}(\mathcal{R}) = (\mathrm{UCP}_n(\mathcal{R}))_n$, where

$$\mathrm{UCP}_n(\mathcal{R}) = \{\Phi : \mathcal{R} \to M_n \mid \Phi \text{ unital completely positive}\},$$

canonically identified as a subset of $M_n(\mathcal{R}^*)$. Given $\varphi \in \mathrm{UCP}_t(\mathcal{R})$ and an isometry $V : \mathbb{C}^s \to \mathbb{C}^t$, the mapping $V^*\varphi V : \mathcal{R} \to M_s$ defined by $V^*\varphi V(R) = V^*\varphi(R)V$ for $R \in \mathcal{R}$ is ucp. In this way, $\mathrm{UCP}(\mathcal{R})$ is a weak*-compact matrix convex set in $\mathbb{M}(\mathcal{R}^*)$. Part (b) of [WW99, Proposition 3.5] states that every compact matrix convex set is the matrix state space of an operator system.

Two operator systems $\mathcal{R}$ and $\mathcal{R}'$ are **isomorphic as operator systems** if there exists a linear isomorphism $G : \mathcal{R} \to \mathcal{R}'$; such that both G and G^{-1} are completely positive. Such a G is a **complete order isomorphism**.

To any compact matrix convex set $\boldsymbol{K}$ we associate the operator system $\mathrm{A}(\boldsymbol{K})$ and to any (abstract) operator system $\mathcal{R}$ we associate the matrix state space $\mathrm{UCP}(\mathcal{R})$, which is a (weak*) compact matrix convex set. By [WW99, Proposition 3.5], the operations UCP and A are dual to each other: $\mathcal{R}$ and $\mathrm{A}(\mathrm{UCP}(\mathcal{R}))$ are isomorphic operator systems and, on the other hand, $\boldsymbol{K}$ and $\mathrm{UCP}(\mathrm{A}(\boldsymbol{K}))$ are matrix affinely homeomorphic compact matrix convex sets.

2.2. **The finite-dimensional case.** By a **finite-dimensional matrix convex set** we mean a matrix convex set $\boldsymbol{K} \subseteq \mathbb{M}(\mathscr{V})$, where $\mathscr{V}$ is a finite-dimensional (locally convex Hausdorff) vector space. Often $\mathscr{V} = \mathbb{C}^{\mathrm{g}}$ so that $\boldsymbol{K} \subseteq \mathbb{M}^{\mathrm{g}}$. Likewise, a **finite-dimensional operator system** $\mathcal{R}$ is an operator system that is finite-dimensional as a vector space. Since $\mathcal{R}$ is finite-dimensional, $\mathcal{R}^*$ is finite-dimensional and locally convex. In finite dimensions topological considerations are trivial, since the topology on a finite-dimensional locally convex space is unique and determined by a norm. See, for instance, [Rud91, Theorem 1.21]. Thus, there is no ambiguity when using terms such as closed and compact that apply level-wise to a matrix convex set $\boldsymbol{K}$. In the finite-dimensional setting of this section, there are relatively simple proofs of the Webster-Winkler duality described in the previous subsection, Subsection 2.1, that we outline here in preparation for establishing similar results in the Γ-convex setting.

Before proceeding we record a lemma. If $\boldsymbol{K}$ is a compact matrix convex set, then each K_n is compact and hence bounded. More is true when $\boldsymbol{K}$ is a subset of a finite-dimensional vector space.

Lemma 2.1 ([HKM17, Proposition 4.3 and Lemma 4.2])**.** *If $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$ is a matrix convex set, then $\boldsymbol{K}$ is bounded if and only if K_1 is bounded and in this case, if R is a bound for K_1, then $\|Y_j\| \le R$ for all n and $Y = (Y_1, \dots, Y_{\mathrm{g}}) \in K_n$, and $\sqrt{\mathrm{g}}\,R$ is a bound for $\boldsymbol{K}$.*

Proof. Suppose R is a bound for K_1 and let n and $Y = (Y_1, \dots, Y_{\mathsf{g}}) \in K_n$ be given. Given a unit vector $h \in \mathbb{C}^n$ the map $V_h : \mathbb{C} \to \mathbb{C}^n$ defined by $V_h\, c = c\, h$ is an isometry. Thus

$$V^*YV = (h^*Y_1h, \dots, h^*Y_{\mathsf{g}}h) \in K_1$$

and consequently

$$|h^*Y_jh|^2 \leq \sum_{j=1}^{\mathsf{g}} h^*Y_jhh^*Y_jh = \|V^*YV\| \leq R^2.$$

Thus $|h^*Y_jh| \leq R$ for each j. Since Y_j is self-adjoint, it follows that $\|Y_j\| \leq R$. Finally,

$$\sum Y_j^2 \leq gR^2$$

and thus $\|Y\| \leq \sqrt{\mathsf{g}}R$ as claimed. ■

2.2.1. *Polar duals.* Recall, given a tuple $B = (B_1, \dots, B_{\mathsf{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathsf{g}}$,

$$L_B(x) = I_{\mathcal{H}} + \sum_{j=1}^{\mathsf{g}} B_j x_j \tag{2.2}$$

denotes the resulting (operator) **monic linear pencil**. In particular, for a tuple $X \in M_n^{\mathsf{g}}$,

$$L_B(X) = I_{\mathcal{H}} \otimes I_n + \sum_{j=1}^{\mathsf{g}} B_j \otimes X_j$$

and the positivity domain of L_B, denoted $\mathcal{D}_B = \big(\mathcal{D}_B(n)\big)_n \subseteq \mathbb{S}^{\mathsf{g}}$, is defined as

$$\mathcal{D}_B(n) = \{X \in \mathbb{S}_n^{\mathsf{g}} \mid L_B(X) \succeq 0\}.$$

Given a tuple $A = (A_1, \dots, A_{\mathsf{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathsf{g}}$, the **polar dual** of $\mathcal{D}_A$ (see [HKM17]) is the graded set $\mathcal{D}_A^\circ = (\mathcal{D}_A^\circ(n))_{n \in \mathbb{N}}$, where

$$\mathcal{D}_A^\circ(n) = \{B \in \mathbb{S}_n^{\mathsf{g}} \mid L_B(X) \succeq 0 \text{ for all } X \in \mathcal{D}_A\}.$$

More generally, the **polar dual** of a graded set $\emptyset \neq \boldsymbol{K} = (K_n)_n \subseteq \mathbb{S}^{\mathsf{g}}$, is the graded set $\boldsymbol{K}^\circ = (K_n^\circ)_n$, where

$$K_n^\circ = \{B \in \mathbb{S}_n^{\mathsf{g}} \mid L_B(X) = I_n \otimes I_{\text{size}(X)} + \sum_{i=1}^{\mathsf{g}} B_i \otimes X_i \succeq 0 \text{ for all } X \in \boldsymbol{K}\}.$$

It is well known that the polar dual $\boldsymbol{K}^\circ$ is matrix convex as we now demonstrate. Suppose $B^{(i)} = (B_1^{(i)}, \dots, B_{\mathsf{g}}^{(i)}) \in K_{n_i}^\circ$ and $V_i \in M_{n_i,n}$ for $i = 1, \dots, k$ and $\sum_{i=1}^k V_i^*V_i = I_n$. To show $\widetilde{B} = \sum_{i=1}^k V_i^* B^{(i)} V_i$ lies in K_n°, let $X \in K_r$ be given and, using the assumption that $L_{B^i}(X) \succeq 0$ for all i, compute

$$L_{\widetilde{B}}(X) = I_n \otimes I_r + \sum_{j=1}^{\mathsf{g}} \left(\sum_{i=1}^{k} V_i^* B_j^{(i)} V_i \right) \otimes X_j$$

$$
\begin{aligned}
&= I_n \otimes I_r + \sum_{i=1}^{k} \sum_{j=1}^{\mathrm{g}} V_i^* B_j^{(i)} V_i \otimes X_j \\
&= \sum_{i=1}^{k} V_i^* V_i \otimes I_r + \sum_{i=1}^{k} (V_i^* \otimes I_n) \left(\sum_{j=1}^{\mathrm{g}} B_j^{(i)} \otimes X_j \right) (V_i \otimes I_n) \\
&= \sum_{i=1}^{k} (V_i \otimes I_n)^* L_{B^{(i)}}(X) (V_i \otimes I_n) \succeq 0.
\end{aligned}
$$

It is now immediate that $\boldsymbol{K}^\circ$ is a closed matrix convex set containing 0. For future use, we state the following result summarizing some basic facts about the relations between $\boldsymbol{K}$ and $\boldsymbol{K}^\circ$ in the finite-dimensional setting.

Lemma 2.2 (cf. [HKM17, Proposition 4.3]). *Suppose $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$.*

(1) *$\boldsymbol{K}^\circ$ is a closed matrix convex set containing* 0;
(2) *If* 0 *is in the interior of K_1, then $\boldsymbol{K}^\circ$ is bounded;*
(3) *If K_1 is bounded, then* 0 *is in the interior of $\boldsymbol{K}^\circ$*;
(4) *If $\boldsymbol{K}$ is a closed matrix convex set and $0 \in K_1$, then $\boldsymbol{K} = \boldsymbol{K}^{\circ\circ}$*;
(5) *Assuming $\boldsymbol{K}$ is a matrix convex set containing* 0, *if* 0 *is in the interior of $(K^\circ)_1$, then $\boldsymbol{K}$ is bounded.*

Proof. A proof of item (1) appeared before the statement of the lemma. By [HKM17, Lemma 4.2], if $0 \in \mathbb{R}^{\mathrm{g}}$ is in the interior of K_1, then 0 is in the interior of $\boldsymbol{K}$. Hence by item (2) of [HKM17, Proposition 4.3], if 0 is in the interior of K_1, then $\boldsymbol{K}^\circ$ is bounded. If K_1 is bounded, then, by Lemma 2.1, $\boldsymbol{K}$ is bounded and thus by item (4) of [HKM17, Proposition 4.3], 0 is in the interior of $\boldsymbol{K}^\circ$. Item (4) is a consequence of the of the Effros-Winkler Bipolar Theorem. Indeed, it is immediate that $\boldsymbol{K} \subseteq \boldsymbol{K}^{\circ\circ}$. On the other hand, if $B \in \boldsymbol{K}^{\circ\circ} \setminus \boldsymbol{K}$, then, by Theorem 1.8, there exists a tuple X such that $L_X(\boldsymbol{K}) \succeq 0$, but $L_X(B) \not\succeq 0$. But $L_X(\boldsymbol{K}) \succeq 0$ is equivalent to $X \in \boldsymbol{K}^\circ$ and then $L_X(B) \not\succeq 0$ leads to the contradiction that $B \notin \boldsymbol{K}^{\circ\circ}$. Finally item (5) flows from items (4) and (3). ∎

Remark 2.3. Since 0 is in the interior of $\mathcal{D}_A$, the closed matrix convex set $\mathcal{D}_A^\circ$ is bounded and hence (level-wise) compact.

When $\boldsymbol{K}$ is a matrix convex set in a finite-dimensional space, by restricting $\boldsymbol{K}$ to the affine span of K_1 followed by a possible translation (see [Bar02, Theorem II.2.4]), it can be assumed, without loss of generality, that 0 is in the interior of $\boldsymbol{K}$. ∎

2.2.2. *The matrix state space and matrix range.* By Remark 2.3, a finite-dimensional operator system

$$\mathcal{R} = \operatorname{span}\{I_{\mathcal{H}}, A_1, \ldots, A_{\mathrm{g}}\}$$

determined by the tuple $A \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ gives rise to the compact matrix convex set $\mathcal{D}_A^\circ$. In Theorem 2.4 below we see that $\mathcal{D}_A^\circ$ is isomorphic, as a matrix convex set, to $\mathrm{UCP}(\mathcal{R})$. Let $\check{\mathcal{R}} = \mathcal{D}_A^\circ$.

The matrix state space $\mathrm{UCP}(\mathcal{R})$ of the operator system $\mathcal{R} = \mathrm{span}\{I_{\mathcal{H}}, A_1, \dots, A_{\mathtt{g}}\}$ is naturally identified with the (joint) **matrix range** $W(A) = (W_n(A))_n$ of the tuple $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ (cf. [Far93, LLPS18]) defined by

$$W_n(A) = \{(\varphi(A_1), \dots, \varphi(A_{\mathtt{g}})) \in \mathbb{S}_n^{\mathtt{g}} : \varphi \in \mathrm{UCP}_n(\mathcal{R})\} \subseteq \mathbb{S}_n^{\mathtt{g}}.$$

The matrix range $W(A)$ is easily seen to be matrix convex. In fact, assuming $\mathcal{D}_A$ is bounded, the matrix range is another description of the polar dual $\mathcal{D}_A^\circ$ of $\mathcal{D}_A$. For the reader's convenience the presentation that follows includes a proof of this fact. See Proposition 2.12. It follows from the work of Arveson [Arv72] that any compact matrix convex set in $\mathbb{C}$ is matrix affinely homeomorphic to the matrix range of a bounded operator on a separable Hilbert space. Theorem 2.4 thus extends this result to the case of a compact matrix convex set in $\mathbb{C}^{\mathtt{g}}$ for any $\mathtt{g} \in \mathbb{N}$.

2.2.3. *Duality.* Using Lemma 2.1, given a compact matrix convex set $\boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ and setting $Y_0 = I_{\mathrm{size}(Y)}$, the direct sums

$$\widehat{Y}_j = \bigoplus_{Y \in \boldsymbol{K}} Y_j,$$

produce a tuple of bounded operators and we associate to $\boldsymbol{K}$ the operator system

$$\widehat{\boldsymbol{K}} = \mathrm{span}\{\widehat{Y}_j : 0 \leq j \leq \mathtt{g}\}.$$

Theorem 2.4 below is a version of [WW99, Proposition 3.5] tailored to the present finite-dimensional setting and incorporating the spectrahedral point of view. It states that the operations $\widehat{\ }$ and $\check{\ }$ are dual to each other. A tuple $A \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ is **semi-finite** if it is an at most countable direct sum of matrix tuples; that is, $A = \oplus_{j \in J} A^{(j)}$, where $J \subseteq \mathbb{N}$, and for each $j \in J$ there is a positive integer n_j such that $A^{(j)} \in \mathbb{S}_{n_j}^{\mathtt{g}}$.

Theorem 2.4. *Suppose $A_1, \dots, A_{\mathtt{g}} \in \mathbb{S}_{\mathcal{H}}$ and let $\mathcal{R} = \mathrm{span}\{I_{\mathcal{H}}, A_1, \dots, A_{\mathtt{g}}\} \subseteq \mathcal{B}(\mathcal{H})$ denote the resulting finite-dimensional operator system and $\mathcal{D}_A$ the corresponding Hilbertian free spectrahedron.*

(1) *If $\mathcal{D}_A$ is bounded, then $\mathcal{R}$ and $\mathrm{A}(\mathcal{D}_A^\circ)$ are isomorphic operator systems;*

(2) *$\mathrm{UCP}(\mathcal{R})$ and $W(A)$ are matrix affinely homeomorphic (isomorphic as matrix convex sets) and, if $\mathcal{D}_A$ is bounded, then $\check{\mathcal{R}} := \mathcal{D}_A^\circ = W(A)$; and*

(3) *If $\mathcal{D}_A$ is bounded, then $\mathcal{R}$ and $\hat{\check{\mathcal{R}}}$ are isomorphic operator systems.*

Suppose $\boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ is a matrix convex set.

(a) *If $\boldsymbol{K}$ is closed and K_1 contains a neighborhood of 0, then there exists a semi-finite tuple $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}$ such that $\boldsymbol{K}$ is the Hilbertian free spectrahedron $\mathcal{D}_A$;*

(b) *If $\boldsymbol{K}$ is compact and K_1 contains a neighborhood of* 0, *then* $\mathrm{A}(\boldsymbol{K})$ *and* $\widehat{\boldsymbol{K}}$ *are isomorphic operator systems; and*

(c) *If $\boldsymbol{K}$ is compact and $0 \in K_1$, then* $\check{\widehat{\boldsymbol{K}}} := \mathcal{D}^{\circ}_{\widehat{Y}} = \boldsymbol{K}$.

Remark 2.5. As it stands, the operators $\widehat{Y}$ generically act on a non-separable Hilbert space $\mathscr{Y}$. However, the resulting operator system admits a representation using a semi-finite tuple; that is, there is a separable Hilbert space $\mathscr{E}$ and a semi-finite tuple $\widehat{E}$ acting on $\mathscr{E}$ such that $\operatorname{span}\{I_{\mathscr{Y}}, \widehat{Y}\}$ and $\operatorname{span}\{I_{\mathscr{E}}, \widehat{E}\}$ are isomorphic as operator systems via the unital map that sends $\widehat{Y}_j$ to $\widehat{E}_j$. Indeed, simply choose a countable dense graded set $\boldsymbol{F} \subseteq \boldsymbol{K}$ (meaning F_m is countable and dense in K_m for each m) and set $\widehat{E} = \oplus_{E \in \boldsymbol{F}} E$. A variation on this construction, along with Theorem 1.8, gives item (a) of Theorem 2.4, proved as Lemma 2.7 below. ■

Remark 2.6. From item (a), there exists a separable Hilbert space $\mathcal{H}$ and a tuple $A = (A_1, \ldots, A_{\mathrm{g}}) \in \mathbb{S}^{\mathrm{g}}_{\mathcal{H}}$ such that $\boldsymbol{K} = \mathcal{D}_A$. However, to identify the operator system $\mathrm{A}(\boldsymbol{K})$ concretely, one first uses item (c) of Theorem 2.4, which gives $\boldsymbol{K} = \mathcal{D}^{\circ}_{\widehat{Y}}$. From Lemma 2.2(5), the assumption that 0 is in the interior of $K_1 = \mathcal{D}^{\circ}_{widehatY}(1)$ implies $\mathcal{D}_{\widehat{Y}}$ is bounded so that, by item (1), $\mathrm{A}(\boldsymbol{K})$ is isomorphic, as an operator system, to $\mathcal{S} = \operatorname{span}\{I, \widehat{Y}_1, \ldots, \widehat{Y}_{\mathrm{g}}\}$. This circuitous route to the duality expressed in item (b) offers some insight into the perspective obtained by viewing matrix convex sets as Hilbertian spectrahedra.

There are various strategies for establishing Theorem 2.4 using Webster-Winkler duality. For instance, using items (b) and (2) and the Webster-Winkler duality between $\mathcal{R}$ and $\mathrm{A}(\mathrm{UCP}(\mathcal{R})$ gives $\hat{\hat{\mathcal{R}}} \cong \mathrm{A}(\check{\mathcal{R}}) \cong \mathrm{A}(\mathrm{UCP}(\mathcal{R})) \cong \mathcal{R}$. We will prove Theorem 2.4 without invoking duality and thus provide an alternate proof of Webster-Winkler duality in this finite-dimensional setting. ■

The proof of Theorem 2.4 occupies the remainder of this subsection. After proving item (a) (see Lemma 2.7), we state Propositions 2.8 and 2.9, which connect completely positive maps between finite-dimensional operator systems with Hilbertian free spectrahedra. Items (3) and (c) are then established as Lemmas 2.10 and 2.11 respectively, followed by items (2) and (1) as Proposition 2.12 and 2.13. As was noted in Remark 2.6, item (b) follows from item (c) ($\boldsymbol{K} = \mathcal{D}^{\circ}_{\widehat{Y}}$) and item (1), which together give

$$\mathrm{A}(\boldsymbol{K}) \cong \mathrm{A}(\mathcal{D}^{\circ}_{\widehat{Y}}) \cong \operatorname{span}\{I, \widehat{Y}_1, \ldots, \widehat{Y}_{\mathrm{g}}\}.$$

Lemma 2.7. *If $\boldsymbol{K}$ is a closed matrix convex set such that 0 is in the interior of K_1, then there is a semi-finite tuple $A = (A_1, \ldots, A_{\mathrm{g}}) \in \mathbb{S}_{\mathcal{H}}$ such that $\boldsymbol{K} = \mathcal{D}_A$.*

Proof. Fix a positive integer n and let $F_n = \mathbb{S}^{\mathrm{g}}_n \setminus K_n$. The set F_n is an open subset of Euclidean space and hence every open cover of F_n admits an at most countable subcover, since every open set in Euclidean space is a countable union of compact sets. For each

$G \in F_n$, there is, by Theorem 1.8, a tuple $A_G \in \mathbb{S}_n^{\mathrm{g}}$ such that $L_{A_G}(\boldsymbol{K}) \succeq 0$, but $L_{A_G}(G) \not\succeq 0$. Let $U_G = \{Y \in \mathbb{S}_n^{\mathrm{g}} : L_{A_G}(Y) \not\succeq 0\} \subseteq F_n$ and note U_G is open. Since $F_n = \cup_{G\in F_n} U_G$, there is a countable set $G_n \subseteq F_n$ such that $F_n = \cup_{G\in G_n} U_G$. Let $\boldsymbol{G} = \cup G_n$ and observe, if $Y \in \mathbb{S}_\ell^{\mathrm{g}} \setminus K_\ell$, then there is a $G \in G_\ell$ such that $Y \in U_G$ so that $L_{A_G}(Y) \not\succeq 0$.

Since 0 is in the interior of K_1, the polar dual K° is bounded by Lemma 2.2. Thus there is an M such that $\|A_G\| \le M$ for all $G \in \boldsymbol{G}$. Hence the operator $A = \oplus_\ell \oplus_{G\in G_\ell} A_G$ acting on the separable Hilbert space $\oplus_\ell \oplus_{G\in G_\ell} \mathbb{C}^\ell$ is bounded. By construction, $\boldsymbol{K} = \mathcal{D}_A$. Indeed, if $Y \notin K_\ell$, then there is $G \in F_\ell$ such that $Y \in U_G$ so that $L_{A_G}(Y) \not\succeq 0$ and thus $L_A(Y) \not\succeq 0$. Hence $Y \notin \mathcal{D}_A$. On the other hand, if $Y \in \boldsymbol{K}$, then $L_{A_G}(Y) \succeq 0$ for all n and $G \in G_n$ and therefore $L_A(Y) \succeq 0$. Hence $Y \in \mathcal{D}_A$. ∎

Proposition 2.8 ([HKM13, Proposition 2.6]). *Let $A = (A_1, \dots, A_{\mathrm{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathrm{g}}$. If the Hilbertian free spectrahedron $\mathcal{D}_A$ is bounded, then $I_{\mathcal{H}}, A_1, \dots, A_{\mathrm{g}}$ are linearly independent.*

Proposition 2.8 is proved, under the assumption that $\mathcal{H}$ is finite-dimensional as [HKM13, Proposition 2.6], but the argument works just as well in the case that $\mathcal{H}$ is infinite-dimensional and the A_j are self-adjoint operators instead of matrices.

Proposition 2.9 ([HKM13, Theorem 3.5] or [Zal17, Theorem 2.5]). *Let H, K denote Hilbert spaces and suppose $A = (A_1, \dots, A_{\mathrm{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathrm{g}}$ and $B = (B_1, \dots, B_{\mathrm{g}}) \in \mathbb{S}_K^{\mathrm{g}}$. Let $\mathcal{S}_A = \operatorname{span}\{I, A_1, \dots, A_n\}$ and $\mathcal{S}_B = \operatorname{span}\{I, B_1, \dots, B_n\}$. If $\mathcal{D}_A$ is bounded, then the unital linear map $\tau : \mathcal{S}_A \to \mathcal{S}_B$ sending*

$$A_i \mapsto B_i$$

for all i is completely positive if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$.

Proposition 2.9 is proved, under the assumption that H, K are finite-dimensional as [HKM13, Theorem 3.5], but the argument works verbatim in infinite dimensions, cf. [Zal17, Theorem 2.5]. Note that the boundedness hypotheses together with Proposition 2.8 implies that the map determined by $A_i \to B_i$ is well defined.

For $A = (A_1, \dots, A_{\mathrm{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathrm{g}}$, the Hilbertian spectrahedron $\mathcal{D}_A$ is a closed matrix convex and 0 is in the interior of $\mathcal{D}_A(1)$. Hence its polar dual $\mathcal{D}_A^\circ$ is bounded by Lemma 2.2. This observation is implicit in the statement of Lemma 2.10 below.

Lemma 2.10. *Fix $A = (A_1, \dots, A_{\mathrm{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathrm{g}}$, let $\mathcal{R} = \operatorname{span}\{I_{\mathcal{H}}, A_1, \dots, A_{\mathrm{g}}\}$ and let*

$$\widehat{B_j} = \bigoplus_{B\in\mathcal{D}_A^\circ} B_j \quad \textit{and} \quad I = \bigoplus_{B\in\mathcal{D}_A^\circ} I_{\operatorname{size}(B)}$$

acting on the Hilbert space $\oplus_{B\in\mathcal{D}_A^\circ} \mathbb{C}^{\operatorname{size}(B)}$.

If the Hilbertian free spectrahedron $\mathcal{D}_A$ is bounded, then $\mathcal{R}$ and $\widehat{\widehat{\mathcal{R}}} = \operatorname{span}\{I, \widehat{B}_1, \dots, \widehat{B}_{\mathrm{g}}\}$ are isomorphic operator systems.

Proof. Since $\mathcal{D}_A$ is assumed bounded, Proposition 2.8 implies the operators $I_{\mathcal{H}}, A_1, \dots, A_{\mathtt{g}}$ are linearly independent. Hence there is a linear map $\tau : \mathcal{R} \to \widehat{\widehat{\mathcal{R}}}$ determined by

$$I_n \mapsto I$$
$$A_i \mapsto \widehat{B}_i.$$

Let $X \in \mathcal{D}_A$ be given. By the definition of the polar dual, $L_B(X) \succeq 0$ for all $B \in \mathcal{D}_A^\circ$. Hence $L_{\widehat{B}}(X) \succeq 0$ and thus $\mathcal{D}_A \subseteq \mathcal{D}_{\widehat{B}}$. By Proposition 2.9, the map τ is ucp. On the other hand, if $X \in \mathcal{D}_{\widehat{B}}$, then $L_B(X) \succeq 0$ for all $B \in \mathcal{D}_A^\circ$ and therefore $X \in \mathcal{D}_A^{\circ\circ} = \mathcal{D}_A$ by Lemma 2.2. Hence $\mathcal{D}_{\widehat{B}} \subseteq \mathcal{D}_A$. Consequently, $\mathcal{D}_{\widehat{B}}$ is bounded and thus $I, \widehat{B}_1, \dots, \widehat{B}_{\mathtt{g}}$ is an independent set by Proposition 2.8. Hence τ is onto and τ^{-1} is also ucp and therefore $\mathcal{R}$ and $\widehat{\widehat{\mathcal{R}}}$ are isomorphic operator systems. ∎

Lemma 2.11. *If $\boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ is a compact matrix convex set and $0 \in K_1$, then $\boldsymbol{K} = \mathcal{D}_{\widehat{Y}}^\circ$, where*

$$\widehat{Y} = \bigoplus_{Y \in \boldsymbol{K}} Y.$$

Note that the compactness assumption ensures that $\widehat{Y}$ is a bounded operator.

Proof. By Lemma 2.2, $\boldsymbol{K}$ equals its bipolar $\boldsymbol{K}^{\circ\circ}$. We will show that $\boldsymbol{K}^\circ$ equals $\mathcal{D}_{\widehat{Y}}$. Letting $\mathscr{Y}$ denote the Hilbert space that $\widehat{Y}$ acts on, note that by definition, for any n and $B \in K_n^\circ$

$$L_B(\widehat{Y}) = I_n \otimes I_{\mathscr{Y}} + \sum_{i=1}^{\mathtt{g}} B_i \otimes \widehat{Y}_i \succeq 0,$$

which (after applying the canonical shuffle) implies that B lies in $\mathcal{D}_{\widehat{Y}}$. Hence, $\boldsymbol{K}^\circ \subseteq \mathcal{D}_{\widehat{Y}}$.

Now suppose $B \in \mathcal{D}_{\widehat{Y}}(n) \setminus K_n^\circ$. Since $\boldsymbol{K}^\circ$ is closed, by the Hahn-Banach Theorem 1.8, there is a monic linear pencil L_A defined by a tuple $A \in \mathbb{S}_n^{\mathtt{g}}$ such that L_A is positive semidefinite on $\boldsymbol{K}^\circ$, but not positive semidefinite at B. Since L_A is positive semidefinite on $\boldsymbol{K}^\circ$, the tuple A belongs to $\boldsymbol{K}^{\circ\circ} = \boldsymbol{K}$. But then A is a direct summand in $\widehat{Y}$, so $L_A(B) \succeq 0$, which is a contradiction. Hence, $\boldsymbol{K}^\circ = \mathcal{D}_{\widehat{Y}}$ and $\boldsymbol{K} = \boldsymbol{K}^{\circ\circ} = \mathcal{D}_{\widehat{Y}}^\circ$. ∎

Proposition 2.12 below interprets Proposition 2.9 as follows.

Proposition 2.12. *For $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$, the (graded) sets $W(A)$ and $\mathrm{UCP}(\mathcal{R})$ are isomorphic as matrix convex sets. Moreover, if $\mathcal{D}_A$ is bounded, then $W(A) = \mathcal{D}_A^\circ$.*

Proof. For notational ease, let $A_0 = I_{\mathcal{H}}$. If $\varphi \in \mathrm{UCP}_t(\mathcal{R})$ and $V : \mathbb{C}^s \to \mathbb{C}^t$ is an isometry, then $V^*\varphi V : \mathcal{R} \to M_s$ defined by $V^*\varphi V(R) = V^*\varphi(R)V$ for $R \in \mathcal{R}$ is easily seen to be unital and completely positive. Indeed, for $R \in M_m(\mathcal{R})$ given by

$$R = \sum Y_j \otimes A_j,$$

where $(Y_0, Y_1, \dots, Y_{\mathtt{g}}) \in \mathbb{S}_m^{\mathtt{g}}$,

$$1_m \otimes (V^*\varphi V)(R) = \sum Y_j \otimes V^*\varphi(A_j)V = (I_m \otimes V)^*(1_m \otimes \varphi)(R)(I_m \otimes V).$$

Thus, if $R \succeq 0$, then, since φ is ucp, $1_m \otimes \varphi(R)$ is positive semidefinite and hence so is $1_m \otimes (V^*\varphi V)(R)$. Therefore, $V^*\varphi V \in \mathrm{UCP}_s(\mathcal{R})$ and $\mathrm{UCP}(\mathcal{R})$ is matrix convex. It now follows that $W(A)$ is matrix convex since $V^*\varphi(A_j)V = V^*\varphi V(A_j)$ for each $1 \le j \le \mathtt{g}$.

The sets $\mathrm{UCP}_t(\mathcal{R})$ are bounded since ucp maps have norm one. It is straightforward to see the (norm) limit of ucp maps $\varphi_k : \mathcal{R} \to M_t$ is again ucp. Hence $\mathrm{UCP}_t(\mathcal{R})$ is closed and hence compact.

Given positive integer t, define $\Psi_t : \mathrm{UCP}_t(\mathcal{R}) \to W_t(A)$ as follows. For $\varphi \in \mathrm{UCP}_t(\mathcal{R})$, let

$$\Psi_t(\varphi) = \big(\varphi(A_1), \dots, \varphi(A_{\mathtt{g}})\big) \in W_t(A).$$

From the definitions, Ψ_t is bijective. It is immediate that Ψ_t is continuous and thus its range, $W_t(A)$, is also compact and Ψ_t is a homeomorphism.

To complete the proof that $\mathrm{UCP}(\mathcal{R})$ and $W(A)$ are isomorphic as matrix convex sets, it remains to show that Ψ is matrix affine. To this end, suppose r is a positive integer and finitely many $V_i : \mathbb{C}^{s_i} \to \mathbb{C}^t$ and $\varphi_i \in \mathrm{UCP}_{s_i}(\mathcal{R})$ such that $\sum V_i^* V_i = I$ are given and observe

$$\Psi_t(\sum V_i^*\varphi_i V) = [\sum V_i^*\varphi_i V](A) = \sum V_i^*\varphi_i(A)V_i = \sum V_i^*\Psi_{s_i}(\varphi_i)V_i.$$

Thus Ψ_t is matrix affine.

Now suppose $\mathcal{D}_A$ is bounded. Using Proposition 2.9, $B \in \mathcal{D}_A^\circ(t)$ if and only if

$$L_B(X) = I_t \otimes I_n + \sum_{i=1}^{\mathtt{g}} B_j \otimes X_j \succeq 0$$

for all n and $X \in \mathcal{D}_A(n)$ if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$ if and only if the unital map $\tau : \mathcal{R} \to M_t$ sending A_i to B_i is completely positive if and only if

$$\big(\tau(A_1), \dots, \tau(A_{\mathtt{g}})\big) = (B_1, \dots, B_{\mathtt{g}}) \in W_t(A).$$

Hence $W(A) = \mathcal{D}_A^\circ$. ■

Proposition 2.13. *Let $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ and define $\mathcal{R} = \mathrm{span}\{I_{\mathcal{H}}, A_1, \dots, A_{\mathtt{g}}\}$. If $\mathcal{D}_A$ is bounded, then $\mathcal{R}$ and the space $\mathrm{A}(\mathcal{D}_A^\circ)$ of matrix affine maps on the polar dual of $\mathcal{D}_A$ are isomorphic operator systems.*

Proof. The boundedness assumption on $\mathcal{D}_A$ implies that $I_{\mathcal{H}}, A_1, \dots, A_{\mathtt{g}}$ are linearly independent by Proposition 2.8. Thus we may construct a map $\Phi : \mathcal{R} \to \mathrm{A}(\mathcal{D}_A^\circ)$ sending a $\tilde{b} \in \mathcal{R}$ to a matrix affine map $\Phi[\tilde{b}] = (\Phi[\tilde{b}]_t)_{t\in\mathbb{N}} : \mathcal{D}_A^\circ \to \mathbb{M}$ as follows. Given

$$\tilde{b} = \lambda\, I_{\mathcal{H}} + \sum_{i=1}^{\mathtt{g}} b_i A_i \in \mathcal{R},$$

a positive integer t and $X \in \mathcal{D}_A^\circ(t) \subseteq \mathbb{S}_t^{\mathsf{g}}$, let

$$\Phi[\tilde{b}]_t(X) = \lambda\, I_t + \sum_{i=1}^{\mathsf{g}} b_i X_i \in M_t.$$

Given finitely many $V_j : \mathbb{C}^{s_i} \to \mathbb{C}^t$ and $X^{(j)} \in \mathcal{D}_A^\circ(s_j)$ such that $\sum V_j^* V_j = I_t$,

$$\Phi[\tilde{b}]_t(\sum V_j^* X^{(j)} V_j) = \lambda I_t + \sum_{i,j} b_i V_j^* X_i^{(j)} V_j = \sum_j V_j^*(\lambda I_t + \sum b_i X_i^{(j)}) V_j = \sum V_j^* \Phi[\tilde{b}]_{s_j}(X) V_j.$$

Thus $\Phi[\tilde{b}]$ is matrix affine. From the construction, Φ is continuous.

To prove injectivity of Φ suppose $\Phi[\tilde{b}]$ is the zero map for some $\tilde{b} = \lambda + \sum_{i=1}^{\mathsf{g}} b_i A_i$. In particular,

$$\lambda + \sum_{i=1}^{\mathsf{g}} b_i x_i = 0$$

for all $x \in \mathcal{D}_A^\circ(1)$. Plugging in $x = 0$ we obtain that $\lambda = 0$. Since $\mathcal{D}_A$ is bounded, we have by Lemma 2.2 that 0 is in the interior of $\mathcal{D}_A^\circ(1)$. Hence $b_i = 0$ for $1 \le i \le \mathsf{g}$ and thus $\tilde{b} = 0$.

Next, we show that Φ is bijective. By inspection of the proof of the duality [WW99, Proposition 3.5], the operator system $\mathrm{A}(\mathcal{D}_A^\circ)$ is isomorphic (as a vector space) to the space of continuous affine maps on $\mathcal{D}_A^\circ(1)$. Hence, $\mathrm{A}(\mathcal{D}_A^\circ)$ and $\mathcal{R}$ have the same dimension $\mathsf{g}+1$, which proves that the injective map Φ is bijective.

We now prove that Φ is completely positive. Accordingly, fix a positive integer m and consider $\Phi_m : M_m(\mathcal{R}) \to M_m(\mathrm{A}(\mathcal{D}_A^\circ)) \cong \mathrm{A}(\mathcal{D}_A^\circ, M_m)$. A self-adjoint $\widetilde{B} \in M_m(\mathcal{R})$ has (after applying the canonical shuffle) a representation of the form

$$\widetilde{B} = B_0 \otimes I_{\mathcal{H}} + \sum_{i=1}^{\mathsf{g}} B_i \otimes A_i$$

for $B_0, \dots, B_{\mathsf{g}} \in \mathbb{S}_m$. Applying Φ_m to $\widetilde{B}$, produces the matrix affine mapping

$$(\Phi_m[\widetilde{B}]_t)_{t\in\mathbb{N}} = \Phi_m[\widetilde{B}] : \mathcal{D}_A^\circ \to \mathbb{M}(M_m),$$

where, for $X \in \mathcal{D}_A^\circ(t)$,

$$\Phi_m[\widetilde{B}]_t(X) = B_0 \otimes I_t + \sum_{i=1}^{\mathsf{g}} B_i \otimes X_i.$$

Now suppose $\widetilde{B} \succeq 0$. By [HKM17, Lemma 3.6], which requires the assumption that $\mathcal{D}_A(1)$ is bounded, $B_0 \succeq 0$. We include the argument to keep the presentation self contained. If $B_0 \not\succeq 0$, then there is a vector v such that $\langle B_0 v, v\rangle < 0$. Letting $V : \mathbb{C}^m \otimes \mathbb{C}v \to \mathbb{C}^m \otimes \mathcal{H}$ denote the inclusion,

$$\langle B_0 v, v\rangle \otimes I_n + \sum_{i=1}^{\mathsf{g}} \langle B_i v, v\rangle \otimes A_i = (V \otimes I_n)^* \Big(B_0 \otimes I_n + \sum_{i=1}^{\mathsf{g}} B_i \otimes A_i \Big) (V \otimes I_n)^* \succeq 0,$$

which implies that $\sum_{i=1}^{\mathrm{g}}\langle B_i v, v\rangle \otimes A_i \succ 0$. Thus the (nonzero) point $t(\langle B_i v, v\rangle)_{i=1}^{\mathrm{g}} \in \mathbb{R}^{\mathrm{g}}$ lies in $\mathcal{D}_A(1)$ for all $t > 0$, contradicting the boundedness of $\mathcal{D}_A(1)$. Hence, $B_0 \succeq 0$.

For $\epsilon > 0$, note that $(B_0 + \epsilon) \otimes I_n + \sum_{i=1}^{\mathrm{g}} B_i \otimes A_i \succ 0$. Since $B_{0,\epsilon} := B_0 + \epsilon \succ 0$, its positive square root $B_{0,\epsilon}^{\frac{1}{2}}$ is invertible and therefore,

$$I_m \otimes I_n + \sum_{i=1}^{\mathrm{g}} B_{0,\epsilon}^{-1/2}\, B_i\, B_{0,\epsilon}^{-1/2} \otimes A_i = (B_{0,\epsilon}^{-1/2} \otimes I_n)\Big(B_{0,\epsilon} \otimes I + \sum_{i=1}^{\mathrm{g}} B_i \otimes A_i\Big)(B_{0,\epsilon}^{-1/2} \otimes I_n) \succeq 0.$$

Thus the tuple $\overline{B}_\epsilon = \big(B_{0,\epsilon}^{-1/2}\, B_i\, B_{0,\epsilon}^{-1/2}\big)_{i=1}^{\mathrm{g}}$ lies in $\mathcal{D}_A(m)$ and hence, by definition of the polar dual, $\Phi_m[\overline{B}_\epsilon]_t(X) \succeq 0$ for all $X \in \mathcal{D}_A^\circ$. Thus

$$\Phi_m[B_\epsilon]_t(X) = (B_{0,\epsilon}^{1/2} \otimes I_n)\, \Phi_m[\overline{B}_\epsilon]_t(X)\, (B_{0,\epsilon}^{1/2} \otimes I_n) \succeq 0$$

for all $X \in \mathcal{D}_A^\circ$. Letting $\epsilon > 0$ tend to 0 gives $\Phi_m[B]_t(X) \succeq 0$, proving that Φ is indeed completely positive.

It remains to prove that the inverse of Φ is also completely positive. So assume $\Phi_m[\widetilde{B}] : \mathcal{D}_A^\circ \to \mathbb{M}(M_m)$ is positive. Equivalently,

$$\Phi_m[\widetilde{B}]_t(X) = B_0 \otimes I_t + \sum_{i=1}^{\mathrm{g}} B_i \otimes X_i \succeq 0$$

for all t and $X \in \mathcal{D}_A^\circ(t)$. If follows from $0 \in \mathcal{D}_A^\circ$ that B_0 is positive semidefinite. By a similar argument as before, the matrix B_0 can be assumed positive definite so that we have

$$I_m \otimes I_t + \sum_{i=1}^{\mathrm{g}} B_0^{-1/2} B_i B_0^{-1/2} \otimes X_i \succeq 0$$

for all $X \in \mathcal{D}_A^\circ$. Hence, the tuple $\overline{B} = \big(B_0^{-1/2} B_i B_0^{-1/2}\big)_{i=1}^{\mathrm{g}}$ lies in the polar dual of $\mathcal{D}_A^\circ$. But then by, Lemma 2.2(4) (the Bipolar Theorem [EW97, Corollary 5.5]), $\overline{B}$ lies in $\mathcal{D}_A$; that is,

$$0 \preceq L_A(\overline{B}) = I_{\mathcal{H}} \otimes I_m + \sum_{i=1}^{\mathrm{g}} A_i \otimes \overline{B}_i \cong I_m \otimes I_{\mathcal{H}} + \sum_{i=1}^{\mathrm{g}} B_0^{-1/2} B_i B_0^{-1/2} \otimes A_i$$

and thus

$$B_0 \otimes I_t + \sum_{i=1}^{\mathrm{g}} B_i \otimes A_i = (B_0^{1/2} \otimes I_n)\Big(I_m \otimes I_n + \sum_{i=1}^{\mathrm{g}} B_0^{-1/2} B_i B_0^{-1/2} \otimes A_i\Big)(B_0^{1/2} \otimes I_n) \succeq 0,$$

Hence $0 \preceq \widetilde{B} \in M_m(\mathcal{R})$ as desired. ■

2.3. **Duality in the Γ-convex setting.** Let again $\Gamma = (\gamma_1, \ldots, \gamma_{\mathtt{r}})$ be a tuple of symmetric noncommutative polynomials with $\gamma_j = x_j$ for $1 \le j \le \mathtt{g} \le \mathtt{r}$. In this section we introduce Γ-analogs of operator systems and ucp maps and prove a categorical duality resembling [WW99, Proposition 3.5] and Theorem 2.4.

Before proceeding, observe that the analog of Lemma 2.1 does not necessarily hold for Γ-convex sets. Hence, compactness (level-wise) of a Γ-convex set $\boldsymbol{K}$ does not necessarily imply it is bounded (uniformly), explaining the need, in many of the results that follow, for the additional boundedness hypothesis.

2.3.1. *The matrix state space, matrix ranges and ucp maps in the* Γ*-convex setting.* Given a tuple $A = (A_1, \ldots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$, let $\mathcal{R} = \mathcal{R}_A^\Gamma$ denote the operator system

$$\operatorname{span}\{\gamma_j(A) \mid j = 0, 1, \ldots, \mathtt{r}\}, \tag{2.3}$$

where, for notational purposes, $\gamma_0(A) = I_{\mathcal{H}}$. We call $\mathcal{R} = \mathcal{R}_A^\Gamma$ a Γ**-operator system**.

Definition 2.14. Given a Hilbert space $\mathcal{K}$, a linear map $\varphi : \mathcal{R}_A^\Gamma \to \mathcal{B}(\mathcal{K})$ is a Γ**-concomitant** provided,

$$\varphi(\gamma_i(A)) = \gamma_i(\operatorname{vec}\varphi(A)),$$

for each $1 \le i \le \mathtt{r}$, where

$$\gamma_i(A) = \gamma_i(A_1, \ldots, A_{\mathtt{g}}), \quad \operatorname{vec}\varphi(A) = (\varphi(A_1), \ldots, \varphi(A_{\mathtt{g}})). \tag{2.4}$$

A linear map $\varphi : \mathcal{R}_A^\Gamma \to M_n$ is a Γ**-ucp map** if it is ucp and a Γ-concomitant. For notational purposes, set $\Gamma_{\operatorname{vec}\varphi(A)} = \big(\gamma_1(\operatorname{vec}\varphi(A)), \ldots, \gamma_{\mathtt{r}}(\operatorname{vec}\varphi(A))\big)$ and

$$L_{\Gamma_{\operatorname{vec}\varphi(A)}}(y) = I_{\mathcal{H}} + \sum_{i=1}^{\mathtt{r}} \varphi(\gamma_i(A))\, y_j. \qquad \blacksquare$$

Remark 2.15. Suppose $A = (A_1, \ldots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$. Assuming, $\{\gamma_j(A) : 0 \le j \le \mathtt{r}\}$ is linearly independent, which, by Proposition 2.8, is the case when $\mathcal{D}_{\Gamma(A)}$ is bounded, and given $(B_1, \ldots, B_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{K}}^{\mathtt{g}}$ there exists a Γ-concomitant uniquely determined by $\gamma_j(A) \mapsto \gamma_j(B)$.

The connection between spectrahedral inclusions $\mathcal{D}_A \subseteq \mathcal{D}_B$ and ucp maps exposed in Proposition 2.9 extends to Γ-ucp maps. Assuming $\mathcal{D}_{\Gamma(A)}$ is bounded, a Γ-concomitant $\varphi : \mathcal{R}_A^\Gamma \to M_m$ is Γ-ucp if and only if $\mathcal{D}_{\Gamma(A)} \subseteq \mathcal{D}_{\Gamma_{\operatorname{vec}\varphi(A)}}$. In particular, φ is ucp if and only if $Y \in \mathbb{S}_m^{\mathtt{r}}$ and

$$L_{\Gamma(A)}(Y) = I_d \otimes I_m + \sum_{i=1}^{\mathtt{r}} \gamma_i(A) \otimes Y_j \succeq 0,$$

implies

$$L_{\Gamma_{\operatorname{vec}\varphi(A)}}(Y) = I_n \otimes I_m + \sum_{i=1}^{\mathtt{r}} \varphi(\gamma_i(A)) \otimes Y_j \succeq 0. \qquad \blacksquare$$

Since they preserve the unit and the positivity structure of Γ-operator systems, Γ-ucp maps are the **morphisms** of Γ-operator systems. An **isomorphism of Γ-operator systems** is a bijective Γ-ucp map whose inverse is also Γ-ucp.

Given a Γ-operator system $\mathcal{R}$ as in (2.3) and a positive integer n, let $\mathrm{UCP}_n^\Gamma(\mathcal{R})$ denote the Γ-ucp maps $\varphi : \mathcal{R} \to M_n$ and let $\mathrm{UCP}^\Gamma(\mathcal{R}) = (\mathrm{UCP}_n^\Gamma(\mathcal{R}))_n$. A pair (φ, V), where $\varphi \in \mathrm{UCP}_t^\Gamma(\mathcal{R})$ and $V : \mathbb{C}^s \to \mathbb{C}^t$ is an isometry, is a **Γ-pair** provided

$$V^*\gamma_i(\operatorname{vec}\varphi(A))V = \gamma_i(V^* \operatorname{vec}\varphi(A)V),$$

for all $1 \le j \le \mathtt{r}$. It is shown in Proposition 2.16 below that $\mathrm{UCP}^\Gamma(\mathcal{R})$ is Γ-convex. That is, if (φ, V) is a Γ-pair, then $V^*\varphi V : \mathcal{R} \to M_s$ defined by $V^*\varphi V(R) = V^*\varphi(R)V$ is Γ-ucp.

The **Γ-matrix range** of a tuple $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ is $\check{\mathcal{R}} = W^\Gamma(A) = (W_n^\Gamma(A))_n$, where

$$W_n^\Gamma(A) = \{(\varphi(A_1), \dots, \varphi(A_{\mathtt{g}})) \in \mathbb{S}_n^{\mathtt{g}} \mid \varphi \in \mathrm{UCP}_n^\Gamma(\mathcal{R}_A^\Gamma)\}. \tag{2.5}$$

Since a ucp map φ is bounded with norm one, it follows that $W^\Gamma(A)$ is bounded.

In the case that $\mathtt{g} = \mathtt{r}$ and thus $\Gamma(x) = x$, the graded set $W(A) = W^\Gamma(A)$ is the matrix range of A from Subsection 2.2.1. A pair (B, V), where $B \in W_t^\Gamma(A)$ and $V : \mathbb{C}^s \to \mathbb{C}^t$ is an isometry, is a **Γ-pair** for $W^\Gamma(A)$ if $B = \operatorname{vec}\varphi(A)$, where $\varphi \in \mathrm{UCP}_t^\Gamma(\mathcal{R})$ and (φ, V) is a Γ-pair for $\mathrm{UCP}^\Gamma(\mathcal{R})$.

Given Γ-convex sets $\boldsymbol{K}$ and $\boldsymbol{K}'$ a sequence $\Phi = (\Phi_n)$ of maps $\Phi_n : K_n \to K_n'$ is **matrix Γ-affine** if $V^*\Phi_n(X)V = \Phi_m(V^*XV)$ for each Γ-pair (X, V) with $X \in K_n$ and $V : \mathbb{C}^m \to \mathbb{C}^n$. The Γ-convex sets $\boldsymbol{K}$ and $\boldsymbol{K}'$ are **isomorphic** if there exists a matrix Γ-affine map Φ such that each Φ_n is a homeomorphism.

Proposition 2.16. *If $(A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$, then $W^\Gamma(A)$ and $\mathrm{UCP}^\Gamma(\mathcal{R}_A^\Gamma)$ are isomorphic closed and bounded Γ-convex sets.*

Proof. Let $\mathcal{R} = \mathcal{R}^\Gamma(\mathcal{R})$. To prove $\mathrm{UCP}(\mathcal{R})$ is Γ-convex, suppose $\varphi \in \mathrm{UCP}_t(\mathcal{R})$ and $V : \mathbb{C}^s \to \mathbb{C}^t$ is an isometry such that (φ, V) is a Γ-pair. It is immediate, from the definition of Γ-pair, that $\psi = V^*\varphi V$ is a Γ-concomitant. Indeed, by definition of a Γ-pair, $V^*\gamma_i(\operatorname{vec}\varphi(A))V = \gamma_i(V^* \operatorname{vec}\varphi(A)V)$ for $0 \le i \le \mathtt{r}$. Hence, as φ is a Γ-concomitant,

$$\psi(\gamma_i(A)) = V^*\varphi(\gamma_i(A))V = V^*\gamma_i(\operatorname{vec}\varphi(A))V = \gamma_i(V^* \operatorname{vec}\varphi(A)V) = \gamma_i(\operatorname{vec}\psi(A)).$$

To see that ψ is ucp, let $R = \sum Y_j \otimes \gamma_j(A) \in M_m(\mathcal{R})$ be given and observe,

$$\psi_m(R) = (I_m \otimes V)^*\varphi_m(R)(I_m \otimes V).$$

Hence if R is positive semidefinite, then so is $\psi_m(R)$. Thus ψ is ucp. Hence $\mathrm{UCP}^\Gamma(\mathcal{R})$ is Γ-convex. That the maps $\Psi_s : \mathrm{UCP}_s^\Gamma(\mathcal{R}) \to W_r^\Gamma(A)$ given by $\Psi_r(\varphi) = \operatorname{vec}\varphi(A)$ are affine bijections follows from the definitions. The details are omitted.

The graded set $\mathrm{UCP}^\Gamma(\mathcal{R})$ is bounded as ucp maps have norm one. It is straightforward to check that $\mathrm{UCP}^\Gamma_s(\mathcal{R})$ is closed so that $\mathrm{UCP}^\Gamma_s(\mathcal{R})$ is compact. Since Ψ_s is evidently continuous, by compactness of its domain, $W^\Gamma(A)$ is also compact and Ψ_s is a homeomorphism. ■

Given a tuple $B \in \mathbb{S}^{\mathtt{r}}_{\mathcal{H}}$, let

$$\mathcal{D}^\Gamma_B = \big(\{X \in \mathbb{S}^{\mathtt{g}}_m : I \otimes I + \sum_{j=1}^{\mathtt{r}} \gamma_j(X) \otimes B_j \succeq 0\}\big)_{m\in\mathbb{N}}.$$

We call $\mathcal{D}^\Gamma_B$ a **Hilbertian Γ-spectrahedron**.

Proposition 2.17. *The graded set $\mathcal{D}^\Gamma_B$ is Γ-convex.*

Suppose $\boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ is closed and Γ-convex and $\boldsymbol{J} \subseteq \mathbb{S}^{\mathtt{r}}$ is a closed matrix convex set that contains $\Gamma(\boldsymbol{K})$. If

(a) $0 \in K_1$;
(b) $\Gamma(0) = 0$;
(c) 0 *is in the interior of* J_1; *and*
(d) $X \in \mathbb{S}^{\mathtt{g}}$ *and* $\Gamma(X) \in \boldsymbol{J}$ *implies* $X \in \boldsymbol{K}$,

then there exists a semi-finite tuple $A \in \mathbb{S}^{\mathtt{r}}_{\mathcal{H}}$ such that $\boldsymbol{K} = \mathcal{D}^\Gamma_A$.

Remark 2.18. Note that $\boldsymbol{J} = \overline{\mathrm{matco}}(\Gamma(\boldsymbol{K}))$ contains $\Gamma(\boldsymbol{K})$. In any case, the condition of item (d) of Proposition 2.17 is, given the assumption that $\Gamma(\boldsymbol{K}) \subseteq \boldsymbol{J}$, equivalent to $\boldsymbol{K} = \Gamma^{-1}(\boldsymbol{J})$. ■

Example 2.19. In [JKMMP21, Theorem 2.6] a condition is given under which $\mathrm{matco}(\Gamma(\boldsymbol{K}))$ contains a neighborhood of 0. Namely, if $\Gamma(0) = 0$ and $\boldsymbol{K}$ is a Γ-convex set containing 0, then 0 is in the interior of $\mathrm{matco}(\Gamma(\boldsymbol{K}))(1)$ if and only if the real span of $\{\gamma_j \mid 1 \le j \le \mathtt{r}\}$ does not contain a polynomial $q \in \mathbb{C}\langle x\rangle$ such that $q(X) \succeq 0$ for all $X \in \boldsymbol{K}$.

This certificate applied to the case of xy-convexity, where $\Gamma = \{x, y, xy+yx, i(xy-yx)\}$, implies if $\boldsymbol{K}$ is xy-convex, K_1 contains a neighborhood of 0 and, for some n, there is a pair $(X,Y) \in K_n$ such that $XY \neq YX$, then $\mathrm{matco}(\Gamma(\boldsymbol{K}))(1)$ contains a neighborhood of 0. In particular, the result holds if K_2 also contains a neighborhood of 0 as we now show.

Suppose $a, b, c \in \mathbb{R}$ and $ax + by + cxy \ge 0$ for all (x, y) in the non-empty interior of K_1. For (x, y) sufficiently small in K_1, we also have $(-x,-y), (x,-y)$ and $(-x, y) \in K_1$. Thus $-ax - ay + c\,xy \ge 0$. Hence $c\,xy \ge 0$ for all (x, y) sufficiently small. Thus $c = 0$. A similar argument gives $a = b = 0$ too. Thus, if we start with $a, b, c, d \in \mathbb{R}$ such that $q(X, Y) = aX + bY + c\,(XY + YX) + i\,d(XY - YX) \succeq 0$ on $\boldsymbol{K}$ and K_1 has an interior, then we conclude that $a = b = c = 0$. Thus $q(x, y) = i\,d(xy - yx)$, for some real number d and now $q(x, y) \succeq 0$ on $\boldsymbol{K}$ implies if $(x, y) \in K_n$ then $i\,d(xy - yx) \succeq 0$. If there is an n such that K_n contains a pair (X, Y) such that $XY - YX \neq 0$, then $i\,(XY - YX)$ has both positive and negative eigenvalues and thus $d = 0$. ■

Example 2.20. Let $K_n = \{(X,Y) \in \mathbb{S}_n^2 : I - X^2 - Y^4 \succeq 0\}$ and $\boldsymbol{K} = (K_n)_n$. It is straightforward to verify that $\boldsymbol{K}$ is Γ-convex, where $\Gamma(x,y) = (x,y,y^2)$; that is, $\boldsymbol{K}$ is y^2-convex.

Let $\boldsymbol{J}$ denote the closure of the matrix convex hull of $\Gamma(\boldsymbol{K})$ and the point $(0,0,-1) \in \mathbb{S}_1^3 = \mathbb{R}^3$. Let $E_k = (0,0,I_k)$ for positive integers k. In particular, $-E_k \in J_k$. Since J_1 contains the points $(\pm 1,0,0)$, $(0,\pm 1,1)$ and $(0,0,-1)$ and is convex, it contains a neighborhood of 0. A routine argument shows that elements of $\boldsymbol{J}$ have the form

$$\begin{pmatrix} V^* & W^* \end{pmatrix} \begin{pmatrix} \Gamma(X,Y) & 0 \\ 0 & -E_k \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix},$$

for some positive integers k, ℓ, tuple $(X,Y) \in K_\ell$, and maps V and W such that $V^*V + W^*W = I$.

Suppose $\Gamma(X,Y) = (X,Y,Y^2)$ is in $\boldsymbol{J}$. Thus there exists a sequence $(X_n,Y_n) \in K_{m_n}$ and $E_{k_n} \in J_{k_n}$ and isometries

$$\mathcal{V}_n = \begin{pmatrix} V_n \\ W_n \end{pmatrix}$$

such that

$$\mathcal{V}_n^* \begin{pmatrix} \Gamma(X_n,Y_n) & 0 \\ 0 & -E_{k_n} \end{pmatrix} \mathcal{V}_n = (V_n^* X_n V_n, V_n^* Y_n V_n, V_n^* Y_n^2 V_n - W_n^* W_n)$$

converges to (X,Y,Y^2). Set

$$Z_n = \mathcal{V}_n^* \begin{pmatrix} Y_n^2 & 0 \\ 0 & -I_{k_n} \end{pmatrix} \mathcal{V}_n = V_n^* Y_n^2 V_n - W_n^* W_n.$$

Thus Z_n converges to Y^2 and therefore Z_n^2 converges to Y^4. Moreover,

$$\begin{aligned} Z_n^2 &\preceq \mathcal{V}_n^* \begin{pmatrix} Y_n^2 & 0 \\ 0 & -I_{k_n} \end{pmatrix}^2 \mathcal{V}_n \\ &= \mathcal{V}_n^* \begin{pmatrix} Y_n^4 & 0 \\ 0 & I_{k_n} \end{pmatrix} \mathcal{V}_n \\ &= V_n^* Y_n^4 V_n + W_n^* W_n. \end{aligned}$$

Thus, as $(X_n,Y_n) \in \boldsymbol{K}$ so that $X_n^2 + Y_n^4 \preceq I$,

$$(V_n^* X_n V_n)^2 + Z_n^2 \preceq V_n^* \big(X_n^2 + Y_n^4\big) V_n + W_n^* W_n \preceq V_n^* V_n + W_n^* W_n = I.$$

Hence, taking the limit on n gives $X^2 + Y^4 \preceq I$. Since $\boldsymbol{K}$ is closed, $(X,Y) \in \boldsymbol{K}$ and consequently $\boldsymbol{J}$ and Γ satisfy the hypotheses of Proposition 2.17. It follows that there exists a semi-finite tuple A such that $\boldsymbol{K} = \mathcal{D}_A^\Gamma$.

Of course, as is easily verified, $\boldsymbol{K}$ is determined by the Γ-linear matrix inequality,

$$I + B_0 x + B_1 y + B_2 y^2 = \begin{pmatrix} 1 & x & y^2 \\ x & 1 & 0 \\ y^2 & 0 & 1 \end{pmatrix}$$

so that $\boldsymbol{K} = \mathcal{D}_B^\Gamma$.

As a further remark, the graded set $\boldsymbol{J} = \overline{\mathrm{matco}}(\Gamma(K))$ satisfies all the hypotheses of the Proposition, save for item (c) since if $(x,y,z) \in \mathrm{matco}(\Gamma(\boldsymbol{K}))(1)$, then $z \geq 0$. ■

Proof of Proposition 2.17. If (X,V) is Γ-pair and $X \in \mathcal{D}_B^\Gamma$, then

$$\begin{aligned} I \otimes I + \sum_{j=1}^{\mathtt{r}} \gamma_j(V^* X V) \otimes B_j &= I \otimes I + \sum_{j=1}^{\mathtt{r}} V^* \gamma_j(X) V \otimes B_j \\ &= [V \otimes I]^* \left(I \otimes I + \sum_{j=1}^{\mathtt{r}} \gamma_j(X) \otimes B_j \right) [V \otimes I] \succeq 0. \end{aligned}$$

Thus $(X,V) \in \mathcal{D}_B^\Gamma$ and thus $\mathcal{D}_B^\Gamma$ is Γ-convex.

The hypotheses imply that $\boldsymbol{J}$ is a closed matrix convex set with 0 in its interior, certifying an application of Lemma 2.7. Hence there exists a semi-finite tuple $A \in \mathbb{S}_{\mathcal{H}}^{\mathtt{r}}$ for some separable Hilbert space $\mathcal{H}$, such that $\boldsymbol{J} = \mathcal{D}_A$. If $X \in \boldsymbol{K}$, then $\Gamma(X) \in \mathrm{matco}(\Gamma(\boldsymbol{K})) \subseteq \boldsymbol{J} = \mathcal{D}_A$ and thus $X \in \mathcal{D}_A^\Gamma$; i.e.,

$$I + \sum \gamma_j(X) \otimes A_j \succeq 0.$$

Conversely, if $X \in \mathbb{S}^{\mathtt{g}} \setminus \boldsymbol{K}$, then the hypothesis of item (d) implies $\Gamma(X) \notin \boldsymbol{J} = \mathcal{D}_A$ and so $X \notin \mathcal{D}_A^\Gamma$. Thus $\boldsymbol{K} = \mathcal{D}_A^\Gamma$ for some semi-finite tuple A as claimed. ■

2.3.2. *Duality.* For any closed and bounded Γ-convex set $\boldsymbol{K}$, let

$$\widehat{Y} = \bigoplus_{Y \in \boldsymbol{K}} Y \quad \text{and} \quad I = \bigoplus_{Y \in \boldsymbol{K}} I_{\mathrm{size}(Y)}, \tag{2.6}$$

and associate to $\boldsymbol{K}$ the Γ-operator system $\widehat{\boldsymbol{K}} = \mathrm{span}\{I = \gamma_0(\widehat{Y}), \gamma_1(\widehat{Y}), \dots, \gamma_{\mathtt{r}}(\widehat{Y})\}$. The assumption that $\boldsymbol{K}$ is bounded ensures each $\widehat{Y}_j$ is a bounded operator.

Theorem 2.21 (**Webster-Winkler duality for Γ-convex sets**). *The operations* $\widehat{\ }$ *of* (2.6) *and* $\check{\ }$ *of* (2.5) *are dual to one another:*

(a) *Suppose $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$ is semi-finite and let $\mathcal{R} = \mathrm{span}\{I_{\mathcal{H}}, \gamma_1(A), \dots, \gamma_{\mathtt{r}}(A)\}$. If $\mathcal{D}_{\Gamma(A)}$ is bounded, then $\mathcal{R}$ and $\widehat{\check{\mathcal{R}}}$ are isomorphic Γ-operator systems.*

(b) *Suppose $\Gamma(0) = 0$ and let $\boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ denote a closed and bounded Γ-convex set with $0 \in K_1$. If $\boldsymbol{K} = \Gamma^{-1}(\overline{\mathrm{matco}}(\Gamma(\boldsymbol{K})))$, then $\boldsymbol{K} = \check{\widehat{\boldsymbol{K}}} = W^\Gamma(\widehat{Y})$ for $\widehat{Y}$ defined as in equation* (2.6).

Remark 2.22. In item (b), if the matrix convex hull of $\Gamma(K)$ is closed, then, by Theorem 1.9, it is the case that $\Gamma(X)$ is in the closure of the matrix convex hull of $\Gamma(\boldsymbol{K})$ if and only if $X \in \boldsymbol{K}$. ■

Proof of Theorem 2.21. Set

$$\widehat{B} = \bigoplus_{B \in W^\Gamma(A)} B \quad \text{and} \quad I = \bigoplus_{B \in W^\Gamma(A)} I_{\operatorname{size}(B)}$$

and let $\mathcal{K}$ denote the space that the $\widehat{B}_j$ act on. From the definitions,

$$\hat{\hat{\mathcal{R}}} = \operatorname{span}\{\gamma_0(\widehat{B}) = I_{\mathcal{K}}, \gamma_1(\widehat{B}), \dots, \gamma_{\mathtt{r}}(\widehat{B})\}.$$

For notational convenience, let $A_0 = I_{\mathcal{H}}$ and $\widehat{B}_0 = I_{\mathcal{K}}$. Since $\mathcal{D}_{\Gamma(A)}$ is assumed bounded, by Proposition 2.8, the operators $I_{\mathcal{H}}, \gamma_1(A), \dots, \gamma_{\mathtt{r}}(A)$ are linearly independent. Hence there is a unital map $\tau : \mathcal{R} \to \hat{\hat{\mathcal{R}}}$ determined by $\tau(\gamma_j(A)) = \gamma_j(\widehat{B})$. By construction, τ is bijective and a Γ-concomitant. Suppose $R = \sum X_j \otimes \gamma_j(A) \in M_m(\mathcal{R})$ is positive semidefinite. Given $B \in W^\Gamma(A)$, there exists a Γ-ucp map φ such that $B = \operatorname{vec}\varphi(A)$. Hence,

$$0 \preceq \varphi(R) = \sum X_j \otimes \varphi(\gamma_j(A)) = \sum X_j \otimes \gamma_j(\operatorname{vec}\varphi(A)) = \sum X_j \otimes \gamma_j(B). \tag{2.7}$$

Therefore,

$$0 \preceq \sum X_j \otimes \gamma_j(\widehat{B}) = \tau(R)$$

and it follows that τ is completely positive. On the other hand, A is an at most countable direct sum $A = \oplus A^{(i)}$ where $A^{(i)} \in \mathbb{S}^{\mathtt{g}}_{n_i}$. Thus if $R \not\succeq 0$, then there is some i such that $R' = \sum X_j \otimes \gamma_j(A^{(i)}) \not\succeq 0$. Since the unital map ψ sending A to $A^{(i)}$ is Γ-ucp, it follows that $\psi(A) = A^{(i)} \in W^\Gamma(A)$ and thus $R' \not\succeq 0$ implies $\tau(R) \not\succeq 0$. Hence the inverse of τ is also completely positive and the proof of item (a) is complete.

Turning to item (b), recall that $\widehat{\boldsymbol{K}} = \operatorname{span}\{I, \gamma_1(\widehat{Y}), \dots, \gamma_{\mathtt{r}}(\widehat{Y}))\}$. For any $n \in \mathbb{N}$ and $B \in K_n$ define the linear map $\varphi : \widehat{\boldsymbol{K}} \to M_n$ by

$$\begin{aligned} I_{\mathscr{Y}} &\mapsto I_n \\ \gamma_i(\widehat{Y}) &\mapsto \gamma_i(B), \end{aligned}$$

where $\mathscr{Y}$ is the space that the $\widehat{Y}_j$ act on. By construction of $\widehat{Y}$, there is an isometry V reducing $\widehat{Y}$ such that $B = V^*\widehat{Y}V$. Thus $(\widehat{Y}, V)$ is a Γ-pair and φ is a Γ-ucp with image B. Thus $B \in W^\Gamma(\widehat{Y})$.

Conversely, let $B \in W^\Gamma_\ell(\widehat{Y})$ be the image of $\widehat{Y}$ by a Γ-ucp map φ. Arguing by contradiction, suppose $B \notin K_\ell$. The hypothesis that $\boldsymbol{K} = \Gamma^{-1}(\overline{\operatorname{matco}}(\Gamma(\boldsymbol{K})))$ implies $\Gamma(B)$ is not in the closed matrix convex set $\boldsymbol{J} = \overline{\operatorname{matco}}(\Gamma(\boldsymbol{K}))$; that is $\Gamma(B) \notin J_\ell$. By Theorem 1.8, there exists a tuple $C = (C_1, \dots, C_{\mathtt{r}}) \in \mathbb{S}^{\mathtt{g}}_\ell$ such that $L_C(\Gamma(Y)) \succeq 0$ for all $Y \in \boldsymbol{K}$, but

$L_C(\Gamma(B)) \not\succeq 0$. Setting

$$L_C^\Gamma(X) = I_\ell \otimes I_{\mathrm{size}(X)} + \sum_{i=1}^{\mathtt{r}} C_i \otimes \gamma_i(X)$$

and applying the canonical shuffle gives,

$$L_{\Gamma(Y)}(C) \underset{u}{\approx} L_C(\Gamma(Y)) = L_C^\Gamma(Y) \succeq 0$$

for all $Y \in \boldsymbol{K}$, but

$$L_{\Gamma(B)}(C) \underset{u}{\approx} L_C(\Gamma(B)) = L_C^\Gamma(B) \not\succeq 0,$$

where $\underset{u}{\approx}$ indicates unitary equivalence. It follows that $L_{\Gamma(\hat{Y})}(C) \succeq 0$, but $L_{\Gamma(B)}(C) \not\succeq 0$, showing that $D_{\Gamma(\hat{Y})} \not\subseteq \mathcal{D}_{\Gamma(B)}$. Thus φ is not Γ-ucp. (See Remark 2.15). Hence, if $B \in W_\ell^\Gamma(\widehat{Y})$, then $B \in \boldsymbol{K}_\ell$ completing the proof that $\boldsymbol{K} = W^\Gamma(A)$. ■

2.3.3. Γ-*polar dual.* Given the tuple $A = (A_1, \dots, A_{\mathtt{g}}) \in \mathbb{S}_{\mathcal{H}}^{\mathtt{g}}$, the Γ-**polar dual** of $\mathcal{D}_{\Gamma(A)}$ is, by definition, the sequence $\mathcal{D}_A^{\Gamma\text{-}\circ} = (\mathcal{D}_A^{\Gamma\text{-}\circ}(n))_n$, where

$$\mathcal{D}_A^{\Gamma\text{-}\circ}(n) = \{X \in \mathbb{S}^{\mathtt{g}} : L_B(\Gamma(X)) = I_{\mathrm{size}\,B} \otimes I_n + \sum_{j=1}^{\mathtt{r}} B_j \otimes \gamma_j(X) \succeq 0 \text{ for all } B \in \mathcal{D}_{\Gamma(A)}\}.$$

Proposition 2.23. *With notations above, if $\mathcal{D}_{\Gamma(A)}$ is bounded, then $\mathcal{D}_A^{\Gamma\text{-}\circ} = W^\Gamma(A)$.*

Proof. Observe, using Proposition 2.9, Remark 2.15 and the boundedness assumption on $\mathcal{D}_{\Gamma(A)}$, that $X \in \mathcal{D}_A^{\Gamma\text{-}\circ}$ if and only if $L_B(\Gamma(X)) \succeq 0$ for all $B \in \mathcal{D}_{\Gamma(A)}$ if and only if the unital Γ-concomitant map φ that sends $\gamma_i(A)$ to $\gamma_i(X)$ is Γ-ucp if and only if $X = \operatorname{vec}\varphi(A) \in W^\Gamma(A)$. ■

Adding a hypothesis produces a bipolar result.

Proposition 2.24. *With notations above, if A is semi-finite and $\mathcal{D}_\Gamma(A)$ is bounded, then*

$$\{B \in \mathbb{S}^{\mathtt{r}} : L_B(\Gamma(X)) \succeq 0 \textit{ for all } X \in \mathcal{D}_A^{\Gamma\text{-}\circ}\} = \mathcal{D}_{\Gamma(A)}. \tag{2.8}$$

Proof. Let S denote the left hand side of (2.8). Suppose $B \in \mathcal{D}_{\Gamma(A)}$. For $X \in \mathcal{D}_A^{\Gamma\text{-}\circ}$, we have $L_C(\Gamma(X)) \succeq 0$ for all $C \in \mathcal{D}_{\Gamma(A)}$. Choosing $C = B$ gives $B \in S$.

Conversely, suppose $B \in S$. Since A is semi-finite, $A = \oplus_{j=1}^\infty A^{(j)}$ for some tuples $A^{(j)} \in \mathbb{S}_{m_j}^{\mathtt{g}}$. Moreover, $L_B(\Gamma(A^{(j)})) \succeq 0$ since each $A^{(j)} \in \mathcal{D}_A^{\Gamma\text{-}\circ}$ and $B \in S$. Thus $L_B(\Gamma(A)) \succeq 0$ and $B \in \mathcal{D}_{\Gamma(A)}$. ■

3. Extreme points of Γ-convex sets

In this section we introduce a notion of an extreme point and establish an analog of the Krein-Milman Theorem for a Γ-convex sets $\boldsymbol{K} = (K_n)$. Doing so requires the addition of a countably infinite level K_∞, even in the case of a matrix convex set ($\Gamma(x) = x$) [Kri19, Eve18, DK+]. The need for this additional level is discussed in further detail as Remark 3.16 at the end of Subsection 3.3. The notion of extreme point here is closely related to that of an Arveson boundary point [Arv69, Arv72, Ham79]. It was introduced by Kleski [Kls14] and was recently studied and adapted in [EHKM18, EH19, DK+, EEHK+] for examples.

3.1. **Pascoe's SOT Bolzano-Weierstraß modulo unitary similarity theorem.** To tackle the problem of potential insufficiency of free extreme points at finite levels we consider matrix convex sets with an added infinite-dimensional component and take advantage of [DK+, Theorem 6.4.2]. This additional level is a subset of $\mathbb{S}^{\mathrm{g}}_{\mathcal{H}} = \mathcal{B}(\mathcal{H})^{\mathrm{g}}_{\mathrm{sa}}$, the set of g-tuples of self-adjoint operators on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$. For convenience, in this section $\mathcal{H}_k$ will denote a Hilbert space of finite dimension $k \in \mathbb{N}$.

Pascoe's SOT Bolzano-Weierstraß Modulo Unitary Similarity Theorem (SOT-BW-MUST) will play an important role in what follows. A version of the result appears in [Man20] (see Lemma 4.5 and Remark 4.6) and is used in [JKMMP21]. As communicated privately to us by Pascoe, the result naturally extends to sequences $(X_j)_j$ as opposed to finite tuples. We use the usual abbreviation **SOT** for the **strong operator topology**.

Theorem 3.1 (Pascoe's SOT-BW-MUST)**.** *Suppose $\mathcal{H}$ is an infinite-dimensional separable Hilbert space and $(X^{(j)})_j$ is a sequence from $\mathcal{B}(\mathcal{H})^{\mathrm{g}}$. If the sequence $(X^{(j)})_j$ is bounded, then there is a subsequence $(Y^{(n)})_n$ of $(X^{(j)})_j$ and a sequence of unitary mapping $(U_n)_n$ on $\mathcal{H}$ such that U_n^* converges SOT to an isometry and $U_n^* Y^{(n)} U_n$ converges SOT.*

In particular, if $S \subseteq \mathcal{B}(\mathcal{H})^{\mathrm{g}}$ is SOT-closed, bounded and closed under isometric conjugation (meaning if $Y \in S$ and $W \in \mathcal{B}(\mathcal{H})$ is an isometry, then $W^*YW \in S$), then S is WOT-compact.

Remark 3.2. Because of Theorem 3.1, the strong operator topology on free subsets of $\mathbb{S}^{\mathrm{g}}$ is often most natural for the purposes here. However, both the **weak* topology** (synonymously **ultra-weak**) and **weak operator topology** (**WOT**) play a role. On norm-bounded sets, the WOT and weak* topology coincide. The SOT is stronger than the WOT and therefore, a (norm) bounded SOT-closed set is automatically both WOT and weak* closed. On the other hand, the SOT and WOT have the same closed convex sets (since they have the same continuous linear functionals). Thus a norm bounded convex set that is closed in any one of the WOT, SOT or weak* topology is closed in all three. In particular, the closure of a norm bounded convex set is the same in all three of these topologies.

For a separable Hilbert space, as is the case here, all three topologies are metrizable on bounded sets. ■

3.2. **Preliminaries on free extreme points.** As a convention in this section, let $\mathbb{S}^{\mathrm{g}}$ denote the graded set $(\mathbb{S}^{\mathrm{g}}_n)_{n\in\mathbb{N}\cup\{\infty\}}$, where $\mathbb{S}^{\mathrm{g}}_n$ is identified with $\mathcal{B}(\mathcal{H}_n)^{\mathrm{g}}_{\mathrm{sa}}$ for $n \in \mathbb{N}$ and $\mathcal{H}_\infty = \mathcal{H}$. A graded set $\boldsymbol{K} = (K_n)_{n\in\mathbb{N}\cup\{\infty\}} \subseteq \mathbb{S}^{\mathrm{g}}$ is **closed** if each K_n is closed in the specified topology. We call $\boldsymbol{K}$ **bounded** if there is $R \in \mathbb{R}_{\geq 0}$ such that $\|X\| \leq R$ (see Definition 1.1) for all $X \in \boldsymbol{K}$. Under mild and natural assumptions (e.g., $\boldsymbol{K}$ is closed under countable direct sums (cf. Remark 3.6) or K_∞ is bounded), $\boldsymbol{K}$ is bounded.

Definition 3.3. Suppose $\boldsymbol{S}$, $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$. Thus $\boldsymbol{K} = (K_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $K_n \subseteq \mathbb{S}^{\mathrm{g}}_n$ for each n and similarly for $\boldsymbol{S}$.

(a) The graded set $\boldsymbol{S}$ is a **free set** if is closed under unitary similarity and at most countable direct sums of norm-bounded families.
(b) A free set $\boldsymbol{S}$ is **fully free** if it is closed under restrictions to reducing subspaces.
(c) Given $n, n_i \in \mathbb{N}\cup\{\infty\}$ and $A^{(1)}, \dots, A^{(k)} \in \mathbb{S}^{\mathrm{g}}$ with $A^{(i)} \in \mathbb{S}^{\mathrm{g}}_{n_i}$ for $1 \leq i \leq k$, an expression of the form

$$\sum_{i=1}^{k} V_i^* A^{(i)} V_i, \tag{3.1}$$

where $V_i \in M_{n_i,n}$ satisfy $\sum_{i=1}^k V_i^* V_i = I_n$, is an **operator convex combination** of $A^{(1)}, \dots, A^{(k)}$.
(d) The **(fully) free hull** of a graded set $\boldsymbol{S}$ is the smallest (fully) free set containing $\boldsymbol{S}$; the **closed** (in a specified topology) **fully free hull** is the smallest closed fully free set containing $\boldsymbol{S}$.
(e) A free set $\boldsymbol{K}$ is **operator convex** if it is closed under operator convex combinations. Observe that each operator convex set is fully free.
(f) The **operator convex hull** of a graded set $\boldsymbol{S} \subseteq \mathbb{S}^{\mathrm{g}}$ is the intersection of all operator convex sets containing $\boldsymbol{S}$, denoted $\mathrm{opco}(\boldsymbol{S})$. Its (level-wise) closure is denoted by $\overline{\mathrm{opco}}(\boldsymbol{S})$, and a superscript such as SOT or w^* may be added to specify the topology if it is not clear from the context. ■

Remark 3.4. For $\infty > R > 0$, the ball $\{X \in \mathbb{S}^{\mathrm{g}} : \|X\| \leq R\}$ is bounded, fully free, closed (in all the relevant topologies), and operator convex. It is clear that any graded set $\boldsymbol{S}$ is contained in a closed fully free operator convex set. Since an intersection of (fully) free sets is (fully) free, the (fully) free hull of a graded set is also (fully) free. The same statement holds with operator convex hull instead of free hull. Thus these hulls exist and are bounded if $\boldsymbol{S}$ is.

Proposition 3.5 below contains some further initial observations. ■

Proposition 3.5. *A free set $\boldsymbol{K}$ is operator convex if and only if for each $m, n \in \mathbb{N}\cup\{\infty\}$, each $X \in K_n$ and each isometry $V : \mathcal{H}_m \to \mathcal{H}_n$, the tuple $V^*XV \in K_m$.*

For a graded set $\boldsymbol{S} \subseteq \mathbb{S}^{\mathrm{g}}$, *and* τ *either the SOT or weak** *topology, the graded set* $\overline{\mathrm{opco}}^{\tau}(\boldsymbol{S})$ *is operator convex and it is the smallest* τ*-closed operator convex set containing* $\boldsymbol{S}$; *that is,* $\overline{\mathrm{opco}}^{\tau}(\boldsymbol{S})$ *is the* τ*-closed convex hull of* $\boldsymbol{S}$.

If $\boldsymbol{F}$ *is a free set, then*

$$\mathrm{opco}(\boldsymbol{F}) = \{V^*FV : F \in \boldsymbol{F},\ V \textit{ is an isometry}\}$$

and if moreover, F *is bounded, then so is* $\mathrm{opco}(\boldsymbol{F})$. *Finally, if* $\boldsymbol{F}$ *is a free set that is bounded and SOT-closed, then* $\mathrm{opco}(\boldsymbol{F})$ *is bounded SOT-closed and hence WOT and weak** *closed, since* $\mathrm{opco}(\boldsymbol{F})$ *is norm bounded and convex.*

The level-wise SOT closure of a norm bounded fully free set is fully free. In particular, if $\boldsymbol{S}$ *is a bounded graded set, then its SOT-closed fully free hull is bounded and the level-wise closure of its fully free hull.*

Proof. The proof of the first statement is routine. Suppose $\boldsymbol{K}$ is free and let an operator convex combination as in (3.1) be given. Since $\boldsymbol{K}$ is closed under direct sums, $X = \oplus A^{(i)}$ is in $\boldsymbol{K}$. Now the operator $V = \mathrm{col}(V_1, \dots, V_k)$ is an isometry and $\sum V_i^* A^{(i)} V_i = V^*XV$.

For the second statement, let $X \in \overline{\mathrm{opco}}^{\tau}(\boldsymbol{S})$, and choose a net $(X^{(i)})_i$ in $\mathrm{opco}(\boldsymbol{S})$ converging to X with respect to the topology τ. Given an isometry V, the net $(V^*X^{(i)}V)_i$ is in $\mathrm{opco}(\boldsymbol{S})$ and converges to $V^*XV \in \overline{\mathrm{opco}}^{\tau}(\boldsymbol{S})$.

The identity of the third statement is a consequence of the first statement since the assumption that $\boldsymbol{F}$ is a free set implies the set $\{V^*FV : F \in \boldsymbol{F}, V \text{ is an isometry}\}$ is easily seen to be operator convex. If $\boldsymbol{F}$ is bounded, then $\boldsymbol{F}$ is contained in a bounded operator convex set and hence $\mathrm{opco}(\boldsymbol{F})$ is bounded. To show that $\mathrm{opco}(\boldsymbol{F})$ is SOT-closed assuming $\boldsymbol{F}$ is bounded and SOT-closed, we proceed as follows. First note that, since $\boldsymbol{F}$ is bounded, so is $\boldsymbol{G} = \mathrm{opco}(\boldsymbol{F})$. Since each G_m is a bounded subset of operators on a separable Hilbert space, the SOT topology on G_m is metrizable. It thus suffices to show, if $Y^{(i)}$ is a sequence from G_m that converges to some Y, then $Y \in G_m$. The proof of this fact will use the following observations for bounded sequences (S_n) and (T_n) of operators on Hilbert space. If (S_n) converges WOT and (T_n) converges SOT to S and T respectively, then (S_nT_n) converges WOT to ST; and if both sequences converge SOT, then (S_nT_n) converges SOT to ST; and if (S_n) converges SOT to S, then (S_n^*) converges SOT to S^*.

Without loss of generality, assume $\boldsymbol{F} \neq \emptyset$. Suppose Y is in the SOT-closure of $\mathrm{opco}(\boldsymbol{F})$ and let $\mathcal{Y}$ denote the separable Hilbert space that Y acts upon. By assumption, there is a sequence $Y^{(i)} = V_i^* F^{(i)} V_i \in \mathrm{opco}(\boldsymbol{F})$ that converges SOT to Y, where $F^{(i)} \in \boldsymbol{F}$ and V_i are isometries. Let $\mathcal{E}_j$ denote the separable Hilbert space that $F^{(j)}$ acts upon. Since the free set $\boldsymbol{F}$ is not empty, there is an $E \in F_\infty$. Let $\mathcal{E}_0$ denote the separable infinite-dimensional Hilbert space that E acts upon. The assumption that $\boldsymbol{F}$ is closed under countable direct sums justifies replacing $F^{(j)}$ and V_j with

$$\begin{pmatrix} F_j & 0 \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} V_j \\ 0 \end{pmatrix}$$

acting on $\mathcal{E}_j \oplus \mathcal{E}_0$ and $\mathcal{Y}$ respectively; and then closure with respect to unitary similarity justifies assuming that the $F^{(j)}$ all act upon the same infinite-dimensional Hilbert space $\mathcal{H}$. Finally, once again using the assumption that $\boldsymbol{F}$ is closed under unitary similarity, the fact that the dimensions of the ranges of V_j in $\mathcal{H}$ are all the same justifies assuming that $V_j = V : \mathcal{Y} \to \mathcal{H}$ for some fixed isometry V and all j.

By Theorem 3.1, passing to a subsequence if needed, there exists a sequence $(U_n)_n$ of unitary operators on $\mathcal{H}$ such that $(U_n^* F^{(n)} U_n)_n$ and $(U_n^*)_n$ each converge SOT to some F and isometry U^*. Moreover, $U_n^* V$ converges SOT to an isometry W and $(U_n^* V)^* = V^* U_n$ converges WOT to W^*. Finally, $[(U_n^* F^{(n)} U_n) U_n^* V]$ is the product of SOT convergent sequences and hence converges SOT to FW. Therefore,

$$\begin{aligned} Y &= \text{SOT - }\lim_n V^* F^{(n)} V \\ &= \text{SOT - }\lim_n V^* U_n U_n^* F^{(n)} U_n U_n^* V \\ &= \text{WOT - }\lim_n V^* U_n [(U_n^* F^{(n)} U_n) U_n^* V] \\ &= W^* F W. \end{aligned}$$

Hence $Y \in \operatorname{opco}(\boldsymbol{F})$ as claimed.

Since the SOT-closed set $\operatorname{opco}(\boldsymbol{F})$ is bounded and convex, it is WOT and weak* closed too.

To prove the fourth, and final statement, let $\boldsymbol{F}$ denote a fully free set and $\overline{\boldsymbol{F}}$ denote its level-wise SOT closure. Since the SOT topology is metrizable on bounded sets in the case of separable Hilbert space, given $F \in \overline{F_n}$ and a unitary U of size n, there exists a sequence $(F^{(m)})$ from F_n that converges SOT to F. Hence $(U^* F^{(m)} U)$ is a sequence from $\boldsymbol{F}$ that converges SOT to $U^* F U$. So $U^* F U \in \overline{\boldsymbol{F}}$ and thus $\overline{\boldsymbol{F}}$ is closed under unitary similarity.

Now suppose $\boldsymbol{F}$ is fully free and bounded by R. Given a sequence $(F^{(m)})$ from $\overline{\boldsymbol{F}}$, for each m there is a sequence $(F^{(m,\ell)})_\ell$ that converges SOT to $F^{(m)}$, since $\boldsymbol{F}$ is a norm bounded subset of operators on a separable Hilbert space (so that the relative SOT topology on $\boldsymbol{F}$ is metrizable). Let $x = \oplus x_m$ denote a vector from the space that $\oplus F^{(m)}$ acts on and observe, for each M,

$$\|(\oplus_m (F^{(m,\ell)} - \oplus F^{(m)}) x\|^2 \le \sum_{m=1}^{M} \|(F^{(m,\ell)} - F^{(m)}) x_m\|^2 + 2R \sum_{m=N+1}^{\infty} \|x_m\|^2,$$

from which it readily follows that $\oplus_m F^{(m,\ell)}$ converges SOT to $\oplus_m F^{(m)}$.

If $\boldsymbol{S} \subseteq \mathbb{S}^{\mathrm{g}}$ is bounded by R, then $\boldsymbol{S}$ is a subset of the fully free set $\boldsymbol{R} = \{X \in \mathbb{S}^{\mathrm{g}} : \|X\| \le R\}$. Hence the fully free hull of $\boldsymbol{S}$ is bounded by R. ■

Remark 3.6. If $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$ is closed under arbitrary (i.e., not only norm-bounded) countable direct sums, then $\boldsymbol{K}$ is bounded. Indeed, arguing the contrapositive, if there does not exists a C such that $\|X\| \le C$ for all $X \in \boldsymbol{K}$, then for each n there exists $X^{(n)} \in \boldsymbol{K}$ with $\|X^{(n)}\| \ge n$, in which case $\oplus_n X^{(n)}$ is not a bounded operator and thus not in $\boldsymbol{K}$. Conversely, assuming $\boldsymbol{K}$ is closed with respect to finite direct sums and is bounded along with some additional closure property implies $\boldsymbol{K}$ is closed with respect to countable direct sums. We leave details to the interested reader and instead consider a similar result for operator convex combinations.

A weak* closed operator convex set $\boldsymbol{K}$ that contains 0 is closed under arbitrary (i.e., not necessarily finite) convex combinations of any norm-bounded family. Recall that if $0 \in K_1$, then $\boldsymbol{K}$ is closed under conjugation by contractions (see, e.g., [HKM16, Lemma 2.3], which remains valid in the infinite dimensional setting with the same proof). Using notation as in equation (3.1), but now with arbitrarily many indices $i \in J$, where J is an index set, for any finite subset of indices $I \subseteq J$ we have $\sum_{i\in I} V_i^* V_i \preceq I$ and the sum

$$a_I = \sum_{i\in I} V_i^* A^{(i)} V_i$$

lies in $\boldsymbol{K}$. The net $(a_I)_I$ is weak* convergent as we now explain. By the norm boundedness assumption on the family $(A^{(i)})_i$, there is an $R > 0$ such that $\|A^{(i)}\| \le R$ for all i. In particular, $R - A_j^{(i)} \ge 0$ for each i and $1 \le j \le \mathrm{g}$. Consequently, with $R - A^{(i)}$ denoting the tuple with j-th entry $R - A_j^{(i)}$, the net $(a'_I)_I$ defined by

$$a'_I = \sum_{i\in I} V_i^*(R - A^{(i)}) V_i$$

is an entrywise increasing bounded net of self-adjoint operators and hence WOT-convergent by Vigier's theorem [Mur90, Theorem 4.1.1]. Since $\sum_i V_i^* V_i = I$, the net $(a_I)_I$ is also WOT-convergent. Now Remark 3.2 implies that $(a_I)_I$ is weak* convergent and hence its limit, denoted by $\sum_i V_i^* A^{(i)} V_i$, lies in $\boldsymbol{K}$.

By [JKMMP21, Proposition 3.7(b)], an SOT-closed operator convex set $\boldsymbol{K}$ containing 0 is the SOT-closure of its matricial levels, $(K_n)_{n\in\mathbb{N}}$. ∎

3.3. **Free extreme points and the Krein-Milman theorem for operator convex sets.** Free extreme points are intuitively those points of a matrix or operator convex set $\boldsymbol{K}$ that cannot be expressed as a nontrivial matrix convex combination of any finite subset of $\boldsymbol{K}$. A notion of an nc convex set is introduced in [DK+] and it turns out that a closed and bounded operator convex set given its weak* topology is a relatively simple example of a compact nc convex set over the dual operator algebra $\mathscr{R}_{\mathrm{g}}$, where $\mathscr{R}_{\mathrm{g}}$ is the operator space known as g-dimensional row Hilbert space. See Appendix A. The [DK+] notion of an **extreme point** of an nc convex set, specialized to operator convex sets follows. For consistency with earlier work, cf. [EPŠ+, EEHK+], we use the terminology **free extreme point**. In finite dimensions (the matrix convex setting) free extreme points also go by absolute extreme points [EHKM18, EH19, Kri19].

Definition 3.7 ([DK+, Definition 6.1.1])**.** Let $\boldsymbol{K} = (K_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be an operator convex set. A tuple $X \in K_n$, where $n \in \mathbb{N}\cup\{\infty\}$, is a **free extreme point** (of $\boldsymbol{K}$) if any expression of X as an operator convex combination

$$X = \sum_{i=1}^{k} V_i^* X^{(i)} V_i, \tag{3.2}$$

where $X^{(i)} \in K_{n_i}$ and the $V_i \in M_{n_i,n}$ are all nonzero, and satisfy $\sum_{i=1}^k V_i^* V_i = I_n$, it is the case that, for each i, the matrix V_i is a scalar multiple of an isometry $W_i \in \mathcal{M}_{n_i,n}$ satisfying $W_i^* X^{(i)} W_i = X$ and with respect to the range of V_i,

$$X^{(i)} = Y^{(i)} \oplus Z^{(i)}$$

for some $Y^{(i)}, Z^{(i)} \in \boldsymbol{K}$, where $Y^{(i)}$ is unitarily equivalent to X. Let $\operatorname{ext}(\boldsymbol{K})$ denote the graded set of free extreme points of $\boldsymbol{K}$. ■

The next lemma follows from the definition of a free extreme point.

Lemma 3.8. *Suppose $\boldsymbol{K}$ is an operator convex set and $X \in \boldsymbol{K}$. If X is a free extreme point, expressed as in (3.2), then $V = \operatorname{col}(V_1, \dots, V_k) = \operatorname{col}(V_i)$ reduces $\oplus_i X^{(i)}$. If $X \in K_n$ for n finite, then X is a free extreme point if and only if it satisfies the conditions of Definition 3.7 with n_i also finite.*

Proof. Using notation in Definition 3.7, note that $W_i W_i^*$ is the projection onto the range of V_i and there are constants λ_i such $V_i = \lambda_i W_i$. Hence $X^{(i)} W_i = W_i W_i^* X^{(i)} W_i = W_i X$ and

$$[\oplus_i X^{(i)}]\, V = \operatorname{col}(\lambda_i X^{(i)} W_i) = \operatorname{col}(\lambda_i W_i X) = V\, X. \tag{3.3}$$

Hence the range of V is invariant, and thus reducing, for $\oplus_i X^{(i)}$.

Another routine argument based on equation (3.3) proves the second statement. ■

Remark 3.9. Any free extreme point $X \in K_n$ with $n \in \mathbb{N}$ is irreducible. Indeed, if X is not irreducible, let V_1 denote the inclusion into $\mathcal{H}_n$ of a proper nontrivial invariant subspace $\mathcal{V} \subseteq \mathcal{H}_n$ of X and V_2 the inclusion of the orthogonal complement of $\mathcal{V}$ into $\mathcal{H}_n$. Thus V_1 and V_2 are isometries and $\sum V_j V_j^* = I_n$. Setting $X^{(j)} = V_j^* X V_j \in \boldsymbol{K}$,

$$X = V_1 X^{(1)} V_1^* + V_2 X^{(2)} V_2^*.$$

Since X is free extreme and the size of $X^{(1)}$ is at most the size of X, it follows that X is unitarily equivalent to $X^{(1)}$. But this contradicts the fact that $\mathcal{V}$ is a proper nontrivial invariant subspace of $\mathbb{C}^n$. ■

Remark 3.10. If $\boldsymbol{K}$ is a matrix convex set, then the condition in Definition 3.7 that V_i must be a scalar multiple of an isometry can be dropped. In fact, in that case a point $X \in \boldsymbol{K}$ is free extreme if and only if (3.2) implies that for each i either $n_i = n$ and $X^{(i)}$ is unitarily equivalent to X or $n_i > n$ and there is a tuple $Y^{(i)} \in \boldsymbol{K}$ such that $X^{(i)}$ is unitarily equivalent to $X \oplus Y^{(i)}$. To see the equivalence of the two definitions (in the absence of an infinite level) note that, for each i, we have as above that $X^{(i)} \underset{u}{\approx} X \oplus Y^{(i,1)}$. Writing $V_i = \operatorname{col}(V_{i,1}, V_{i,2})$ gives

$$X = \sum_{i=1}^{k} V_i^* X^{(i)} V_i = \sum_{i=1}^{k} V_{i,1}^* X V_{i,1} + \sum_{i=1}^{k} V_{i,2}^* Y^{(i,1)} V_{i,2},$$

which is again an expression of X as a matrix convex combination with nonzero $V_{i,j}$. Since X is free extreme, each of the $Y^{(i,1)}$ must be unitarily equivalent to $X \oplus Y^{(i,2)}$. We proceed by induction until, for some m, the size of $Y^{(i,m)}$ is at most n, forcing $Y^{(i,m)} = U_{i,m}^* X U_{i,m}$ for unitary matrices $U_{i,m}$. Hence there are unitary matrices $U_{i,j}$ such that

$$X = \sum_{j=1}^{m} \sum_{i=1}^{k} V_{i,j}^* U_{i,j}^* X U_{i,j} V_{i,j}$$

with $V_i = \operatorname{col}(V_{i,1} \dots, V_{i,m})$. Now by [EHKM18, Proposition 4.6], for each i, j, the matrix $U_{i,j} V_{i,j}$ is a scalar multiple of the identity, say $U_{i,j} V_{i,j} = t_{i,j} I_n$, which implies that each V_i is a scalar multiple of an isometry as

$$V_i^* V_i = \sum_{j=1}^{m} V_{i,j}^* V_{i,j} = \sum_{j=1}^{m} V_{i,j}^* U_{i,j}^* U_{i,j} V_{i,j} = \sum_{j=1}^{m} t_{i,j}^2 \, I_n.$$

It is immediate that $V_i^* X_i V_i = \sum_{j=1}^{m} t_{i,j}^2 \, X$. ■

The following result is [DK+, Theorem 6.4.2] specialized to operator convex sets.

Theorem 3.11 (**Krein-Milman theorem for free extreme points**). *A weak* closed and bounded operator convex set $\boldsymbol{K}$ is the weak* closed operator convex hull of its free extreme points; that is every weak* closed and bounded operator convex set that contains the extreme point of $\boldsymbol{K}$ contains $\boldsymbol{K}$.*

A class of matrix convex sets where free extreme points at finite levels do span the whole set (even without taking closures) are real free spectrahedra [EH19]; i.e., spectrahedra closed under complex conjugation. Given a tuple $A \in \mathbb{S}_d^{\mathrm{g}}$, the spectrahedron $\mathcal{D}_A$ is **closed under complex conjugation** if $X = (X_1, \dots, X_{\mathrm{g}}) \in \mathcal{D}_A$ implies $\overline{X} \in \mathcal{D}_A$, where $\overline{X}$ is the tuple obtained from X by entry-wise conjugating each X_j. Let $\boldsymbol{K}$ denote the operator convex set

$$\boldsymbol{K} = \{X \in \mathbb{S}^{\mathrm{g}} : L_A(X) \succeq 0\}.$$

Thus $K_n = \mathcal{D}_A(n)$ for $n \in \mathbb{N}$. In this case, the main result (Theorem 1.3) of [EH19] yields a stronger conclusion than that of Theorem 3.11.

Proposition 3.12. *With notations as above (and assuming $\mathcal{D}_A$ is closed under complex conjugation), if $\boldsymbol{K}$ is bounded, then $\boldsymbol{K}$ is the SOT-closure of the operator convex hull of* $\operatorname{ext}(\boldsymbol{K}) \cap \mathcal{D}_A$.

Proof. An application of [EH19, Theorem 1.3] together with the second part of Lemma 3.8 yields that $\mathcal{D}_A$ (the matricial levels of $\boldsymbol{K}$) is the matrix convex hull of $\operatorname{ext}(\boldsymbol{K}) \cap \mathcal{D}_A$.

By [JKMMP21, Proposition 3.7(b)], the SOT-closure of the fully free set $\boldsymbol{S}$ generated by $\mathcal{D}_A$ is $\boldsymbol{K}$. ■

3.4. **Weak converses to the Krein-Milman theorem.** This subsection consists of several variations on weak converses to Theorem 3.11 of independent interest. We begin with a partial converse inspired by Agler's abstract approach to model theory [Agl88].

The analog of a boundary for an Agler family of operators adapted to operator convex sets reads as follows. A **boundary** of an operator convex set $\boldsymbol{K}$ is an SOT-closed fully free set $\boldsymbol{B} \subseteq \boldsymbol{K}$ (so closed under norm-bounded countable direct sums, unitary similarities and restrictions to reducing subspaces and level-wise SOT-closed) such that $\operatorname{opco}(\boldsymbol{B}) = \boldsymbol{K}$. Note that by Proposition 3.5 $\operatorname{opco}(\boldsymbol{B})$ is automatically SOT-closed and bounded. Also note that $\boldsymbol{K}$ is a boundary for itself. In particular, the set $\mathscr{B}$, of boundaries for $\boldsymbol{K}$ is not empty and we call the set

$$\partial^A \boldsymbol{K} = \bigcap \{\boldsymbol{B} : \boldsymbol{B} \in \mathscr{B}\} \tag{3.4}$$

the **Agler boundary** for $\boldsymbol{K}$ by analogy with the notion of the Agler boundary for a family of operators [Agl88].

Adapting Agler's notion of an extremal element of a family of operators [Agl88] yields the following definition. A point $X \in K_n$ is an **Agler extreme point** of $\boldsymbol{K}$, if $Y \in K_m$ and $V : \mathcal{H}_n \to \mathcal{H}_m$ is an isometry such that $X = V^*YV$, then the range of V reduces Y. Let $\operatorname{ext}^A(\boldsymbol{K})$ denote the Agler extreme points of $\boldsymbol{K}$. By Lemma 3.8, $\operatorname{ext}(\boldsymbol{K}) \subseteq \operatorname{ext}^A(\boldsymbol{K})$. Theorem 3.11 says that the SOT-closed **fully free hull** of the free extreme points of a closed and bounded operator convex set $\boldsymbol{K}$ is a boundary for $\boldsymbol{K}$.

Proposition 3.13. *If $\boldsymbol{B}$ is a boundary of a bounded SOT-closed operator convex set $\boldsymbol{K}$, then $\operatorname{ext}^A(\boldsymbol{K}) \subseteq \boldsymbol{B}$. In particular, $\operatorname{ext}(\boldsymbol{K}) \subseteq \boldsymbol{B}$.*

The SOT-closed fully free hull of $\operatorname{ext}(\boldsymbol{K})$ is a boundary that is contained in every other boundary of $\boldsymbol{K}$; that is, the Agler boundary for $\boldsymbol{K}$ that is a boundary that is contained in all other boundaries.

Proof. Let $\boldsymbol{B}$ be a boundary of $\boldsymbol{K}$. By definition $\operatorname{opco}(\boldsymbol{B}) = \boldsymbol{K}$ and is SOT-closed and fully free. So given $X \in \operatorname{ext}^A(\boldsymbol{K})$, there exists a $B \in \boldsymbol{B}$ and an isometry V such that $X = V^*BV$. Since X is extreme in $\boldsymbol{K}$, the range of V reduces B by Lemma 3.8. Since $\boldsymbol{B}$ is closed with respect to reducing subspaces, $X \in \boldsymbol{B}$. Thus, if $\boldsymbol{B}$ is a boundary for $\boldsymbol{K}$, then $\operatorname{ext}(\boldsymbol{K}) \subseteq \boldsymbol{B}$.

Now let $\boldsymbol{E}$ denote the SOT-closed fully free hull of $\operatorname{ext}(\boldsymbol{K})$. Since a boundary for $\boldsymbol{B}$ for $\boldsymbol{K}$ is SOT-closed and contains $\operatorname{ext}(\boldsymbol{K})$, as was just proved, it follows that $\boldsymbol{E} \subseteq \boldsymbol{B}$. Since $\boldsymbol{K}$ is a boundary, $\boldsymbol{E} \subseteq \boldsymbol{K}$ and thus $\operatorname{opco}(\boldsymbol{E}) \subseteq \boldsymbol{K}$. By Proposition 3.5, $\operatorname{opco}(\boldsymbol{E})$ is bounded and SOT-closed. Since $\operatorname{opco}(\boldsymbol{E})$ is bounded, SOT-closed and operator convex, it is also weak*-closed. By the Krein Milman Theorem for operator convex sets, Theorem 3.11, $\boldsymbol{K} \subseteq \operatorname{opco}(\boldsymbol{E})$. Thus $\operatorname{opco}(\boldsymbol{E}) = \boldsymbol{K}$ and therefore $\boldsymbol{E}$ is a boundary for $\boldsymbol{K}$ that is contained in every other boundary for $\boldsymbol{K}$. ■

Theorem 6.4.3 from [DK+] specialized to the present setting gives the following result. A graded set $\boldsymbol{S} \subseteq \mathbb{S}^{\mathrm{g}}$ is **closed under compressions** if $V^*XV \in \boldsymbol{S}$ whenever $X \in \boldsymbol{S}$ and V is an isometry. We call V^*XV a **compression** of X.

Proposition 3.14 (Special case of [DK+, Theorem 6.4.3])**.** *Suppose $\boldsymbol{K}$ is a closed and bounded operator convex set. If $\boldsymbol{S} \subseteq \boldsymbol{K}$ is weak* closed and closed under compressions and if the closed operator convex hull of $\boldsymbol{S}$ is $\boldsymbol{K}$, then* $\operatorname{ext}(\boldsymbol{K}) \subseteq \boldsymbol{S}$.

In Proposition 3.13, $\boldsymbol{B}$ is assumed closed with respect to only reducing subspaces, and not necessarily compressions like $\boldsymbol{S}$ from Proposition 3.14. On the other hand, $\boldsymbol{B}$ is assumed to be a free set, in particular closed with respect to norm-bounded countable direct sums unlike $\boldsymbol{S}$. Proposition 3.15 below gives a couple of variants of Proposition 3.14 obtained by imposing slightly different hypotheses on $\boldsymbol{S}$.

Proposition 3.15. *Let $\boldsymbol{K}$ be an operator convex set.*

(a) *Suppose $\boldsymbol{S} \subseteq \boldsymbol{K}$ is a subset of irreducible tuples that is closed under unitary conjugation. If the operator convex hull of $\boldsymbol{S}$ equals $\boldsymbol{K}$, then $\boldsymbol{S}$ contains all the free extreme points of $\boldsymbol{K}$.*
(b) *Suppose $\boldsymbol{S} \subseteq \boldsymbol{K}$ is closed under compressions. If the operator convex hull of $\boldsymbol{S}$ equals $\boldsymbol{K}$, then $\boldsymbol{S}$ contains all the free extreme points of $\boldsymbol{K}$.*

Proof. Suppose $X \in \boldsymbol{K}$ is a free extreme point. Since $X \in \boldsymbol{K}$, by assumption, it can be written as an operator convex combination (3.1),

$$X = \sum_{i=1}^{k} V_i^* X^{(i)} V_i,$$

of points $X^{(i)}$ from $\boldsymbol{S}$.

In the case of item (a), since X is free extreme and the tuples $X^{(i)}$ are irreducible, we deduce that $X^{(i)}$ is unitarily equivalent to X for each i. As $\boldsymbol{S}$ is closed under unitary conjugation, X lies in $\boldsymbol{S}$.

For item (b), since X is free extreme, each V_i is a scalar multiple of an isometry W_i and $X = W_i^* X^{(i)} W_i$. As $\boldsymbol{S}$ is closed under compressions, this implies $X \in \boldsymbol{S}$ as desired. ■

Remark 3.16. We now return to the need for the addition of an infinite level, alluded to at the outset of this section, to obtain a Krein-Milman Theorem for operator convex sets. Suppose $\boldsymbol{K}$ is a matrix convex set. What is desired is a notion of extreme point with the property that (1) the closed matrix convex hull of the graded set $\boldsymbol{E}$ of extreme points of $\boldsymbol{K}$ is $\boldsymbol{K}$; and (2) if $\boldsymbol{S}$ is any closed graded set whose closed matrix convex hull is $\boldsymbol{K}$, then $\boldsymbol{S}$

contains $\boldsymbol{E}$. In this sense, $\boldsymbol{E}$ (or its closure) is *the smallest spanning set* for $\boldsymbol{K}$. From what we have seen, the only possibility for $\boldsymbol{E}$ is the graded set of free extreme points. On the other hand, it can happen that a matrix convex set does not contain any free extreme points. The simplest example is exhibited by the Cuntz [Cun77] isometries, cf. [Kri19, Example 6.30]; an alternate self-contained construction is given in [Eve18]. Namely, given a tuple A of compact self-adjoint operators with no nontrivial finite-dimensional reducing subspace, the matrix convex set generated by $\{A\}$ has no (finite-dimensional) free extreme points.

There is a notion of, and a version of the Krein-Milman for, matrix extreme points [WW99, Mor94]. See also [FHL18, Fis96, Far04] for some additional references. However, the graded set of matrix extreme points of a matrix convex set is not necessarily a minimal spanning set, because a matrix extreme point of a matrix convex set could be expressible as a compression of a matrix extreme point at a higher level. That is, the graded set of matrix extreme points satisfies condition (1), but not condition (2), above – see for instance [EEHK+] where it is shown (2) can fail even for a free spectrahedron. ■

3.5. **Operator Γ-convex sets.** The notion of an operator convex set naturally extends to that of an **operator Γ-convex set** by appending a countably infinite-dimensional level $K_\infty \subseteq \mathcal{B}(\mathcal{H})^{\mathtt{g}}_{\mathrm{sa}} = \mathbb{S}^{\mathtt{g}}_\infty$ for an infinite-dimensional separable Hilbert space $\mathcal{H}$. For a tuple $\Gamma = (\gamma_1, \dots, \gamma_{\mathtt{r}})$ of symmetric noncommutative polynomials with $\gamma_j = x_j$ for $1 \le j \le \mathtt{g} \le \mathtt{r}$, the mapping $\Gamma : \mathbb{S}^{\mathtt{g}} \to \mathbb{S}^{\mathtt{r}}$ naturally extends to the operator level.

Lemma 3.17. *The mapping* Γ *is (level-wise) SOT-SOT continuous on bounded sets. Moreover, if* $\boldsymbol{B} \subseteq \mathbb{S}^{\mathtt{g}}$ *is bounded and SOT-closed, then so is* $\Gamma(\boldsymbol{B})$.

Proof. Continuity is immediate at finite levels. Suppose $B \subseteq \mathcal{B}(\mathcal{H})$ is bounded. Thus the (product) SOT topology on B is metrizable. Hence SOT-SOT-continuity of $\Gamma|_B$ is equivalent to sequential SOT-SOT-continuity, which is immediate since products and sums of bounded SOT convergent sequences converge SOT.

Now suppose $\boldsymbol{B}$ is closed and bounded. It is immediate that $\Gamma(\boldsymbol{B})$ is bounded. To prove $\Gamma(\boldsymbol{B})$ is closed, it suffices to show $\Gamma(\boldsymbol{B})$ is SOT-sequentially closed. Given an SOT-convergent sequence $\Gamma(X^{(n)}) = (X^{(n)}, \gamma(X^{(n)})$, there is some X such that $X^{(n)}$ converges SOT to X. Since $\boldsymbol{B}$ is SOT-closed, $X \in \boldsymbol{B}$. By continuity of Γ on bounded sets, $\Gamma(X^{(n)})$ converges SOT to $\Gamma(X) \in \Gamma(\boldsymbol{B})$. ■

The notion of a **Γ-pair** is extended to include any pair (X, V) satisfying $V^*\Gamma(X)V = \Gamma(V^*XV)$, where $X \in \mathbb{S}^{\mathtt{g}}_n$ and $V : \mathcal{H}_m \to \mathcal{H}_n$ is an isometry (with $\mathcal{H}_\infty = \mathcal{H}$). As before, denote the graded set of all Γ-pairs by $\mathcal{C}_\Gamma$.

Definition 3.18. Let $\boldsymbol{S}, \boldsymbol{K} \subseteq \mathbb{S}^{\mathtt{g}}$ denote graded sets. Thus $S_n, K_n \subseteq \mathbb{S}^{\mathtt{g}}_n$ for each $n \in \mathbb{N} \cup \{\infty\}$.

(1) The graded set $\boldsymbol{K} = (K_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is **operator Γ-convex** if it is free and $V^*XV \in \boldsymbol{K}$ whenever $X \in \boldsymbol{K}$ and $(X, V) \in \mathcal{C}_\Gamma$.

(2) The **operator Γ-convex hull** of $\boldsymbol{S}$ is the intersection of all operator Γ-convex sets containing $\boldsymbol{S}$, denoted by $\Gamma\text{-}\operatorname{opco}(\boldsymbol{S})$. For a topology τ, the τ-closed operator Γ-convex hull of $\boldsymbol{S}$ is the intersection of all τ-closed operator Γ-convex sets containing $\boldsymbol{S}$, denoted by $\Gamma\text{-}\overline{\operatorname{opco}}_\tau(\boldsymbol{S})$. ■

Remark 3.19. Note, in the case $\Gamma(x) = x$, that $\Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\boldsymbol{S}) = \overline{\operatorname{opco}}^{\mathrm{sot}}(\boldsymbol{S})$ and similarly with SOT replaced by weak*; that is, in these cases the closure of $\operatorname{opco}(\boldsymbol{K})$ is the smallest closed operator convex set containing $\boldsymbol{S}$ by Lemma 3.5. Thus the notation $\Gamma\text{-}\overline{\operatorname{opco}}_\tau$, instead of $\Gamma\text{-}\overline{\operatorname{opco}}^\tau$, to distinguish between the smallest closed Γ-convex set containing $\boldsymbol{S}$ and the closure of $\Gamma\text{-}\operatorname{opco}(\boldsymbol{S})$.

It is easy to see that a free set $\boldsymbol{K}$ is operator Γ-convex if and only if it is closed under **operator Γ-convex combinations**; that is, operator convex combinations of the form

$$\sum_{i=1}^{k} V_i^* X^{(i)} V_i$$

as in (3.1), where (X, V) with

$$X = \oplus_{i=1}^{k} X^{(i)} \quad \text{and} \quad V = \operatorname{col}(V_1, \dots, V_k) = \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix},$$

is a Γ-pair. ■

A Hahn-Banach separation theorem for operator Γ-convex subsets of $\mathbb{S}^{\mathrm{g}}$ was proved in [JKMMP21]. Assuming $0 \in \boldsymbol{K}$, this theorem holds in our setting as well after identifying the finite levels K_n of $\boldsymbol{K}$ with $K_n \oplus 0 \subseteq K_\infty$.

Theorem 3.20. [JKMMP21, Theorem 3.8] *Suppose $\Gamma(0) = 0$ and let $\boldsymbol{K}$ be an SOT-closed bounded operator Γ-convex set such that $0 \in \boldsymbol{K}$. If $X \notin \boldsymbol{K}$, then there is an $n \in \mathbb{N}$ and a monic linear pencil L of size n such that the Γ-pencil $L^\Gamma = L \circ \Gamma$ is positive semidefinite on $\boldsymbol{K}$, but not at X.*

Propositions 1.4 and 1.5 generalize immediately to the operator setting. Recall the definition of the fully free hull of a graded set $\boldsymbol{S} \subseteq \mathbb{S}^{\mathrm{g}}$ from Definition 3.3.

Proposition 3.21. *If $\boldsymbol{J} = (J_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is not empty and $\boldsymbol{F}$ is its free hull, then*

(a) $\Gamma\text{-opco}(\boldsymbol{J}) = \Gamma\text{-opco}(\boldsymbol{F})$ *and* $\operatorname{opco}(\Gamma(\boldsymbol{J})) = \operatorname{opco}(\Gamma(\boldsymbol{F}))$;
(b) $\Gamma\text{-opco}(\boldsymbol{J}) = \{V^*XV : X \in \boldsymbol{F},\ (X, V) \in \mathcal{C}_\Gamma\}$;
(c) $\Gamma^{-1}(\operatorname{opco}(\Gamma(\boldsymbol{J}))) = \Gamma\text{-}\operatorname{opco}(\boldsymbol{J})$;
(d) $\Gamma^{-1}(\overline{\operatorname{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J})))$ *is Γ-convex;*

(e) *if* $\Gamma(0) = 0$, *if the graded set* $\boldsymbol{J}$ *is bounded and if* $0 \in J_1$, *then*

$$\Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{J}) = \Gamma^{-1}(\overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J}))).$$

In particular, if $\boldsymbol{K}$ *is operator convex and* $X \in \boldsymbol{K} \setminus \Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{J})$, *then* $\Gamma(X) \notin \overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J}))$.

Proof. It is immediate that Γ-opco$(\boldsymbol{J}) \subseteq \Gamma$-opco$(\boldsymbol{F})$. On the other hand, since Γ-opco$(\boldsymbol{J})$ is a free set (since it is Γ-convex) that contains $\boldsymbol{J}$, it follows that $\boldsymbol{F} \subseteq \Gamma$-opco$(\boldsymbol{J})$. Hence Γ-opco$(\boldsymbol{F}) \subseteq \Gamma$-opco$(\boldsymbol{J})$. Similarly, since $\Gamma(\boldsymbol{J}) \subseteq \Gamma(\boldsymbol{F})$, it follows that opco$(\Gamma(\boldsymbol{J})) \subseteq$ opco$(\Gamma(\boldsymbol{F}))$. On the other hand, it is readily checked that opco$(\Gamma(\boldsymbol{J}))$ is a free set that contains $\Gamma(\boldsymbol{F})$ giving the reverse inclusion and completing the proof of item (a).

From item (a), it suffices to prove the remaining items with $\boldsymbol{J}$ replaced by a free set $\boldsymbol{F}$. For such an $\boldsymbol{F}$, items (b) and (c) are readily verified (cf. Propositions 1.4 and 1.5). For instance, for item (b), it suffices to show that the right hand side is operator Γ-convex. To show that the right-hand side is a free set, i.e., closed under countable direct sums, we proceed as follows. Suppose $(Y^{(j)})_j$ is a sequence from the right-hand size of (b). Thus there exists a sequence $(X^{(j)})_j$ from $\boldsymbol{F}$ and isometries V_j such that $Y^{(j)} = V_j^* X^{(j)} V_j$ and $(X^{(j)}, V_j) \in \mathcal{C}_\Gamma$. Since $\boldsymbol{F}$ is closed with respect to countable direct sums, $X = \oplus X^{(j)} \in \boldsymbol{F}$. Let $V = \oplus V_j$. Thus V is an isometry, $(X, V) \in \mathcal{C}_\Gamma$ and $V^*XV = \oplus_j Y^{(j)}$. Hence $\oplus_j Y^{(j)}$ is in the right-hand size of (b). Similar arguments show the right hand side is closed under unitary similarity and operator Γ-convex combinations.

To show that $\boldsymbol{L} = \Gamma^{-1}(\overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J}))$ is Γ-convex, suppose $X \in \boldsymbol{L}$ and $(X, V) \in \mathcal{C}_\Gamma$. Thus $\Gamma(V^*XV) = V^*\Gamma(X)V$ and $\Gamma(X) \in \overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J}))$. Since $\overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J}))$ is operator convex by Proposition 3.5, it follows that $\Gamma(V^*XV) = V^*\Gamma(X)V \in \overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J}))$. Thus $V^*XV \in \boldsymbol{L}$ and hence $\boldsymbol{L}$ is Γ-convex and item (d) is proved.

Turning to item (e), from item (c), $\Gamma\text{-}\operatorname{opco}(\boldsymbol{F}) = \Gamma^{-1}(\operatorname{opco}(\Gamma(\boldsymbol{F})))$. By the SOT-continuity of Γ on bounded sets, Lemma 3.17 implies that the graded set $\Gamma^{-1}(\overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{F})))$ is SOT-closed. From item (d) it is Γ-convex. Therefore,

$$\Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{F}) \subseteq \Gamma^{-1}(\overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{F}))).$$

To prove the reverse inclusion, suppose $W \notin \Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{F})$. By Theorem 3.20 applied to $\Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{F})$, there is an $n \in \mathbb{N}$ and a Γ-pencil $L^\Gamma = L \circ \Gamma$, where L is a monic linear pencil of size n, such that $L^\Gamma(W) \not\succeq 0$ and $L^\Gamma(Y) = L(\Gamma(Y)) \succeq 0$ for all $Y \in \Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{F})$. If $Z \in$ opco$(\Gamma(\boldsymbol{F}))$, then $Z = V^*\Gamma(Y)V$ for some $Y \in \boldsymbol{F}$ and isometry V. Hence, as $Y \in \boldsymbol{F} \subseteq \Gamma\text{-}\overline{\mathrm{opco}}_{\mathrm{sot}}(\boldsymbol{F})$, it follows that $L^\Gamma(Y) \succeq 0$ and therefore,

$$L(Z) = (I \otimes V)^* L(\Gamma(Y))(I \otimes V) = (I \otimes V)^* L^\Gamma(Y)(I \otimes V) \succeq 0.$$

Thus $L \succeq 0$ on opco$(\Gamma(\boldsymbol{F}))$ and thus on $\overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{F}))$. Consequently, $\Gamma(W) \notin \overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{F}))$. Equivalently, $W \notin \Gamma^{-1}(\overline{\mathrm{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{F}))$ and the proof is complete. ∎

Proposition 3.22. *Suppose* $\boldsymbol{F} = (F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ *is a free set. If* $\boldsymbol{F}$ *is SOT-closed and bounded, then so is* $\operatorname{opco}(\Gamma(\boldsymbol{F}))$*. In particular,* $\operatorname{opco}(\Gamma(\boldsymbol{F}))$ *is also weak** *closed.*

Proposition 3.22 extends [JKMMP21, Theorem 3.3], which gave a similar statement for subsets of tuples of operators $\mathcal{B}(\mathcal{H})^{\mathrm{g}}_{\mathrm{sa}}$.

Proof. Without loss of generality, we assume throughout the proof that $\boldsymbol{F}$ is nonempty. It is immediate that $\operatorname{opco}(\Gamma(\boldsymbol{F}))$ is bounded.

Since $\boldsymbol{F}$ is bounded and SOT-closed, $\Gamma(\boldsymbol{F})$ is bounded and SOT-closed by Lemma 3.17. Since $\boldsymbol{F}$ is free, so is $\Gamma(\boldsymbol{F})$. An application of Proposition 3.5 with $\Gamma(\boldsymbol{F})$ in place of $\boldsymbol{F}$ shows $\operatorname{opco}(\Gamma(\boldsymbol{F}))$ is SOT and weak* closed. ∎

3.6. **Γ-extreme points.** We are now ready to define the Γ-analog of a free extreme point.

Definition 3.23. Let $\boldsymbol{K} = (K_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be an operator Γ-convex set (that is not assumed closed). A tuple $X \in K_n$ for $n \in \mathbb{N}\cup\{\infty\}$ is a **Γ-extreme point** if any expression of X as an operator Γ-convex combination

$$X = \sum_{i=1}^{k} V_i^* X^{(i)} V_i,$$

where $X^{(i)} \in K_{n_i}$, the $V_i \in M_{n_i,n}$ are all nonzero and $\sum_{i=1}^k V_i^*V_i = I_n$ and (X, V), where $X = \oplus_i X^{(i)}$ and $V = \operatorname{col}(V_1,\dots,V_k)$, is a Γ-pair, implies that for each i, the matrix V_i is a scalar multiple of an isometry $W_i \in M_{n_i,n}$ such that $(X^{(i)}, W_i)$ is a Γ-pair and, with respect to the range of V_i,

$$X^{(i)} = Y^{(i)} \oplus Z^{(i)}$$

for some $Y^{(i)}, Z^{(i)} \in \boldsymbol{K}$ with $Y^{(i)}$ unitarily equivalent to X. Denote the graded set of Γ-extreme points of $\boldsymbol{K}$ by $\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})$. ∎

We first establish the existence and then a spanning property, i.e., a Krein-Milman type theorem, for Γ-extreme points. For that we first show that all the free extreme points of $\operatorname{opco}(\Gamma(\boldsymbol{K}))$ lie in $\Gamma(\boldsymbol{K})$ and explain the correspondence between Γ-extreme points of $\boldsymbol{K}$ and free extreme points of $\operatorname{opco}(\Gamma(\boldsymbol{K}))$.

Lemma 3.24. *If* $\boldsymbol{F} \subseteq \mathbb{S}^{\mathrm{g}}$ *is a fully free set, then every free extreme point of* $\operatorname{opco}(\Gamma(\boldsymbol{F}))$ *is of the form* $\Gamma(X)$ *for some* $X \in \boldsymbol{F}$*; that is,* $\operatorname{ext}(\operatorname{opco}(\Gamma(\boldsymbol{F}))) \subseteq \Gamma(\boldsymbol{F})$.

Proof. Let Y be a free extreme point of $\operatorname{opco}(\Gamma(\boldsymbol{F}))$. Since $Y \in \operatorname{opco}(\Gamma(\boldsymbol{F}))$, by Lemma 3.5 there exists $X^{(i)} \in \boldsymbol{F}$ and V_i such that Y can be expressed as the operator convex combination

$$Y = \sum_{i=1}^{k} V_i^*\Gamma(X^{(i)})V_i = V^*\Gamma(X)V,$$

where $X = \oplus X^{(i)} \in \boldsymbol{F}$ and the operator $V = \mathrm{col}(V_1, \dots, V_k)$ is an isometry. Since Y is a free extreme of the operator convex set $\mathrm{opco}(\Gamma(\boldsymbol{F}))$, Lemma 3.8 applied to $\mathrm{opco}(\Gamma(\boldsymbol{F}))$ implies that V reduces $\Gamma(X)$ and hence X, since $\boldsymbol{F}$ is fully free and hence closed with respect to restriction to reducing subspaces. Thus $V^*XV \in \boldsymbol{F}$, whence $Y = \Gamma(V^*XV) \in \Gamma(\boldsymbol{F})$. ■

3.7. **Krein-Milman for operator Γ-convex sets.**

Proposition 3.25. *Let $\boldsymbol{K}$ be an operator Γ-convex set. If Z is a free extreme point of* $\mathrm{opco}(\Gamma(\boldsymbol{K}))$, *then there is a Γ-extreme point $X \in \boldsymbol{K}$ such that $Z = \Gamma(X)$. Thus,*

$$\mathrm{ext}(\mathrm{opco}(\Gamma(\boldsymbol{K}))) \subseteq \Gamma(\Gamma\text{-}\mathrm{ext}(\boldsymbol{K})).$$

In particular, if $X \in \mathbb{S}^{\mathrm{g}}$ and $\Gamma(X)$ is a free extreme point of $\mathrm{opco}(\Gamma(\boldsymbol{K}))$, *then X is a Γ-extreme point of $\boldsymbol{K}$.*

Proof. Let T be a free extreme point of $\mathrm{opco}(\Gamma(\boldsymbol{K}))$. By Lemma 3.24, there is an $n \in \mathbb{N}\cup\{\infty\}$ and $X \in K_n$ such that $T = \Gamma(X)$. To prove X is a Γ-extreme point of $\boldsymbol{K}$, suppose

$$X = \sum_{i=1}^{k} V_i^* X^{(i)} V_i \tag{3.5}$$

for a Γ-pair $(\oplus_i X^{(i)}, \mathrm{col}(V_1, \cdots, V_k))$ with $X^{(i)} \in K_{n_i}$ and $n_i \in \mathbb{N} \cup \{\infty\}$, and nonzero $V_i \in M_{n_i,n}$ satisfying $\sum_{i=1}^k V_i^* V_i = I_n$. Applying Γ on both sides of (3.5) and using the defining property of a Γ-pair gives

$$\Gamma(X) = \sum_{i=1}^{k} V_i^* \Gamma(X^{(i)}) V_i.$$

Since $\Gamma(X)$ is a free extreme point of $\mathrm{opco}(\Gamma(\boldsymbol{K}))$, each of the V_i is a scalar multiple of an isometry $W_i \in M_{n_i,n}$ satisfying $W_i^*\Gamma(X^{(i)})W_i = \Gamma(X)$ and with respect to the range of V_i,

$$\Gamma(X^{(i)}) = Y^{(i)} \oplus Z^{(i)},$$

for some $Y^{(i)}, Z^{(i)} \in \mathrm{opco}(\Gamma(\boldsymbol{K}))$, where $Y^{(i)}$ is unitarily equivalent to $\Gamma(X)$. From the identity $W_i^*\Gamma(X^{(i)})W_i = \Gamma(X)$, it follows that $W_i^* X^{(i)} W_i = X$ and thus $W_i^*\Gamma(X^{(i)})W_i = \Gamma(W_i^* X W_i)$; that is, $(X^{(i)}, W_i)$ is a Γ-pair. Further, setting $\mathcal{Y}^{(i)} = (Y_1^{(i)}, \dots, Y_{\mathrm{g}}^{(i)})$ and $\mathcal{Z}^{(i)} = (Z_1^{(i)}, \dots, Z_{\mathrm{g}}^{(i)})$, it follows that $X^{(i)} = \mathcal{Y}^{(i)} \oplus \mathcal{Z}^{(i)}$ with respect to the range of V_i and $\mathcal{Y}^{(i)}$ is unitarily equivalent to X. In particular, since $X^{(i)} \in \boldsymbol{K}$ and $\boldsymbol{K}$ is Γ-convex and therefore fully free, both $\mathcal{Y}^{(i)}$ and $\mathcal{Z}^{(i)}$ are in $\boldsymbol{K}$. Hence X is a Γ-extreme point of $\boldsymbol{K}$. ■

We offer the following partial converse to Proposition 3.25, which does not figure in future developments.

Proposition 3.26. *Let $\boldsymbol{K}$ be an SOT-closed and bounded operator Γ-convex set. If* $\mathrm{opco}(\mathrm{ext}(\mathrm{opco}(\Gamma(\boldsymbol{K}))))$ *is SOT-closed and $X \in \Gamma\text{-}\mathrm{ext}(\boldsymbol{K})$, then $\Gamma(X) \in \mathrm{ext}(\mathrm{opco}(\Gamma(\boldsymbol{K})))$.*

Proof. Suppose $n \in \mathbb{N} \cup \{\infty\}$ and $X \in K_n$ is a Γ-extreme point of $\boldsymbol{K}$. To prove $\Gamma(X)$ is a free extreme point in $\operatorname{opco}(\Gamma(\boldsymbol{K}))$, let $n_i \in \mathbb{N} \cup \{\infty\}$, tuples $Y^{(i)} \in \operatorname{opco}(\Gamma(\boldsymbol{K}))_{n_i}$ and nonzero $V_i \in M_{n_i,n}$ such that $\sum_{i=1}^k V_i^* V_i = I_n$ and

$$\Gamma(X) = \sum_{i=1}^{k} V_i^* Y^{(i)} V_i$$

be given. By Proposition 3.22, $\boldsymbol{J} := \operatorname{opco}(\Gamma(\boldsymbol{K}))$ is weak* closed. Thus, by the Krein-Milman Theorem 3.11, $\boldsymbol{J}$ is the weak* closed hull of its free extreme points. Hence $\boldsymbol{J} = \overline{\operatorname{opco}}^{w^*}(\operatorname{ext}(\boldsymbol{J})) = \operatorname{opco}(\operatorname{ext}(\boldsymbol{J}))$, since it is assumed that $\operatorname{opco}(\operatorname{ext}(\boldsymbol{J}))$ is SOT-closed and the SOT and weak* closures of the bounded convex set $\operatorname{opco}(\operatorname{ext}(\boldsymbol{J}))$ are the same. Thus each of the $Y^{(i)}$ is an operator convex combination of free extreme points of $\operatorname{opco}(\Gamma(\mathbf{K}))$. By Lemma 3.24, every free extreme point of $\operatorname{opco}(\Gamma(\mathbf{K}))$ is of the form $\Gamma(X)$ for some $X \in \boldsymbol{K}$. So each $Y^{(i)}$ is an operator convex combination

$$Y^{(i)} = \sum_{j=1}^{m_i} W_{i,j}^* \Gamma(X^{(i,j)}) W_{i,j},$$

where $\Gamma(X^{(i,j)}) \in \mathbb{S}_{r_i}^{\mathbf{r}}$ is a free extreme point of $\operatorname{opco}(\Gamma(\mathbf{K}))$, and

$$\Gamma(X) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} V_i^* W_{i,j}^* \Gamma(X^{(i,j)}) W_{i,j} V_i. \tag{3.6}$$

Now a comparison of the first $\mathtt{g}$ coordinates in (3.6) implies that X is expressed as an operator convex combination

$$X = \sum_{i=1}^{k} V_i^* W_{i,j}^* X^{(i,j)} W_{i,j} V_i.$$

Plugging this expression into (3.6) we see that $(\oplus_{i,j} X^{(i,j)}, \operatorname{col}(W_{i,j}V_i))$ is a Γ-pair. As X is Γ-extreme, for each i,j, every $W_{i,j}V_i = t_{i,j}Q_{i,j}$ is a scalar multiple of an isometry $Q_{i,j} \in M_{r_i,n}$ with $Q_{i,j}^* \Gamma(X^{(i,j)}) Q_{i,j} = \Gamma(X)$. Moreover, for each i,j, there are tuples $A^{(i,j)}, B^{(i,j)} \in \boldsymbol{K}$ such that with respect to the range of $Q_{i,j}$, the tuple $X^{(i,j)}$ decomposes as $A^{(i,j)} \oplus B^{(i,j)}$ with $A^{(i,j)} \underset{u}{\approx} X$. Now

$$V_i^* V_i = V_i^* \sum_{j=1}^{m_i} W_{i,j}^* W_{i,j} V_i = \sum_{j=1}^{m_i} V_i^* W_{i,j}^* W_{i,j} V_i = \sum_{j=1}^{m_i} t_{i,j}^2 \, Q_{i,j}^* Q_{i,j} = \sum_{j=1}^{m_i} t_{i,j}^2 \, I_n,$$

so V_i is a scalar multiple of an isometry. Since $Q_{i,j}^* \Gamma(X^{(i,j)}) Q_{i,j} = \Gamma(X)$ for all i,j, we have that

$$V_i^* Y_i V_i = \sum_{j=1}^{m_i} V_i^* W_{i,j}^* \Gamma(X^{(i,j)}) W_{i,j} V_i = \sum_{j=1}^{m_i} t_{i,j}^2 \, Q_{i,j}^* \Gamma(X^{(i,j)}) Q_{i,j}$$

is a scalar multiple of $\Gamma(X)$. Since Γ respects unitary conjugation, the decomposition of $X^{(i,j)}$ as $A^{(i,j)} \oplus B^{(i,j)}$ for each i, j implies

$$\Gamma(X^{(i,j)}) = \Gamma(A^{(i,j)} \oplus B^{(i,j)}) = \Gamma(A^{(i,j)}) \oplus \Gamma(B^{(i,j)})$$

with $\Gamma(A^{(i,j)})$ unitarily equivalent to $\Gamma(X)$ proving that $\Gamma(X)$ is free extreme. ∎

The previous proposition implies that the existence of Γ-extreme points is implied by the existence of free extreme points. Proposition 3.22 gives a natural condition under which $\operatorname{opco}(\Gamma(\boldsymbol{K}))$ is closed and bounded and hence, ensuring the latter exist.

Theorem 3.27 (**Krein-Milman theorem for Γ-convex sets**). *Suppose* $\Gamma(0) = 0$. *Let* $\boldsymbol{K}$ *be an SOT-closed and bounded operator* Γ*-convex set, which contains* 0. *Then* $\Gamma\text{-}\operatorname{ext}(\boldsymbol{K}) \neq \emptyset$ *and*

$$\boldsymbol{K} = \Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})). \tag{3.7}$$

Remark 3.28. Assuming $\boldsymbol{K}$ is bounded and weak* closed it follows that $\boldsymbol{K}$ is WOT-closed and hence SOT-closed. (See Remark 3.2.) Thus Theorem 3.27 applies. Thus, if $\boldsymbol{K}$ is weak* closed and bounded, operator Γ-convex and contains 0, then $\boldsymbol{K} = \Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K}))$. Since $\Gamma\text{-}\operatorname{opco}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})) \subseteq \boldsymbol{K}$ and $\boldsymbol{K}$ is weak* closed,

$$\Gamma\text{-}\overline{\operatorname{opco}}_{w^*}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})) \subseteq \boldsymbol{K} = \Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})) \subseteq \Gamma\text{-}\overline{\operatorname{opco}}_{w^*}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})).$$

Thus, when $\boldsymbol{K}$ is weak* closed,

$$\boldsymbol{K} = \Gamma\text{-}\overline{\operatorname{opco}}_{w^*}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})) = \Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})).$$

In particular, letting $\boldsymbol{B}$ denote the weak* closed fully free hull of $\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})$, one obtains

$$\Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\boldsymbol{B}) = \Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K})) = \boldsymbol{K}.$$

Proof of Theorem 3.27. Since $\boldsymbol{K}$ is SOT-closed and bounded, by Proposition 3.22 the operator convex set $\operatorname{opco}(\Gamma(\boldsymbol{K}))$ is also SOT-closed and bounded. Since $\operatorname{opco}(\Gamma(\boldsymbol{K}))$ is convex, Remark 3.2 implies that it is also weak* closed. Hence, an application of the Banach-Alaoglu theorem shows that it is weak* compact. By Theorem 3.11, the compact operator convex set $\operatorname{opco}(\Gamma(\boldsymbol{K}))$ has a free extreme point Y. By Lemma 3.24, there is a $X \in \boldsymbol{K}$ such that $Y = \Gamma(X)$ and thus, by Proposition 3.25, X is a Γ-extreme point of $\boldsymbol{K}$.

To prove (3.7) assume there is an $X \in \boldsymbol{K}$ not lying in $\Gamma\text{-}\overline{\operatorname{opco}}_{\mathrm{sot}}(\Gamma\text{-}\operatorname{ext}(\boldsymbol{K}))$. An application of item (e) in Proposition 3.21 to $\boldsymbol{J} = \Gamma\text{-}\operatorname{ext}(\boldsymbol{K})$ gives

$$\Gamma(X) \notin \overline{\operatorname{opco}}^{\mathrm{sot}}(\Gamma(\boldsymbol{J})).$$

Since $\operatorname{opco}(\Gamma(\boldsymbol{K}))$ is SOT-closed, and thus weak* closed, Theorem 3.11 gives

$$\overline{\operatorname{opco}}^{w^*}(\operatorname{ext}(\operatorname{opco}(\Gamma(\boldsymbol{K}))) = \overline{\operatorname{opco}}^{w^*}(\Gamma(\boldsymbol{K})) = \operatorname{opco}(\Gamma(\boldsymbol{K})).$$

On the other hand, Proposition 3.25 implies $\operatorname{ext}(\operatorname{opco}(\Gamma(\boldsymbol{K})) \subseteq \Gamma(\boldsymbol{J})$. Thus

$$\operatorname{opco}(\Gamma(\boldsymbol{K})) = \overline{\operatorname{opco}}^{w^*}(\Gamma(\boldsymbol{K})) \subseteq \overline{\operatorname{opco}}^{w^*}(\Gamma(\boldsymbol{J})) \subseteq \overline{\operatorname{opco}}^{w^*}(\Gamma(\boldsymbol{K})) = \operatorname{opco}(\Gamma(\boldsymbol{K})),$$

as, for bounded convex sets, SOT and weak* closures are the same. Thus

$$\operatorname{opco}(\Gamma(\boldsymbol{K})) = \overline{\operatorname{opco}}^{\rm sot}(\Gamma(\boldsymbol{J})).$$

Since $X \in \boldsymbol{K}$, we obtain the contradiction, $\Gamma(X) \in \operatorname{opco}(\Gamma(\boldsymbol{K})) = \overline{\operatorname{opco}}^{\rm sot}(\Gamma(\boldsymbol{J})) \not\ni \Gamma(X)$. Hence $\Gamma\text{-}\overline{\operatorname{opco}}^{\rm sot}(\boldsymbol{J}) = \boldsymbol{K}$ as claimed. ■

We close this subsection with a weak converse to the Γ-version of the Krein-Milman Theorem, Theorem 3.27.

The analog of an boundary for an Agler family of operators adapted to operator Γ-convex sets reads as follows. A **Γ-boundary** of an operator Γ-convex set $\boldsymbol{K}$ is a SOT-closed fully free set $\boldsymbol{B} \subseteq \boldsymbol{K}$ such that $\Gamma\text{-}\overline{\operatorname{opco}}_{\rm sot}(\boldsymbol{B}) = \boldsymbol{K}$. Let $\mathscr{B}_\Gamma$ denote the set of Γ-boundaries for $\boldsymbol{K}$ and set

$$\partial_\Gamma^A \boldsymbol{K} = \bigcap \{\boldsymbol{B} : \boldsymbol{B} \in \mathscr{B}_\Gamma\}.$$

A point $X \in K_n$ is an **Γ-Agler extreme point** of $\boldsymbol{K}$, if $Y \in K_m$ and $V : \mathcal{H}_n \to \mathcal{H}_m$ is an isometry such that $(Y, V) \in \mathcal{C}_\Gamma$ and $X = V^*YV$, then the range of V reduces Y. Let $\Gamma\text{-}\operatorname{ext}^A(\boldsymbol{K})$ denote the Γ-Agler extreme points of $\boldsymbol{K}$. Evidently, $\Gamma\text{-}\operatorname{ext}(\boldsymbol{K}) \subseteq \Gamma\text{-}\operatorname{ext}^A(\boldsymbol{K})$. Theorem 3.27 (also see Remark 3.28) implies that the SOT-closed *fully free hull* of the free Γ-extreme points of a closed and bounded operator Γ-convex set $\boldsymbol{K}$ is a Γ-boundary for $\boldsymbol{K}$.

Proposition 3.29. *If $\boldsymbol{B}$ is a Γ-boundary of a bounded and SOT-closed operator Γ-convex set $\boldsymbol{K}$, then $\Gamma\text{-}\operatorname{ext}^A(\boldsymbol{K}) \subseteq \boldsymbol{B}$. Hence $\Gamma\text{-}\operatorname{ext}(\boldsymbol{K}) \subseteq \boldsymbol{B}$ and therefore the SOT-closed fully free hull of $\Gamma\text{-}\operatorname{ext}(K)$ is $\partial_\Gamma^A K$ and is a boundary for $\boldsymbol{K}$. In particular, $\partial_\Gamma^A \boldsymbol{K}$ is the smallest boundary $\boldsymbol{K}$.*

Proof. Suppose $\boldsymbol{B} \subseteq \boldsymbol{K}$ is fully free SOT-closed and $\Gamma\text{-}\overline{\operatorname{opco}}_{\rm sot}(\boldsymbol{B}) = \boldsymbol{K}$. By Proposition 3.22, $\operatorname{opco}(\Gamma(\boldsymbol{B}))$ is SOT-closed and bounded. By Proposition 3.21(e),

$$\boldsymbol{K} = \Gamma\text{-}\overline{\operatorname{opco}}_{\rm sot}(\boldsymbol{B}) = \Gamma^{-1}(\overline{\operatorname{opco}}^{\rm sot}(\Gamma(\boldsymbol{B}))) = \Gamma^{-1}(\operatorname{opco}(\Gamma(\boldsymbol{B}))). \tag{3.8}$$

Now suppose $X \in \Gamma\text{-}\operatorname{ext}^A(\boldsymbol{K})$. Since $X \in \boldsymbol{K}$, equation (3.8) implies $\Gamma(X) \in \operatorname{opco}(\Gamma(\boldsymbol{B}))$. Hence there is an isometry V and $B \in \boldsymbol{B}$ such that $\Gamma(X) = V^*\Gamma(B)V$ and thus $X = V^*BV$ and $(B, V) \in \mathcal{C}_\Gamma$. Since X is Agler extreme, it follows that the range of B reduces V and since $\boldsymbol{B}$ is closed under restrictions to reducing subspaces, $X \in \boldsymbol{B}$. ■

3.8. **The free parabola.** We close this section with the example of the free parabola.

Example 3.30. Let $\boldsymbol{K} = (K_n)_{n \in \mathbb{N} \cup \{\infty\}}$ denote the operator y^2-convex set

$$\boldsymbol{K} = \{(X, Y) \mid -(Y^2 + I) \preceq X \preceq Y^2 + I,\ Y^2 \preceq I\}$$

and let $\boldsymbol{K}^{\rm mat} = (K_n)_{n \in \mathbb{N}}$ denote the matrix convex set consisting of the finite levels of $\boldsymbol{K}$. It is straightforward to check that $\boldsymbol{K}$, and hence $\boldsymbol{K}^{\rm mat}$, is bounded and SOT-compact and Γ-convex. By Lemmas 3.17 and 3.5, $\overline{\operatorname{opco}}^{w^*}(\Gamma(\boldsymbol{K})) = \operatorname{opco}(\Gamma(\boldsymbol{K}))$. Below we show that (1) $\overline{\operatorname{opco}}^{w^*}(\Gamma(\boldsymbol{K})) = \operatorname{opco}(\Gamma(\boldsymbol{K}))$ is described by a linear matrix inequality; (2) $\operatorname{matco}(\Gamma(\boldsymbol{K}^{\rm mat}))$

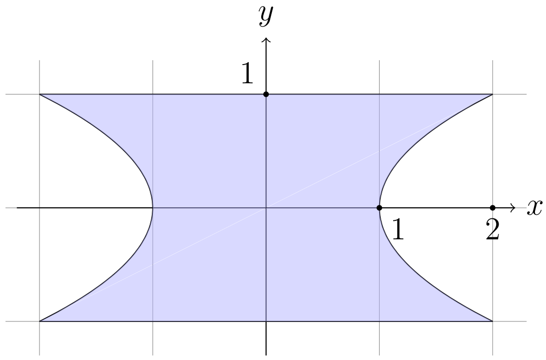

FIGURE 1. The first level component K_1 of $\boldsymbol{K}$.

is a spectrahedron and (3) it is the matrix convex hull of its finite level free extreme points; (4) $\boldsymbol{K}^{\text{mat}}$ is the Γ-convex hull of its finite level Γ-extreme points.

Let $\boldsymbol{J}$ denote the bounded SOT-closed operator convex set consisting of of all tuples (X, Y, Z) described by the linear matrix inequality

$$-(Z+I) \preceq X \preceq Z+I$$
$$0 \preceq Z \preceq I$$
$$0 \preceq \begin{pmatrix} I & Y \\ Y & Z \end{pmatrix},$$

and note it contains $\Gamma(\boldsymbol{K})$. The affine linear transformation $(X, Y, Z) \mapsto (X, Y, W = Z - \frac{1}{2}I)$ sends $\boldsymbol{J}$ to the free spectrahedron $\mathcal{D}_A$, where

$$L_A(x, y, w) = \left(\tfrac{3}{2} + w - x\right) \oplus \left(\tfrac{3}{2} + x + w\right) \oplus \left(1 - 2w\right) \oplus \left(1 + 2w\right) \oplus \begin{pmatrix} 1 & \sqrt{2}y \\ \sqrt{2}y & 1 + 2w \end{pmatrix}.$$

Since $\mathcal{D}_A$ is closed under complex conjugation, it is spanned by its finite level free extreme points by [EH19, Theorem 1.3] (see Proposition 3.12). Thus so is $\boldsymbol{J}^{\text{mat}}$; that is,

$$\operatorname{matco}(\operatorname{ext}(\boldsymbol{J}) \cap \boldsymbol{J}^{\text{mat}}) = \boldsymbol{J}^{\text{mat}}. \tag{3.9}$$

Claim: If $(X, Y, Z) \in \boldsymbol{J}^{\text{mat}}$ is a free extreme point of $\boldsymbol{J}$, then $(X, Y, Z) = (X, Y, Y^2) \in \Gamma(\boldsymbol{K}^{\text{mat}})$.

Before proceeding, we collect some consequences of this claim. First,

$$\operatorname{ext}(\boldsymbol{J}) \cap \boldsymbol{J}^{\text{mat}} \subseteq \Gamma(\boldsymbol{K}^{\text{mat}}) \subseteq \operatorname{matco}(\Gamma(\boldsymbol{K}^{\text{mat}})) \subseteq \boldsymbol{J}^{\text{mat}}. \tag{3.10}$$

Combining the identities of equations (3.9) and (3.10) gives

$$\operatorname{matco}(\Gamma(\boldsymbol{K}^{\text{mat}})) = \boldsymbol{J}^{\text{mat}}.$$

Thus $\operatorname{matco}(\Gamma(\boldsymbol{K}^{\text{mat}}))$ is a spectrahedron described by a linear matrix inequality so that (2) holds. Moreover, by [JKMMP21, Proposition 3.7(b)], $\overline{\operatorname{opco}}^{\text{sot}}(\boldsymbol{J}^{\text{mat}}) = \boldsymbol{J}$. Since for convex sets SOT and weak* closures coincide, $\overline{\text{opco}}^{w^*}(\boldsymbol{J}^{\text{mat}}) = \boldsymbol{J}$ and we conclude that $\overline{\text{opco}}^{w^*}(\Gamma(\boldsymbol{K}^{\text{mat}})) = \boldsymbol{J}$. Since also

$$\boldsymbol{J} = \overline{\text{opco}}^{w^*}(\Gamma(\boldsymbol{K}^{\text{mat}})) \subseteq \overline{\text{opco}}^{w^*}(\Gamma(\boldsymbol{K})) = \text{opco}(\Gamma(\boldsymbol{K})) \subseteq \boldsymbol{J},$$

it follows $\text{opco}(\Gamma(\boldsymbol{K})) = \boldsymbol{J}$ so that (1) holds. Further by equation (3.9), (3) holds.

By Proposition 3.25 and the Claim, if $(X,Y,Z) \in \text{ext}(\boldsymbol{J}) \cap \boldsymbol{J}^{\text{mat}}$, then $(X,Y,Z) = \Gamma(X,Y)$ and $(X,Y) \in \Gamma\text{-}\text{ext}(\boldsymbol{K})$.

Now let $(X,Y) \in \boldsymbol{K}^{\text{mat}}$ be given. Since $\Gamma(X,Y) \in \boldsymbol{J}^{\text{mat}}$, by another application of [EH19, Theorem 1.3], there exist free extreme points $(X_i,Y_i,Z_i) \in \boldsymbol{J}^{\text{mat}}$ of $\boldsymbol{J}$ and V_i such that $\sum_i V_i^* V_i = I$ and $\sum V_i^*(X_i,Y_i,Z_i)V = \Gamma(X,Y)$. Since $(X_i,Y_i,Z_i) \in \text{ext}(\boldsymbol{J}) \cap \boldsymbol{J}^{\text{mat}}$, we have $(X_i,Y_i,Z_i) = \Gamma(X_i,Y_i)$ and $(X_i,Y_i) \in \Gamma\text{-}\text{ext}(\boldsymbol{K})$. Finally, setting $V^* = \begin{pmatrix} V_1^* & \dots & V_N^* \end{pmatrix}$ and $(\widetilde{X},\widetilde{Y}) = \oplus(X_i,Y_i)$,

$$\Gamma(\widetilde{X},\widetilde{Y}) = \Gamma(X,Y) = V^*\Gamma(\widetilde{X},\widetilde{Y})V.$$

Thus $((\widetilde{X},\widetilde{Y}),V)$ is a Γ-pair and $(X,Y) = V^*(\widetilde{X},\widetilde{Y})V$. So $(X,Y) \in \Gamma\text{-}\text{conv}(\Gamma\text{-}\text{ext}(\boldsymbol{K}) \cap \boldsymbol{K}^{\text{mat}})$ and therefore $\boldsymbol{K}^{\text{mat}} = \Gamma\text{-}\text{conv}(\Gamma\text{-}\text{ext}(\boldsymbol{K}) \cap \boldsymbol{K}^{\text{mat}})$ as claimed; that is, (4) holds.

Turning to the proof of the Claim, we first argue that if $(X,Y,Z) \in \boldsymbol{J}^{\text{mat}}$ is a free extreme point of $\boldsymbol{J}$, then Y and Z commute. To do so, it suffices to show, given $(X,Y,Z) \in \boldsymbol{J}^{\text{mat}}$, that either Y and Z commute or there exists a tuple $(\varphi,\alpha,u) \neq 0$ and β, w, ψ such that

$$X_* = \begin{pmatrix} X & \varphi \\ \varphi^* & \psi \end{pmatrix} \quad Y_* = \begin{pmatrix} Y & \alpha \\ \alpha^* & \beta \end{pmatrix}, \quad Z_* = \begin{pmatrix} Z & u \\ u^* & w \end{pmatrix}$$

satisfies $I \succeq Z_* \succeq Y_*^2$ and $-(Z_*+I) \preceq X_* \preceq Z_* + I$; that is, $(X_*,Y_*,Z_*) \in \boldsymbol{K}^{\text{mat}}$.

Suppose Y and Z do not commute and let $\mathcal{H}$ denote the finite-dimensional space (X,Y,Z) that acts upon. Let $\mathcal{K} = \ker(I-Z) \cap \ker(Z-Y^2)$. If $\mathcal{K} = \mathcal{H}$, then Z is the identity. Thus $\mathcal{K}^\perp \neq \{0\}$.

From the definition of $\mathcal{K}$,

$$\text{range}(I-Z) + \text{range}(Z-Y^2) = \mathcal{K}^\perp \neq \{0\}. \tag{3.11}$$

Let $\Delta^2 = Z - Y^2$. If $Y\Delta^2 = 0$, then $YZ = Y^3$ and hence $YZ = Y^3 = (YZ)^* = ZY$. Thus Y and Z and commute, contradicting our earlier assumption. It follows that $Y\Delta^2 \neq 0$. Choose h such that $Y\Delta^2 h \neq 0$ and let $\alpha = \Delta^2 h$. From the spanning condition of equation (3.11), there exists $f, g \in \mathcal{K}^\perp$ such that

$$(I-Z)f + \Delta^2 g = Y\Delta^2 h = Y\alpha. \tag{3.12}$$

Let

$$u = (I-Z)f.$$

In particular,

$$u - Y\alpha = (1-Z)f - Y\alpha = -\Delta^2 g, \tag{3.13}$$

Since $u \in \operatorname{range}(I-Z)$, there exist $s_0, w > 0$ such that

$$\begin{pmatrix} I-Z & -su \\ -su^* & 1-w \end{pmatrix} \succeq 0.$$

With this w fixed and $0 < s \leq s_0$, let

$$Y_*(s) = \begin{pmatrix} Y & s\alpha \\ s\alpha^* & 0 \end{pmatrix}, \quad Z_*(s) = \begin{pmatrix} Z & su \\ su^* & w \end{pmatrix}$$

and observe from equation (3.13) that

$$\begin{aligned} Z_*(s) - Y_*^2(s) &= \begin{pmatrix} Z - [Y^2 + s^2\alpha\alpha^*] & s[u - Y\alpha] \\ * & w - s^2\alpha^*\alpha \end{pmatrix} = \begin{pmatrix} \Delta^2 - s^2\alpha\alpha^* & -s\Delta^2 g \\ * & w - s^2\alpha^*\alpha \end{pmatrix} \\ &= \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I - s^2\Delta hh^*\Delta & -s\Delta g \\ * & w - s^2\alpha^*\alpha \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & w - s^2\alpha^*\alpha \end{pmatrix}. \end{aligned} \tag{3.14}$$

Without loss of generality, assume $0 < s_0$ is small enough so that

$$Z_*(s) - Y_*^2(s) \succeq \begin{pmatrix} \Delta^2 & 0 \\ 0 & w \end{pmatrix} - \begin{pmatrix} -s^2\Delta hh^*\Delta & -s\Delta g \\ -sg^*\Delta & -s^2\alpha^*\alpha \end{pmatrix} \succeq 0,$$

which is possible since the matrix in the middle of the right hand side of equation (3.14) is positive definite for s_0 small.

Finally, we construct a suitable dilation X_* of X. If $v \in \ker(Z+I-X) \cap \ker(Z+I+X)$, then $Xv = (Z+I)v = -(Z+I)v$ and hence v lies in the kernel of $Z+I$. Now $Z \succeq 0$ implies $v = 0$. So $\ker(Z+I-X) \cap \ker(Z+I+X) = \{0\}$, or alternatively, $\operatorname{range}(Z+I-X) + \operatorname{range}(Z+I+X) = \mathcal{H}$, which implies there are vectors $u_1, u_2 \in \mathcal{H}$ such that

$$u = (Z+I-X)u_1 + (Z+I+X)u_2.$$

Now set

$$\phi = (Z+I-X)u_1 - (Z+I+X)u_2$$

and note that $u - \phi = 2(Z+I+X)u_2 \in \operatorname{range}(Z+I+X)$ and $u + \phi = 2(Z+I-X)u_1 \in \operatorname{range}(Z+I-X)$. Hence, setting

$$X_*(s) = \begin{pmatrix} X & s\phi \\ s\phi^* & 0 \end{pmatrix}$$

and choosing $0 < s \leq s_0$ sufficiently small,

$$Z_*(s) + I \pm X_*(s) = \begin{pmatrix} Z + I \pm X & s(u \pm \phi) \\ s(u \pm \phi)^* & w+1 \end{pmatrix}$$

are both positive semidefinite. We conclude, if Y and Z do not commute, then (X, Y, Z) is not an Agler extreme point, and hence by Proposition 3.13, not a free extreme point of $\boldsymbol{J}^{\mathrm{mat}}$.

Now suppose Y and Z commute. To show either $Z = Y^2$ or (X, Y, Z) is not an extreme point of $\boldsymbol{J}$, first note that as Y and Z commute and they are both self-adjoint, they are, without loss of generality, (simultaneously) diagonal. Let $Y_{j,j}$ and $Z_{j,j}$ denote their diagonal entries and note that $Y^2 \preceq Z$ is equivalent to $|Y_{j,j}| \leq \sqrt{Z_{j,j}}$. Suppose $Y^2 \neq Z$. In this case, without loss of generality, $|Y_{1,1}| < \sqrt{Z_{1,1}}$. For notational convenience, let $y = Y_{1,1}$ and $z = Z_{1,1}$. Hence there exists an $0 < \lambda < 1$ such that $y = \lambda\sqrt{z} - (1-\lambda)\sqrt{z}$. Let $Y^{\pm}$ denote the diagonal matrices with $Y^{\pm}_{1,1} = \pm\sqrt{z}$ and $Y^{\pm}_{j,j} = Y_{j,j}$ for $j > 1$. Thus $(Y^{\pm})^2 \preceq Z$ and therefore $(X, Y^{\pm}, Z) \in \boldsymbol{K}$. Moreover,

$$(X, Y, Z) = \lambda(X, Y^+, Z) + (1 - \lambda)(X, Y^-, Z),$$

and hence (X, Y, Z) is not free extreme. Thus, if (X, Y, Z) is extreme in $\boldsymbol{J}$, then $Z = Y^2$ as claimed. ∎

4. Free analog of the Lasserre-Parrilo construction in the Γ-convex setting

In the commutative setting, a construction due to Lasserre [Las09] (see also Parrilo [Par06]) assigns to a semialgebraic set $D_p = \{x \in \mathbb{R}^g \mid p(x) \geq 0\}$ a sequence of spectrahedra whose projections give a decreasing family of convex semialgebraic sets, called *relaxations*, approximating the convex hull of D_p. Under mild assumptions [HN09, HN10], such an approximation scheme is *exact*, in which case the convex hull of D_p is presented as a projection of a spectrahedron, called *spectrahedrop* [BPR13].

4.1. **Introduction and basic notation.** A free analog of the Lasserre-Parrilo relaxations was introduced and studied in [HKM16]. By adapting those methods, we give a construction of a sequence $\mathcal{D}^{\Gamma}_{A(d)}$ of free Γ-spectrahedra in increasingly larger spaces whose projections give better and better approximations of the operator Γ-convex hull of the positivity domain $\mathcal{D}_p$ of a symmetric matrix-valued noncommutative polynomial p,

$$\mathcal{D}_p = (\mathcal{D}_p(n))_n = (\{X \in \mathbb{S}^{\mathtt{g}}_n \mid p(X) \succeq 0\})_n.$$

Fix a symmetric matrix-valued noncommutative polynomial $p \in M_\mu(\mathbb{C}\langle x\rangle)$ of degree $\leq \delta$ in $\mathtt{g}$ variables. Thus p takes the form

$$p(x) = \sum_{|\alpha| \leq \delta} p_\alpha \alpha, \tag{4.1}$$

where $p_\alpha = p_{\alpha^*}$.

In the first part of this section free analogs of moment sequences and Hankel matrices adapted to the Γ-convex setting are used to construct an infinite free Γ-spectrahedron $\mathfrak{L}_p$ and a canonical projection of $\mathfrak{L}_p$ onto the matricial levels of the operator Γ-convex hull of $\mathcal{D}_p$. In the final part of this section we explain how $\mathfrak{L}_p$ naturally determines a sequence of finite free Γ-spectrahedra whose projections, i.e., *free Γ-spectrahedrops*, are increasingly finer outer approximations to the matricial levels of the operator Γ-convex hull of $\mathcal{D}_p$.

4.1.1. *Free* Γ*-spectrahedra and their projections.* Fix $\Gamma(x) = (\gamma_1, \dots, \gamma_{\mathtt{r}})$ with $\gamma_i = x_i$ for $i = 1, \dots, \mathtt{g}$. We extend the notion of a Γ-pencil to any matrix-valued polynomial of the form

$$L^\Gamma(x,y) = A_0 + \sum_{i=1}^{\mathtt{g}} A_i\, x_i + \sum_{j=\mathtt{g}+1}^{\mathtt{r}} A_j \gamma_j(x) + \sum_{k=1}^{\mathtt{h}} B_k y_k \tag{4.2}$$

for some $\mathtt{h} \in \mathbb{N}$. Here the y_k are new symmetric noncommutative variables (that do not appear in any of the γ_j). In other words, we infer that the terms that determine Γ are γ_i for $i = \mathtt{g}+1, \dots, \mathtt{r}$ and refer to the polynomial obtained by adding new variables as linear terms to a Γ-pencil as a Γ-pencil as well.

Given a Γ-pencil L^Γ as in (4.2), the free set $\operatorname{proj}_x(\mathcal{D}_{L^\Gamma}) = (\operatorname{proj}_x(\mathcal{D}_{L^\Gamma})_n$ with

$$\operatorname{proj}_x(\mathcal{D}_{L^\Gamma})_n = \{X \in \mathbb{S}_n^{\mathtt{g}} \mid \exists Y \in \mathbb{S}_n^{\mathtt{h}} \text{ such that } L^\Gamma(X,Y) \succeq 0\} \tag{4.3}$$

is called a **free Γ-spectrahedrop**. Note that by definition, we project the positivity domain of L^Γ onto the first $\mathtt{g}$ variables in the linear part of L^Γ. It is easy to check that a free Γ-spectrahedron is a Γ-convex set, or see Proposition 2.17.

4.1.2. *Free operator semialgebraic sets.* Recall the definitions of an operator convex set (hull) and an operator Γ-convex set (hull) from Section 3. Similarly, we extend the matricial positivity domain $\mathcal{D}_p$ of p by including an operator level in $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space. Recall from Section 3 that $\mathbb{S}^{\mathtt{g}}$ is the graded set $(\mathbb{S}_n^{\mathtt{g}})_{n\in\mathbb{N}\cup\{\infty\}}$ where $\mathbb{S}_n^{\mathtt{g}}$ is identified with $\mathcal{B}(\mathcal{H}_n)_{\mathrm{sa}}^{\mathtt{g}}$ for an n-dimensional Hilbert space $\mathcal{H}_n$ for $n \in \mathbb{N}$, and $\mathcal{H}_\infty = \mathcal{H}$.

Given a symmetric free matrix-valued polynomial p let

$$\mathcal{D}_p^\infty = \{X \in \mathbb{S}^{\mathtt{g}} \mid p(X) \succeq 0\}. \tag{4.4}$$

The set $\mathcal{D}_p^\infty$ is the **free operator semialgebraic set** defined by the polynomial p. We extend the notion of a Γ-convex hull as in Definition 1.1 to accept free operator semialgebraic sets as follows. The **Γ-convex hull** of $\mathcal{D}_p^\infty$ is the graded set

$$\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty) = \big(\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)(n)\big)_n,$$

where $n \in \mathbb{N}$. (Most of what follows also works for *operator* Γ-convex hulls, $\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty)$. See Subsection 4.7.) Here a tuple $X \in \mathbb{S}_n^{\mathtt{g}}$ lies in $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)(n)$ if there exists a tuple $Y \in \mathcal{D}_p^\infty$ acting on some (finite-dimensional or separable) Hilbert space $\mathcal{H}$ and an isometry $V : \mathbb{C}^n \to \mathcal{H}$ such that (Y, V) is a Γ-pair and $X = V^*YV$. By construction this Γ-convex hull is the smallest Γ-convex set *spanned* by $\mathcal{D}_p^\infty$.

Example 4.1. From [BP65] (see also [Tao19]), if H is an infinite-dimensional separable Hilbert space, then the operator

$$C = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

acting on $K = H \oplus H$, where I is the identity on H, is a commutator; that is, there exists operators X, Y on K such that $[X, Y] = XY - YX = C$. Let $X_1 = \frac{1}{2}(X + X^*)$ and $X_2 = \frac{1}{2i}(X - X^*)$ denote the real and imaginary parts of X and similarly for Y. In particular, $X = X_1 + iX_2$ and likewise for Y. Observe,

$$\begin{aligned} 4\left([X,Y]+[X,Y]^*\right) &= 4\left([X,Y]-[X^*,Y^*]\right) = [X_1+iX_2, Y_1+iY_2] - [X_1-iX_2, Y_1-iY_2] \\ &= 2i\left([X_2,Y_1]+[X_1,Y_2]\right). \end{aligned}$$

Hence, setting $p(x_1, x_2, y_1, y_2) = 2i([x_2, y_1] + [x_1, y_2]) - 1$ we have

$$p(X_1, X_2, Y_1, Y_2) = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \succ 0.$$

On the other hand, for any tuple (W_1, W_2, Z_1, Z_2) of self-adjoint matrices acting on a Hilbert space of finite dimension n we have $p(W_1, W_2, Z_1, Z_2)$ has trace $-n$ and thus is not positive semidefinite. Thus

$$\mathcal{D}_p^\infty \neq \emptyset = \mathcal{D}_p. \tag{4.5}$$

By direct summing $N \pm x_i$ and $N \pm y_i$ for an appropriate $N \in \mathbb{N}$, we can obtain an Archimedean polynomial p satisfying (4.5).

In the setting of xy-convexity, where Γ consists of (the real and imaginary parts of) $x_i y_j$, the constructed polynomial p is a Γ-pencil. Thus $\mathcal{D}_p^\infty$ is a operator Γ-convex set with empty finite levels $\mathcal{D}_p$. ■

4.2. **Free Hankel matrices and moment sequences.** The main tool for constructing the spectrahedral lifts and the Γ-convex projections approximating $\mathcal{D}_p$ are free analogs of Hankel matrices as in [HKM16]. In the classical matrix-valued single variable theory, a Hankel matrix $H = (H_{i,j})$ with entries from M_n for some n has the property of being constant on anti-diagonals, that is, there is a *moment sequence* $(A_k)_k$ of self-adjoint matrices from M_n such that $H_{i,j} = A_{i+j}$.

Definition 4.2. Let n be a positive integer and $Y = (Y_\alpha)_\alpha$ a sequence of $n \times n$ matrices indexed by words α in the free symmetric variables $x_1, \dots, x_{\mathtt{g}}$. Let $\Gamma = (\gamma_1, \dots, \gamma_{\mathtt{r}})$ be a tuple of symmetric free polynomials with $\gamma_j = x_j$ for $1 \le j \le \mathtt{g} \le \mathtt{r}$ and monomial expansions $\gamma_j = \sum_k \gamma_{j,k} m_{j,k}$, where $\gamma_{j,k} \in \mathbb{C}$ and $m_{j,k}$ are words in x.

(a) The sequence Y is a **Γ-moment sequence** if $Y_\emptyset = I$, if, for each α,

$$Y_{\alpha^*} = (Y_\alpha)^*, \tag{4.6}$$

and for each $j = 1, \dots, \mathtt{r}$,

$$\sum_k \gamma_{j,k}\, Y_{m_{j,k}(x)} = \sum_k \gamma_{j,k}\, m_{j,k}(Y_{x_1}, \dots, Y_{x_{\mathtt{g}}}) = \gamma_j(Y_{x_1}, \dots, Y_{x_{\mathtt{g}}}). \tag{4.7}$$

In particular, property (4.6) implies each of the Y_{x_j} is self-adjoint.

(b) Let $\mathfrak{M}^\Gamma(n)$ denote the set of Γ-moment sequences $(Y_\alpha)_\alpha$ with $Y_\alpha \in M_n$.

(c) The **free Γ-Hankel matrix** associated to $Y \in \mathfrak{M}^\Gamma$ is defined as

$$H(Y) = \big(Y_{\alpha^*\beta}\big)_{\alpha,\beta}, \tag{4.8}$$

while for a positive integer d, the corresponding **truncated free Γ-Hankel matrix** is

$$H_d(Y) = \big(Y_{\alpha^*\beta}\big)_{|\alpha|,|\beta|\le d}. \tag{4.9}$$

These definitions are taken from [HKM16], but are here only applied to Γ-moment sequences Y.

(d) For a $\mu \times \mu$ symmetric matrix-valued polynomial p as in equation (4.1) and $Y \in \mathfrak{M}^\Gamma(n)$, the **$p$-localizing matrix** $H_p^{\Uparrow}(Y) = (H^{\Uparrow}(Y)_{\alpha,\beta})_{\alpha,\beta}$ with $n\mu \times n\mu$ matrix with block entry at position (α,β)

$$H_p^{\Uparrow}(Y)_{\alpha,\beta} = \sum_{|\gamma|\le\delta} p_\gamma \otimes Y_{\alpha^*\gamma\beta}.$$

For a positive integer d, the **d-truncated localizing matrix** of p is

$$H_{p,d}^{\Uparrow}(Y) = \big(H_p^{\Uparrow}(Y)_{\alpha,\beta}\big)_{|\alpha|,|\beta|\le d}.$$ ■

Example 4.3. To illustrate the structure of a Γ-Hankel matrix, let $\Gamma(x_1,x_2) = (x_1,x_2,x_2^2)$. To shorten the notation we write Y_j instead of Y_{x_j} and substitute the index α in Y_α by the indices of the variables in α, e.g., we write Y_{121} instead of $Y_{x_1x_2x_1}$. Equation (4.7) then says that $Y_{22} = X_2^2$ and the truncated Γ-Hankel matrix for $d = 2$ equals

$$H_2(Y) = \begin{pmatrix} 1 & X_1 & X_2 & Y_{11} & Y_{12} & Y_{21} & X_2^2 \\ X_1 & Y_{11} & Y_{12} & Y_{111} & Y_{112} & Y_{121} & Y_{122} \\ X_2 & Y_{21} & X_2^2 & Y_{211} & Y_{212} & Y_{221} & Y_{222} \\ Y_{11} & Y_{111} & Y_{112} & Y_{1111} & Y_{1112} & Y_{1121} & Y_{1122} \\ Y_{21} & Y_{211} & Y_{212} & Y_{2111} & Y_{2112} & Y_{2121} & Y_{2122} \\ Y_{12} & Y_{121} & Y_{122} & Y_{1211} & Y_{1212} & Y_{1221} & Y_{1222} \\ X_2^2 & Y_{221} & Y_{222} & Y_{2211} & Y_{2212} & Y_{2221} & Y_{2222} \end{pmatrix},$$

which is a x_2^2-pencil. ■

Remark 4.4. Any $Z \in \mathcal{D}_p^\infty(n)$, where $n \in \mathbb{N} \cup \{\infty\}$, together with an isometry $V \in M_{n,m}$ such that (Z,V) is a Γ-pair, determines a Γ-moment sequence

$$Y_\alpha = V^*Z^\alpha V \in M_m, \tag{4.10}$$

where $Z^\alpha = \alpha(Z)$. For example, if $\alpha = x_1x_2^2x_1$, then $Y_\alpha = V^*Z_1Z_2^2Z_1V$. It is easy to verify that $(Y_\alpha)_\alpha$ is indeed a Γ-moment sequence and direct computation shows

$$H(Y) \succeq 0 \quad \text{and} \quad H_p^{\Uparrow}(Y) \succeq 0. \tag{4.11}$$ ■

4.3. **Step 1: construction of the lift.** Fix a symmetric matrix-valued free polynomial p. Let $\mathfrak{L}_p^\Gamma = (\mathfrak{L}_p^\Gamma(n))_n$ be defined by

$$\mathfrak{L}_p^\Gamma(n) = \{Y = (Y_\alpha)_\alpha \in \mathfrak{M}^\Gamma(n) \mid \;\; H(Y) \succeq 0,\;\; H_p^\Uparrow(Y) \succeq 0\}. \tag{4.12}$$

Given $Y \in \mathfrak{L}_p^\Gamma(n)$ let

$$\hat{Y} = (Y_{x_1}, Y_{x_2}, \dots, Y_{x_{\mathrm{g}}}) \in \mathbb{S}_n^{\mathrm{g}} \tag{4.13}$$

and denote

$$\hat{\mathfrak{L}}_p^\Gamma = \{\hat{Y} \mid Y \in \mathfrak{L}_p^\Gamma\}.$$

Remark 4.5. With the notation just introduced, Remark 4.4 states that the Γ-moment sequence Y of equation (4.10) belongs to $\mathfrak{L}_p^\Gamma$ if Z is in $\mathcal{D}_p^\infty$. ■

We now introduce a boundedness assumption on the polynomial p that will ensure our construction of the Γ-convex lifts clamps down on the Γ-convex hull of $\mathcal{D}_p^\infty$.

Definition 4.6. A matrix-valued polynomial p is **Archimedean** if there is a $k > 0$ and finitely many matrix-valued polynomials s_j and t_j such that

$$k^2 - \sum_{i=1}^{\mathrm{g}} x_i^2 = \sum_j s_j^* s_j + \sum_j t_j^* p\, t_j. \tag{4.14}$$

In this case we that p is k-**Archimedean**. Observe that (4.14) implies each $X \in \mathcal{D}_p^\infty$ satisfies $\|X\| \leq k$. ■

We now prove the main result of this subsection: a spectrahedral realization of the Γ-convex hull of the free operator semialgebraic set defined by an Archimedean polynomial. It is the Γ-analog to [HKM16, Theorem 5.4].

Theorem 4.7. *If p is Archimedean, then*

$$\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty) = \hat{\mathfrak{L}}_p^\Gamma.$$

Moreover, if p is k-Archimedean, then $\hat{\mathfrak{L}}_p^\Gamma$ is bounded by k and if $Y \in \mathfrak{L}_p^\Gamma$, then $\|Y_\alpha\| \leq k^{|\alpha|}$ for each word α.

The proof of Theorem 4.7 will use the following proposition.

Proposition 4.8. *If p is Archimedean, then for each $Y \in \mathfrak{L}_p^\Gamma(n)$ there exists a Γ-pair (T, V) with $T \in \mathcal{D}_p^\infty$ and V an isometry such that $Y_\alpha = V^* T^\alpha V$ for all α.*

Proof. Let $Y \in \mathfrak{L}_p^\Gamma(n)$ be given. Define a sesquilinear form $[\cdot,\cdot]$ on $\mathcal{V} = \mathbb{C}\langle x\rangle \otimes \mathbb{C}^n$ by

$$[s,t]_Y = \sum_{\alpha,\beta} \langle Y_{\beta^*\alpha} s_\alpha, t_\beta\rangle \tag{4.15}$$

where $s = \sum \alpha \otimes s_\alpha$ and $t = \sum \beta \otimes t_\beta$. The assumption $H(Y) \succeq 0$ implies $[s,s]_Y \geq 0$ for all $s \in \mathcal{V}$, so that this sesquilinear form is positive semidefinite. A standard argument (using the Cauchy-Schwarz inequality) shows that

$$\mathcal{N} = \{s : [s,s]_Y = 0\}$$

is a subspace of $\mathcal{V}$. Modding out $\mathcal{N}$ produces a well-defined and positive semidefinite form

$$[s,t]_Y = [s+\mathcal{N}, t+\mathcal{N}]_Y$$

on the quotient $\mathcal{W}$ of $\mathcal{V}$ by $\mathcal{N}$, where, as is standard practice, $s \in \mathcal{V}$ is identified with its image $s + \mathcal{N}$ in the quotient.

Similar to the standard GNS construction we now show that $\mathcal{N}$ is a left $\mathbb{C}\langle x\rangle$-submodule: if $s \in \mathcal{N}$ and $1 \leq j \leq \mathsf{g}$, then $r = x_j s \in \mathcal{N}$. Indeed, if $s \in \mathcal{N}$, then $H(Y) \succeq 0$ implies

$$\sum_\alpha Y_{\beta^*\alpha} s_\alpha = 0$$

for each β (and conversely). Hence

$$\sum_\gamma Y_{\beta^*\gamma} r_\gamma = \sum_\alpha Y_{\beta^* x_j \alpha} s_\alpha = \sum_\gamma Y_{(x_j\beta)^*\alpha} s_\alpha = 0$$

so that $r \in \mathcal{N}$. It now follows that the multiplication mapping Z_j sending s to $x_j s$ is well defined on $\mathcal{W}$. The computation above also shows that (even without the condition $s \in \mathcal{N}$),

$$[x_j s, t]_Y = [s, x_j t]_Y. \tag{4.16}$$

Further, for each word γ we obtain an operator $Z^\gamma = \gamma(Z)$ on $\mathcal{W}$ satisfying $Z^\gamma s = ws = \sum \gamma\alpha \otimes s_\alpha$.

To prove $p(Z) = \sum p_\eta \otimes Z^\eta$ is positive definite on $\mathcal{W}$, let $s = \sum e_j \otimes \alpha \otimes s_{\alpha,j}$, where $\{e_1, \dots, e_\mu\}$ is the standard orthonormal basis for $\mathbb{C}^\mu$ (with $p_\eta \in M_\mu$) and $s_{\alpha,j} \in \mathbb{C}^n$ and observe

$$\begin{aligned}
\langle p(Z)s, s\rangle &= \sum_{\alpha,\beta,\eta,j,k} \langle (p_\eta \otimes Z^\eta) e_j \otimes \alpha \otimes s_{\alpha,j}, e_k \otimes \beta \otimes s_{\beta,k}\rangle \\
&= \sum \langle p_\eta e_j, e_k\rangle \, \langle Z^\eta \alpha \otimes s_{\alpha,j}, \beta \otimes s_{\beta,k}\rangle \\
&= \sum \langle p_\eta e_j, e_k\rangle \, \langle Y_{\beta^*\eta\alpha} s_{\alpha,j}, s_{\beta,k}\rangle \\
&= \sum_{\alpha,\beta} \Big\langle \Big(\sum_\eta p_\eta \otimes Y_{\beta^*\eta\alpha}\Big) \sum_j e_j \otimes s_{\alpha,j}, \sum_k e_k \otimes s_{\beta,k} \Big\rangle \\
&= \langle H_p^{\Uparrow}(Y)\vec{s}, \vec{s}\rangle,
\end{aligned} \tag{4.17}$$

where $\vec{s} = (s_\alpha)_\alpha$ is the vector with $s_\alpha = \sum_j e_j \otimes s_{\alpha,j}$. The assumption $H_p^{\Uparrow}(Y) \succeq 0$ implies $p(Z) \succeq 0$ as desired.

Define $Q : \mathbb{C}^n \to \mathcal{W}$ by

$$Qv = \emptyset \otimes v.$$

It is straightforward to see that Q is an isometry and by construction,

$$Q^* Z^\alpha Q = Y_\alpha \tag{4.18}$$

for all α, where Z^α is defined as $\alpha(Z_1, \dots, Z_{\mathtt{g}})$. Moreover, (Z, Q) is a Γ-pair, where $Z = (Z_1, \dots, Z_{\mathtt{g}})$. Indeed, if $\gamma_j = \sum_k \gamma_{j,k} m_{j,k}$ with $\gamma_{j,k} \in \mathbb{C}$ and words $m_{j,k}$, then (4.18) together with the fact that Y is a Γ-moment sequence implies that

$$\begin{aligned} \gamma_j(Q^* Z Q) = \gamma_j(\hat{Y}) = \sum_k \gamma_{j,k} m_{j,k}(\hat{Y}) = \sum_k \gamma_{j,k} Y_{m_{j,k}} = \sum_k \gamma_{j,k} Q^* Z^{m_{j,k}} Q \\ = Q^* \left(\sum_k \gamma_{j,k} Z^{m_{j,k}} \right) Q = Q^* \gamma_j(Z) Q \end{aligned} \tag{4.19}$$

for every $j = 1, \dots, \mathtt{r}$.

Since p is Archimedean, the Z_j are bounded operators. Indeed, since p is Archimedean,

$$k^2 - \sum_j x_j^2 = \sum_i f_i^* f_i + \sum_k g_k^* p \, g_k$$

for some $k > 0$ and noncommutative polynomials f_i, g_k; and $p(Z) \succeq 0$ since $Z \in \mathcal{D}_p^\infty$. Hence

$$k^2 - \sum_j Z_j^2 \succeq 0.$$

So $\|Z\| \leq k$ and in particular $\|Z_j\|^2 \leq k^2$. Thus Z_j is bounded for each $1 \leq j \leq \mathtt{g}$. It now follows that, for each word γ, the operator Z^γ on $\mathcal{W}$ extends to a bounded operator $\widetilde{Z}_\gamma$ on the on the completion $\mathcal{H}$ of $\mathcal{W}$. Setting $T_j = \widetilde{Z}_{x_j}$, the identity of equation (4.16) implies $T_j^* = T_j$. In particular, if (h_n) is a sequence from $\mathcal{W}$ that converges to $h \in \mathcal{H}$, then $(Z_j h_n) = (T_j h_n)$ is a sequence from $\mathcal{W}$ that converges to $T_j h$ and thus, by the invariance of $\mathcal{W}$ under Z_j, it follows that $(Z^w h_n)$ converges to $\widetilde{Z}_w h = T^w h$. For instance, $(Z_1 Z_2 h_n = Z_1[Z_2 h_n])$ converges to $T_1[T_2 h]$ since $(Z_2 h_n)$ converges to $\widetilde{Z}_{x_2} h$. Consequently, $q(Z)$ is the restriction of $q(T)$ to $\mathcal{W}$ for $q \in \mathbb{C}\langle x \rangle$.

It now follows that $p(T) \succeq 0$ since $p(Z) \succeq 0$. Hence $T \in \mathcal{D}_p^\infty$. Further, Q is a isometry into $\mathcal{H}$ such that $Q^* \gamma_j(T) Q = Q^* \gamma_j(Z) Q$. Hence and $\gamma_j(Q^* T Q) = Q^* \gamma_j(T) Q$ from equation (4.19). Thus (T, Q) is a Γ-pair. Finally, equation (4.18) gives $Q^* T^\alpha Q = Y_\alpha$ for each word α. ■

Proof of Theorem 4.7. To prove the inclusion $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty) \subseteq \hat{\mathfrak{L}}_p^\Gamma$, note that every $X \in \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$ is of the form $V^* Z V$ for some $Z \in \mathcal{D}_p^\infty$ and isometry V such that (Z, V) is a Γ-pair. Remark 4.4 now implies the moment sequence $Y = (Y_\alpha)_\alpha$ with $Y_\alpha = V^* Z^\alpha V$ as in (4.10) belongs to $\mathfrak{L}_p^\Gamma$, hence $X = \hat{Y} \in \hat{\mathfrak{L}}_p^\Gamma$.

To prove the converse, let $\widehat{Y} \in \hat{\mathfrak{L}}_p^\Gamma$ be given. Thus there exists a Γ-moment sequence $(Y_\alpha)_\alpha$ from $\mathfrak{L}_p^\Gamma(n)$ such that $\widehat{Y}$ is given by equation (4.13). From Proposition 4.8, there is a Γ-pair (T, V) such that $T \in \mathcal{D}_p^\infty$ and V is an isometry such that $\widehat{Y} = V^*TV$ so that $\widehat{Y} \in \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$.

Finally, since $\|T_j\| \le k$ for each j, it follows that $\|T^\alpha| \le k^{|\alpha|}$ for each word α. Hence $\|Y_\alpha\| = \|V^*T^\alpha V\| \le \|T^\alpha\| \le k^{|\alpha|}$. Similarly, $\|\hat{Y}\| \le k$ since $\|T\| \le k$. ∎

Corollary 4.9. *If p is Archimedean, then $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$ is closed and bounded.*

Proof. Boundedness of $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$ follows from Theorem 4.7.

To prove that it is closed, suppose $(X^{(k)})_k$ is a sequence from $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$ that converges to some $X \in \mathbb{S}_n^{\mathrm{g}}$. For each k there is an isometry V_k and an element $Y^{(k)} \in \mathcal{D}_p^\infty$ such that $(Y^{(k)}, V_k)$ is a Γ-pair and $X^{(k)} = V_k^* Y^{(k)} V_k$. As noted in Remark 4.5, for each k, the moment sequence $(Z_\alpha^{(k)})_\alpha$ defined by

$$Z_\alpha^{(k)} = V_k^*(Y^{(k)})^\alpha V_k$$

lies in $\mathfrak{L}_p^\Gamma$. Now for any α, Theorem 4.7 implies that the sequence $(Z_\alpha^{(k)})_k$ is bounded. It thus has a convergent subsequence and by passing to a subsequence, we can assume that for each α, the sequence $(Z_\alpha^{(k)})$ converges to some Z_α. Since membership in $\mathfrak{L}_p^\Gamma$ is determined by the positivity conditions of equation (4.11), if follows that $(Z_\alpha)_\alpha \in \mathfrak{L}_p^\Gamma$. By construction, $X = (Z_{x_1}, \dots, Z_{x_{\mathrm{g}}}) \in \hat{\mathfrak{L}}_p^\Gamma = \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$, where the last equality is guaranteed by Theorem 4.7. ∎

4.4. **Step 2: truncated lifts and Γ-moment sequences.** Here we show that the degree-bound truncations of $\mathfrak{L}_p^\Gamma$ form a sequence of finite free Γ-spectrahedral lifts of $\mathcal{D}_p^\infty$ whose projections give better and better outer approximations of the Γ-convex hull of $\mathcal{D}_p^\infty$.

4.4.1. *The clamping down theorem.* Let δ be the maximum degree of the polynomials γ_j. For $n \in \mathbb{N}$ and $d \ge \delta$ denote the n-th level of the d-truncation of $\mathfrak{L}_p^\Gamma$ by

$$\begin{aligned}\mathfrak{L}_p^\Gamma(n, d) = \Big\{ Y = (Y_\alpha)_{|\alpha| \le 2d + \deg p + 1} \mid Y_\alpha \in M_n,\ Y_\emptyset = I,\ Y_{\alpha^*} = Y_\alpha^*, \\ Y \text{ satisfies } (4.7),\ H_{d + \lceil \frac{1}{2} \deg p \rceil}(Y) \succeq 0,\ H_{p,d}^{\Uparrow}(Y) \succeq 0 \Big\}\end{aligned} \tag{4.20}$$

and let $\mathfrak{L}_p^\Gamma(\cdot, d) = (\mathfrak{L}_p^\Gamma(n, d))_n$. As in the previous section define

$$\hat{\mathfrak{L}}_p^\Gamma(n, d) = \{\hat{Y} = (Y_{x_1}, Y_{x_2}, \dots, Y_{x_{\mathrm{g}}}) \in \mathbb{S}_n^{\mathrm{g}} \mid Y \in \mathfrak{L}_p^\Gamma(n, d)\}. \tag{4.21}$$

The next theorem asserts that the d-truncations $\hat{\mathfrak{L}}_p^\Gamma(\cdot, d)$ are projections of free Γ-spectrahedra and that they intersect precisely at $\hat{\mathfrak{L}}_p^\Gamma$, which, by Theorem 4.7, is $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$.

Theorem 4.10. *If p is an Archimedean symmetric matrix-valued noncommutative polynomial, then*

(a) *for every* n,

$$\bigcap_{d=0}^{\infty} \hat{\mathfrak{L}}_p^{\Gamma}(n,d) = \hat{\mathfrak{L}}_p^{\Gamma}(n) = \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^{\infty})(n); \tag{4.22}$$

(b) *for every* d *there is a* Γ*-pencil* L_d^{Γ} *given by a tuple* $A^{(d)}$ *such that* $\hat{\mathfrak{L}}_p^{\Gamma}(\cdot,d)$ *is the projection of* $\mathcal{D}_{A^{(d)}}^{\Gamma}$.

Hence, the free Γ*-spectrahedrops* $\hat{\mathfrak{L}}_p^{\Gamma}(\cdot,d)$ *give increasingly finer outer approximations of the* Γ*-convex hull of* $\mathcal{D}_p^{\infty}$.

Lemma 4.11. *Let* p *be an Archimedean polynomial. Then there is a natural number* ν *and a* $c>0$ *such that for every* $Y \in \mathfrak{L}_p^{\Gamma}(n,d)$ *and word* α *of length* $|\alpha| \le 2(d-\nu)$,

$$\|Y_\alpha\| \le c^{|\alpha|}.$$

The proof follows follows along the lines of [HKM16, Lemma 6.5], since the definition of our localizing matrices $H_p^{\Uparrow}(Y)$ coincides with the one there.

Proof of Theorem 4.10. To prove item (a) first observe that if $(Y_\alpha)_\alpha$ is a moment sequence, then

$$H(Y) \succeq 0 \quad \text{and} \quad H_p^{\Uparrow}(Y) \succeq 0 \tag{4.23}$$

if and only if for all d,

$$H_{d+\lceil \frac{1}{2}\deg p\rceil}(Y) \succeq 0 \quad \text{and} \quad H_{p,d}^{\Uparrow}(Y) \succeq 0. \tag{4.24}$$

The second set equality in (4.22) is given in Theorem 4.7. To prove the nontrivial inclusion $\subseteq$ in the first equality of (4.22) let $Z \in \bigcap_d \hat{\mathfrak{L}}_p^{\Gamma}(n,d)$. For every d there is a (truncated) moment sequence $Y^{(d)} = (Y_\alpha^{(d)}) \in \hat{\mathfrak{L}}_p^{\Gamma}(n;d)$ such that

$$(Y_{x_1}^{(d)},\dots,Y_{x_g}^{(d)}) = Z.$$

By Lemma 4.11, for a given word α, the sequence $(Y_\alpha^{(d)})_{|\alpha|\le 2d+\deg(p)}$ is bounded. Since there are countably many such sequences, there exists a moment sequence (Y_α) from $\mathbb{S}_n$ and a subsequence $(d_k)_k$ of indices such that $(Y_\alpha^{(d_k)})$ converges to $Y_\alpha \in \mathbb{S}_n$ for each word α.

For each k the sequence $Y^{(d_k)}$ satisfies (4.24) with $d=d_k$ and hence for all $d \le d_k$. Hence the limit Γ-moment sequence $Y=(Y_\alpha)_\alpha$ satisfies (4.24) for all indices d and thus Y satisfies (4.23) and hence belongs to $\mathfrak{L}_p^{\Gamma}(n)$. Thus, $Z \in \hat{\mathfrak{L}}_p^{\Gamma}(n)$ since

$$Z = (Y_{x_1},\dots,Y_{x_g}) = \hat{Y}.$$

Item (b) is proved in the next Subsection 4.4.2, where we show that the $\hat{\mathfrak{L}}_p^{\Gamma}(\cdot,d)$ are in fact free Γ-spectrahedrops. ■

4.4.2. *Free Γ-spectrahedral lifts of the* $\hat{\mathfrak{L}}_p^\Gamma(\cdot,d)$. By the Lasserre-Parrilo construction in Section 4.3, the $\hat{\mathfrak{L}}_p^\Gamma(\cdot,d)$ are projections of the approximate lifts $\mathfrak{L}_p^\Gamma(\cdot,d)$ and each of the latter is the positivity set of a matrix-valued polynomial of the form

$$L^\Gamma(x,y)=\mathcal{A}(x)+\sum_{j=1}^{\mathtt{h}}(B_jy_j+B_j^*y_j^*),$$

where $\mathcal{A}$ is a (not necessarily monic) Γ-pencil of size k and $B_j\in M_k$. The following procedure, described in [HKM16, Lemma 6.3], replaces the non-self-adjoint coefficients B_j and non-symmetric variables y_j in $L^\Gamma(x,y)$ with (twice as many) self-adjoint coefficients and symmetric variables such that the resulting Γ-pencil, denoted $\widetilde{L^\Gamma}$, provides a lift of $\hat{\mathfrak{L}}_p^\Gamma(\cdot,d)$ to a Γ-spectrahedron:

$$\operatorname{proj}_x\mathcal{D}^\Gamma_{\widetilde{L^\Gamma}}=\operatorname{proj}_x\mathcal{D}^\Gamma_{L^\Gamma}=\hat{\mathfrak{L}}_p^\Gamma(\cdot,d).$$

Decomposing $B_j=C_j+\mathrm{i}D_j$ and $y_j=w_j+\mathrm{i}w_{-j}$ into self-adjoint matrices C_j,D_j and free symmetric variables $w_{-\mathtt{h}},\dots,w_{-1},w_1,\dots,w_{\mathtt{h}}$ gives

$$\begin{aligned}B_jy_j+B_j^*y_j^*&=(C_j+\mathrm{i}D_j)(w_j+\mathrm{i}w_{-j})+(C_j-\mathrm{i}D_j)(w_j-\mathrm{i}w_{-j})\\&=2(C_jw_j-D_jw_{-j})\end{aligned}$$

and

$$\widetilde{L^\Gamma}(x,w)=\mathcal{A}(x)+2\sum_{j=1}^{h}\big(C_jw_j-D_jw_{-j}\big)$$

is a Γ-pencil with self-adjoint coefficients and symmetric variables satisfying $\operatorname{proj}_x\mathcal{D}^\Gamma_{\widetilde{L^\Gamma}}=\operatorname{proj}_x\mathcal{D}^\Gamma_{L^\Gamma}$.

4.5. **Examples.** We next give a few examples of the Lasserre-Parrilo lifting construction in the Γ-setting.

Example 4.12. Denote the variables by x,y instead of x_1,x_2 and let $\Gamma=(x,y,y^2)$. Consider $p=(1-2y^2+x^2)\oplus(1-x^2)$. Then

$$\mathcal{D}_p=\{(X,Y)\mid 2Y^2\preceq 1+X^2,\ X^2\preceq I\}$$

is not y^2-convex as seen from Figure 2 representing $\mathcal{D}_p(1)$.

Let us to prove that the y^2-convex hull of $\mathcal{D}_p$, i.e., the convex hull with respect to the x-coordinate, is

$$\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p)=\{(X,Y)\mid Y^2\preceq I,\ X^2\preceq I\}. \tag{4.25}$$

Denoting the right hand side in (4.25) by $\boldsymbol{K}=(K_n)_n$, we immediately see that $\boldsymbol{K}$ is free and convex (in X for each fixed Y). We now prove that, for each $n\in\mathbb{N}$, any point $(X,Y)\in K_n$ is a y^2-convex combination of points from $\mathcal{D}_p$.

Note that by combining the two defining inequalities for $\mathcal{D}_p$, we obtain $Y^2\preceq I$ on $\mathcal{D}_p$. That is, $\mathcal{D}_p\subseteq\boldsymbol{K}$. Let $(X,Y)\in\boldsymbol{K}$, i.e., X and Y are contractions. Then X can be

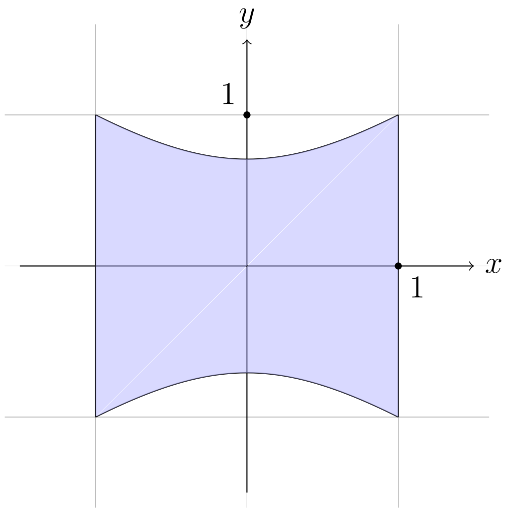

FIGURE 2. The first level component $\mathcal{D}_p(1)$ is clearly not convex in x, hence $\mathcal{D}_p$ is not y^2-convex.

diagonalized as $X = U^*DU$, where D is diagonal with diagonals $d_1, \dots, d_n \in [-1, 1]$ and U is unitary. Write d_1 as a convex combination of the points ± 1 and express $X = tX_1 + (1-t)X_2$ as a convex combination of matrices X_1, X_2, where $X_i = U^*D_iU$ and D_i has diagonals $(-1)^i, d_2, \dots, d_n$. Repeat this process on X_1 and X_2 to obtain an expression of X as a convex combination

$$X = \sum_{i=1}^{k} t_i X_i$$

of matrices $X_1, \dots, X_k$ with $X_i^2 = I$. Clearly, for each i, the tuple (X_i, Y) lies in $\mathcal{D}_p$ since

$$1 + X_i^2 = 2 \succeq 2Y^2.$$

We can now write

$$(X, Y) = \sum_{i=1}^{k} t_i (X_i, Y) \in \Gamma\text{- conv}(\mathcal{D}_p).$$

We deduce that $\boldsymbol{K}$ must be contained in every other free set which is y^2-convex and contains $\mathcal{D}_p$. Hence $\boldsymbol{K} = \Gamma\text{- conv}(\mathcal{D}_p)$. The level one of (4.25) is the square $[-1, 1]^2$.

We now show that the projection of the first Lasserre-Parrilo lift $\mathfrak{L}_p^\Gamma(\cdot, 0)$, given by

$$H_1(Z) = \begin{pmatrix} I & X & Y \\ X & Z_{11} & Z_{12} \\ Y & Z_{21} & Y^2 \end{pmatrix} \succeq 0,$$

$$H_{p,0}^{\Uparrow}(Z) = \begin{pmatrix} I - 2Y^2 + Z_{11} & 0 \\ 0 & I - Z_{11} \end{pmatrix} \succeq 0,$$

equals $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p)$, i.e., the lift is exact. Indeed, for any $(X,Y) \in \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p)$, the canonical moment sequence (4.10) lies in $\mathfrak{L}_p^\Gamma(\cdot,0)$. For the other inclusion let $(X,Y) \in \hat{\mathfrak{L}}_p^\Gamma(\cdot,0)$. Then combining the inequalities

$$I - 2Y^2 + Z_{11} \succeq 0 \quad \text{and} \quad I - Z_{11} \succeq 0$$

gives $Y^2 \preceq I$, while taking Schur complements in $H_1(Z)$ implies

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Y^2 \end{pmatrix} - \begin{pmatrix} X \\ Y \end{pmatrix} \begin{pmatrix} X & Y \end{pmatrix} = \begin{pmatrix} Z_{11} - X^2 & Z_{12} - XY \\ Z_{21} - YX & 0 \end{pmatrix} \succeq 0.$$

The latter forces $X^2 \preceq Z_{11}$ and hence

$$X^2 \preceq Z_{11} \preceq I,$$

so the tuple (X,Y) lies in $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p)$.

Finally, using Theorem 4.10, we can deduce

$$\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p) \subseteq \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty) = \bigcap_{d=0}^\infty \hat{\mathfrak{L}}_p^\Gamma(\cdot,d) = \hat{\mathfrak{L}}_p^\Gamma \subseteq \hat{\mathfrak{L}}_p^\Gamma(\cdot,0) = \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p), \tag{4.26}$$

whence we have equalities throughout. ■

Example 4.13. Consider the free semialgebraic set $\mathcal{D}_p$ defined by

$$p(x,y) = 1 - x^2 - y^4,$$

the so-called TV screen [JKMMP21, Example 1.1]. Figure 3 depicts $\mathcal{D}_p(1)$.

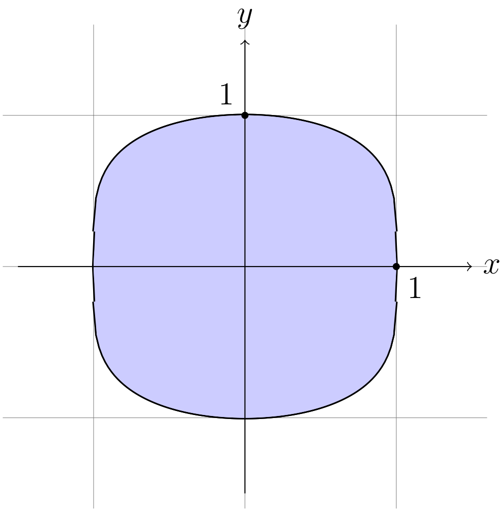

FIGURE 3. Bent TV screen $\mathcal{D}_p(1) = \{(x,y) \in \mathbb{R}^2 : 1 - x^2 - y^4 \geq 0\}$.

Since $p(X,Y) \succeq 0$ iff

$$\begin{pmatrix} I & X & Y^2 \\ X & I & 0 \\ Y^2 & 0 & I \end{pmatrix} \succeq 0, \tag{4.27}$$

$\mathcal{D}_p$ is y^2-convex. Let us show that the projection of the first Lasserre-Parrilo lift $\mathfrak{L}_p^\Gamma(\cdot,0)$, given by

$$\begin{aligned} H_2(Z) &= \begin{pmatrix} I & X & Y & Z_{11} & Z_{12} & Z_{21} & Y^2 \\ X & Z_{11} & Z_{12} & Z_{111} & Z_{112} & Z_{121} & Z_{122} \\ Y & Z_{21} & Y^2 & Z_{211} & Z_{212} & Z_{221} & Z_{222} \\ Z_{11} & Z_{111} & Z_{112} & Z_{1111} & Z_{1112} & Z_{1121} & Z_{1122} \\ Z_{21} & Z_{211} & Z_{212} & Z_{2111} & Z_{2112} & Z_{2121} & Z_{2122} \\ Z_{12} & Z_{121} & Z_{122} & Z_{1211} & Z_{1212} & Z_{1221} & Z_{1222} \\ Y^2 & Z_{221} & Z_{222} & Z_{2211} & Z_{2212} & Z_{2221} & Z_{2222} \end{pmatrix} \succeq 0, \\ H_{p,0}^{\Uparrow}(Z) &= \big(I - Z_{11} - Z_{2222}\big) \succeq 0, \end{aligned} \tag{4.28}$$

of $\mathcal{D}_p$ is exact, i.e., $\hat{\mathfrak{L}}_p^\Gamma(\cdot,0) = \mathcal{D}_p$.

Suppose (X,Y,Z) satisfy (4.28). By considering the top left 2×2 submatrix of $H_2(Z)$ we get

$$Z_{11} \succeq X^2, \tag{4.29}$$

and by considering the 2×2 principal submatrix on the first and last column we deduce

$$Z_{2222} \succeq Y^4. \tag{4.30}$$

Using (4.29) and (4.30) in $H_{p,0}^{\Uparrow}(Z) \succeq 0$ yields

$$0 \preceq I - Z_{11} - Z_{2222} \preceq I - X^2 - Y^4,$$

whence $(X,Y) \in \mathcal{D}_p$. We can now conclude as in (4.26) that

$$\mathcal{D}_p = \bigcap_{d=0}^{\infty} \hat{\mathfrak{L}}_p^\Gamma(\cdot,d) = \hat{\mathfrak{L}}_p^\Gamma = \hat{\mathfrak{L}}_p^\Gamma(\cdot,0). \qquad \blacksquare$$

Example 4.14. The situation changes if we consider the even more bent TV screen, that is, the free semialgebraic set $\mathcal{D}_p$ defined by

$$p(x,y) = 1 - x^2 - y^6.$$

Figure 4 depicts $\mathcal{D}_p(1)$. The graded set $\mathcal{D}_p$ is again y^2-convex as is easily seen. In fact, it is even a Γ-spectrahedron [JKMMP21, Proposition 4.2].

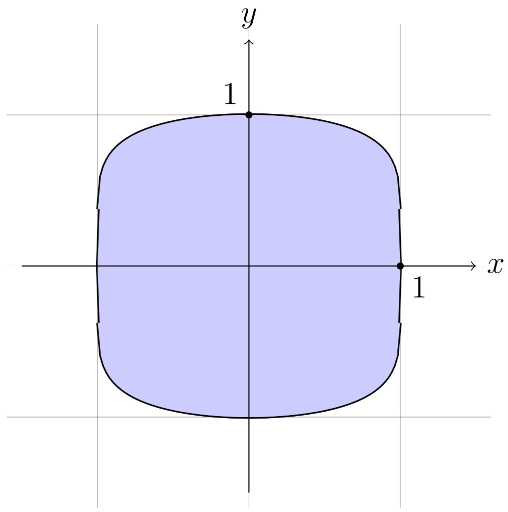

FIGURE 4. Bent TV screen $\mathcal{D}_p(1) = \{(x, y) \in \mathbb{R}^2 : 1 - x^2 - y^6 \geq 0\}$.

However, in contrast to the above examples, in this case the projection of the first Lasserre-Parrilo lift $\mathfrak{L}_p^\Gamma(\cdot, 0)$ is not exact. Consider

$$X = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\det(p(tX, tY)) = \det(I - t^2X^2 - t^6Y^6) = \frac{t^8}{2} - t^6 - t^2 + 1, \tag{4.31}$$

so $t(X, Y) \in \mathcal{D}_p$ for $|t| \leq t_0 \approx 0.861724$, where t_0 is the smallest positive root of (4.31).

Now consider the first Lasserre-Parrilo lift $\mathfrak{L}_p^\Gamma(\cdot, 0)$ described by

$$H_3(Z) = \begin{pmatrix}
1 & X & Y & Z_{11} & Z_{12} & Z_{21} & Y^2 & Z_{111} & Z_{112} & Z_{121} & Z_{122} & Z_{211} & Z_{212} & Z_{221} & Z_{222} \\
X & Z_{11} & Z_{12} & Z_{111} & Z_{112} & Z_{121} & Z_{122} & Z_{1111} & Z_{1112} & Z_{1121} & Z_{1122} & Z_{1211} & Z_{1212} & Z_{1221} & Z_{1222} \\
Y & Z_{21} & Y^2 & Z_{211} & Z_{212} & Z_{221} & Z_{222} & Z_{2111} & Z_{2112} & Z_{2121} & Z_{2122} & Z_{2211} & Z_{2212} & Z_{2221} & Z_{2222} \\
Z_{11} & Z_{111} & Z_{112} & Z_{1111} & Z_{1112} & Z_{1121} & Z_{1122} & Z_{11111} & Z_{11112} & Z_{11121} & Z_{11122} & Z_{11211} & Z_{11212} & Z_{11221} & Z_{11222} \\
Z_{21} & Z_{211} & Z_{212} & Z_{2111} & Z_{2112} & Z_{2121} & Z_{2122} & Z_{21111} & Z_{21112} & Z_{21121} & Z_{21122} & Z_{21211} & Z_{21212} & Z_{21221} & Z_{21222} \\
Z_{12} & Z_{121} & Z_{122} & Z_{1211} & Z_{1212} & Z_{1221} & Z_{1222} & Z_{12111} & Z_{12112} & Z_{12121} & Z_{12122} & Z_{12211} & Z_{12212} & Z_{12221} & Z_{12222} \\
Y^2 & Z_{221} & Z_{222} & Z_{2211} & Z_{2212} & Z_{2221} & Z_{2222} & Z_{22111} & Z_{22112} & Z_{22121} & Z_{22122} & Z_{22211} & Z_{22212} & Z_{22221} & Z_{22222} \\
Z_{111} & Z_{1111} & Z_{1112} & Z_{11111} & Z_{11112} & Z_{11121} & Z_{11122} & Z_{111111} & Z_{111112} & Z_{111121} & Z_{111122} & Z_{111211} & Z_{111212} & Z_{111221} & Z_{111222} \\
Z_{211} & Z_{2111} & Z_{2112} & Z_{21111} & Z_{21112} & Z_{21121} & Z_{21122} & Z_{211111} & Z_{211112} & Z_{211121} & Z_{211122} & Z_{211211} & Z_{211212} & Z_{211221} & Z_{211222} \\
Z_{121} & Z_{1211} & Z_{1212} & Z_{12111} & Z_{12112} & Z_{12121} & Z_{12122} & Z_{121111} & Z_{121112} & Z_{121121} & Z_{121122} & Z_{121211} & Z_{121212} & Z_{121221} & Z_{121222} \\
Z_{221} & Z_{2211} & Z_{2212} & Z_{22111} & Z_{22112} & Z_{22121} & Z_{22122} & Z_{221111} & Z_{221112} & Z_{221121} & Z_{221122} & Z_{221211} & Z_{221212} & Z_{221221} & Z_{221222} \\
Z_{112} & Z_{1121} & Z_{1122} & Z_{11211} & Z_{11212} & Z_{11221} & Z_{11222} & Z_{112111} & Z_{112112} & Z_{112121} & Z_{112122} & Z_{112211} & Z_{112212} & Z_{112221} & Z_{112222} \\
Z_{212} & Z_{2121} & Z_{2122} & Z_{21211} & Z_{21212} & Z_{21221} & Z_{21222} & Z_{212111} & Z_{212112} & Z_{212121} & Z_{212122} & Z_{212211} & Z_{212212} & Z_{212221} & Z_{212222} \\
Z_{122} & Z_{1221} & Z_{1222} & Z_{12211} & Z_{12212} & Z_{12221} & Z_{12222} & Z_{122111} & Z_{122112} & Z_{122121} & Z_{122122} & Z_{122211} & Z_{122212} & Z_{122221} & Z_{122222} \\
Z_{222} & Z_{2221} & Z_{2222} & Z_{22211} & Z_{22212} & Z_{22221} & Z_{22222} & Z_{222111} & Z_{222112} & Z_{222121} & Z_{222122} & Z_{222211} & Z_{222212} & Z_{222221} & Z_{222222}
\end{pmatrix} \succeq 0, \tag{4.32}$$

$$H^{\Uparrow}_{p,0}(Z) = \big(I - Z_{11} - Z_{222222}\big) \succeq 0.$$

Pick $t' = \frac{173}{200} = 0.865 > t_0$. Then $p(t'X, t'Y)$ is not positive semidefinite; its smallest eigenvalue is

$$\frac{53304846667911 - 29929\sqrt{3362359178476091681}}{128000000000000} \approx -0.012306,$$

whence $(t'X, t'Y) \notin \mathcal{D}_p$. However, we claim that $(t'X, t'Y) \in \hat{\mathfrak{L}}_p^{\Gamma}(\cdot, 0)$. Replace X, Y in (4.32) by $t'X, t'Y$ leading to a pair of semidefinite constraints. We ran this SDP with the built-in solver in Wolfram Mathematica (with the trivial objective function 1 which tends to produce high rank solutions) to obtain floating point solutions for the 2×2 matrix variables Z. The smallest obtained eigenvalues for $H_3(Z)$ and $H_{p,0}^{\Uparrow}(Z)$ were approximately $5 \cdot 10^{-4}$ and $2 \cdot 10^{-3}$, respectively. Since these are far enough from 0, routine numerical analysis (see, e.g. [PP08, CKP15]) shows that a fine enough rationalization will lead to exact symbolic feasible points. Indeed, rationalizing leads to the following feasible point (as can be checked, e.g., with the Cholesky decomposition):[5]

$$(Z_{11})_{11} \to \frac{63}{82},\ (Z_{11})_{12} \to -\frac{2}{39},\ (Z_{11})_{22} \to \frac{53}{319},\ (Z_{12})_{11} \to \frac{46}{123},\ (Z_{12})_{12} \to \frac{55}{147},\ (Z_{12})_{21} \to 0,\ (Z_{12})_{22} \to 0,$$
$$(Z_{22})_{11} \to \frac{37}{69},\ (Z_{22})_{12} \to \frac{5}{34},\ (Z_{22})_{22} \to \frac{82}{117},\ (Z_{111})_{11} \to \frac{112}{165},\ (Z_{111})_{12} \to -\frac{4}{97},\ (Z_{111})_{22} \to -\frac{1}{140},$$
$$(Z_{112})_{11} \to \frac{29}{94},\ (Z_{112})_{12} \to \frac{24}{77},\ (Z_{112})_{21} \to \frac{4}{75},\ (Z_{112})_{22} \to \frac{2}{47},\ (Z_{121})_{11} \to \frac{43}{133},\ (Z_{121})_{12} \to 0,\ (Z_{121})_{22} \to 0,$$
$$(Z_{122})_{11} \to \frac{32}{69},\ (Z_{122})_{12} \to \frac{7}{55},\ (Z_{122})_{21} \to 0,\ (Z_{122})_{22} \to -\frac{1}{8348},\ (Z_{212})_{11} \to \frac{16}{99},\ (Z_{212})_{12} \to \frac{6}{37},\ (Z_{212})_{22} \to \frac{16}{99},$$
$$(Z_{222})_{11} \to \frac{55}{166},\ (Z_{222})_{12} \to \frac{18}{73},\ (Z_{222})_{22} \to \frac{8}{15},\ (Z_{1111})_{11} \to \frac{51}{32},\ (Z_{1111})_{12} \to -\frac{65}{61},\ (Z_{1111})_{22} \to \frac{571}{143},$$
$$(Z_{1112})_{11} \to \frac{43}{156},\ (Z_{1112})_{12} \to \frac{22}{79},\ (Z_{1112})_{21} \to -\frac{1}{47},\ (Z_{1112})_{22} \to -\frac{1}{50},\ (Z_{1121})_{11} \to \frac{67}{251},\ (Z_{1121})_{12} \to -\frac{1}{982},$$
$$(Z_{1121})_{21} \to \frac{5}{114},\ (Z_{1121})_{22} \to \frac{1}{232},\ (Z_{1122})_{11} \to \frac{40}{99},\ (Z_{1122})_{12} \to \frac{3}{38},\ (Z_{1122})_{21} \to -\frac{1}{630},\ (Z_{1122})_{22} \to \frac{7}{67},$$
$$(Z_{1212})_{11} \to \frac{13}{93},\ (Z_{1212})_{12} \to \frac{7}{50},\ (Z_{1212})_{21} \to 0,\ (Z_{1212})_{22} \to \frac{1}{4614},\ (Z_{1221})_{11} \to \frac{113}{81},\ (Z_{1221})_{12} \to -\frac{67}{65},$$
$$(Z_{1221})_{22} \to \frac{639}{160},\ (Z_{1222})_{11} \to \frac{43}{150},\ (Z_{1222})_{12} \to \frac{19}{89},\ (Z_{1222})_{21} \to 0,\ (Z_{1222})_{22} \to 0,\ (Z_{2112})_{11} \to \frac{130}{49},$$
$$(Z_{2112})_{12} \to -\frac{159}{61},\ (Z_{2112})_{22} \to \frac{394}{85},\ (Z_{2122})_{11} \to \frac{59}{294},\ (Z_{2122})_{12} \to \frac{7}{127},\ (Z_{2122})_{21} \to \frac{46}{229},\ (Z_{2122})_{22} \to \frac{7}{128},$$
$$(Z_{2222})_{11} \to \frac{21}{64},\ (Z_{2222})_{12} \to \frac{13}{95},\ (Z_{2222})_{22} \to \frac{87}{136},\ (Z_{11111})_{11} \to \frac{83}{69},\ (Z_{11111})_{12} \to -\frac{101}{143},\ (Z_{11111})_{22} \to -\frac{33}{97},$$
$$(Z_{11112})_{11} \to -\frac{1}{2668},\ (Z_{11112})_{12} \to \frac{7}{102},\ (Z_{11112})_{21} \to \frac{120}{109},\ (Z_{11112})_{22} \to \frac{61}{80},\ (Z_{11121})_{11} \to \frac{20}{91},$$
$$(Z_{11121})_{12} \to -\frac{1}{142},\ (Z_{11121})_{21} \to -\frac{2}{97},\ (Z_{11121})_{22} \to -\frac{1}{2946},\ (Z_{11122})_{11} \to \frac{69}{193},\ (Z_{11122})_{12} \to \frac{22}{309},$$
$$(Z_{11122})_{21} \to -\frac{2}{93},\ (Z_{11122})_{22} \to -\frac{1}{75},\ (Z_{11211})_{11} \to \frac{13}{58},\ (Z_{11211})_{12} \to \frac{6}{175},\ (Z_{11211})_{22} \to \frac{1}{645},\ (Z_{11212})_{11} \to \frac{12}{103},$$
$$(Z_{11212})_{12} \to \frac{25}{213},\ (Z_{11212})_{21} \to \frac{1}{47},\ (Z_{11212})_{22} \to \frac{1}{51},\ (Z_{11221})_{11} \to \frac{32}{47},\ (Z_{11221})_{12} \to -\frac{49}{64},\ (Z_{11221})_{21} \to \frac{5}{54},$$
$$(Z_{11221})_{22} \to -\frac{29}{168},\ (Z_{11222})_{11} \to \frac{27}{112},\ (Z_{11222})_{12} \to \frac{17}{104},\ (Z_{11222})_{21} \to \frac{1}{38},\ (Z_{11222})_{22} \to \frac{6}{89},\ (Z_{12112})_{11} \to \frac{31}{17},$$
$$(Z_{12112})_{12} \to -\frac{111}{56},\ (Z_{12112})_{21} \to \frac{24}{107},\ (Z_{12112})_{22} \to -\frac{27}{113},\ (Z_{12121})_{11} \to \frac{16}{129},\ (Z_{12121})_{12} \to -\frac{1}{4415},$$

[5] Since $H_3(Z)$ is positive semidefinite, it is self-adjoint. Hence $Z_{i_1 i_2 \cdots i_r} = Z^*_{i_r i_{r-1} \cdots i_1}$ for all indices i_j. We thus omit one of such a pair in the display below.

$(Z_{12121})_{22} \to \frac{1}{3301}$, $(Z_{12122})_{11} \to \frac{17}{98}$, $(Z_{12122})_{12} \to \frac{1}{21}$, $(Z_{12122})_{21} \to 0$, $(Z_{12122})_{22} \to \frac{1}{7768}$, $(Z_{12212})_{11} \to -\frac{5}{74}$, $(Z_{12212})_{12} \to -\frac{1}{104}$, $(Z_{12212})_{21} \to \frac{81}{73}$, $(Z_{12212})_{22} \to \frac{95}{117}$, $(Z_{12221})_{11} \to \frac{18}{95}$, $(Z_{12221})_{12} \to -\frac{1}{90}$, $(Z_{12221})_{22} \to -\frac{1}{449}$, $(Z_{12222})_{11} \to \frac{32}{113}$, $(Z_{12222})_{12} \to \frac{8}{67}$, $(Z_{12222})_{21} \to \frac{1}{6553}$, $(Z_{12222})_{22} \to -\frac{1}{3675}$, $(Z_{21112})_{11} \to \frac{14}{117}$, $(Z_{21112})_{12} \to \frac{22}{175}$, $(Z_{21112})_{22} \to \frac{11}{86}$, $(Z_{21122})_{11} \to \frac{221}{295}$, $(Z_{21122})_{12} \to -\frac{268}{153}$, $(Z_{21122})_{21} \to -\frac{43}{117}$, $(Z_{21122})_{22} \to \frac{169}{83}$, $(Z_{21212})_{11} \to \frac{5}{82}$, $(Z_{21212})_{12} \to \frac{3}{49}$, $(Z_{21212})_{22} \to \frac{7}{114}$, $(Z_{21222})_{11} \to \frac{12}{97}$, $(Z_{21222})_{12} \to \frac{4}{43}$, $(Z_{21222})_{21} \to \frac{15}{121}$, $(Z_{21222})_{22} \to \frac{8}{87}$, $(Z_{22122})_{11} \to \frac{49}{197}$, $(Z_{22122})_{12} \to \frac{5}{73}$, $(Z_{22122})_{22} \to \frac{1}{55}$, $(Z_{22222})_{11} \to \frac{16}{69}$, $(Z_{22222})_{12} \to \frac{19}{116}$, $(Z_{22222})_{22} \to \frac{23}{41}$, $(Z_{111111})_{11} \to \frac{19025}{73}$, $(Z_{111111})_{12} \to -\frac{150}{47}$, $(Z_{111111})_{22} \to \frac{22307}{82}$, $(Z_{111112})_{11} \to \frac{86}{115}$, $(Z_{111112})_{12} \to \frac{30}{49}$, $(Z_{111112})_{21} \to -\frac{101}{138}$, $(Z_{111112})_{22} \to -\frac{37}{63}$, $(Z_{111121})_{11} \to \frac{5}{72}$, $(Z_{111121})_{12} \to -\frac{1}{113}$, $(Z_{111121})_{21} \to \frac{62}{67}$, $(Z_{111121})_{22} \to \frac{23}{275}$, $(Z_{111122})_{11} \to \frac{51}{98}$, $(Z_{111122})_{12} \to -\frac{25}{113}$, $(Z_{111122})_{21} \to -\frac{3}{119}$, $(Z_{111122})_{22} \to \frac{251}{96}$, $(Z_{111211})_{11} \to \frac{38}{241}$, $(Z_{111211})_{12} \to -\frac{5}{64}$, $(Z_{111211})_{21} \to -\frac{4}{115}$, $(Z_{111211})_{22} \to \frac{5}{74}$, $(Z_{111212})_{11} \to \frac{4}{39}$, $(Z_{111212})_{12} \to \frac{11}{126}$, $(Z_{111212})_{21} \to -\frac{1}{112}$, $(Z_{111212})_{22} \to -\frac{1}{141}$, $(Z_{111221})_{11} \to \frac{11371}{127}$, $(Z_{111221})_{12} \to -\frac{406}{123}$, $(Z_{111221})_{21} \to -\frac{349}{138}$, $(Z_{111221})_{22} \to \frac{8227}{81}$, $(Z_{111222})_{11} \to \frac{17}{80}$, $(Z_{111222})_{12} \to \frac{21}{137}$, $(Z_{111222})_{21} \to -\frac{1}{68}$, $(Z_{111222})_{22} \to -\frac{1}{63}$, $(Z_{112112})_{11} \to \frac{201}{98}$, $(Z_{112112})_{12} \to -\frac{139}{70}$, $(Z_{112112})_{21} \to -\frac{13}{37}$, $(Z_{112112})_{22} \to \frac{40}{47}$, $(Z_{112121})_{11} \to \frac{13}{83}$, $(Z_{112121})_{12} \to \frac{1}{3352}$, $(Z_{112121})_{21} \to \frac{4}{105}$, $(Z_{112121})_{22} \to 0$, $(Z_{112122})_{11} \to \frac{29}{119}$, $(Z_{112122})_{12} \to \frac{5}{71}$, $(Z_{112122})_{21} \to \frac{7}{109}$, $(Z_{112122})_{22} \to \frac{1}{44}$, $(Z_{112211})_{11} \to \frac{14181}{83}$, $(Z_{112211})_{12} \to \frac{3}{103}$, $(Z_{112211})_{22} \to \frac{14321}{84}$, $(Z_{112212})_{11} \to -\frac{3}{64}$, $(Z_{112212})_{12} \to \frac{1}{467}$, $(Z_{112212})_{21} \to -\frac{1}{33}$, $(Z_{112212})_{22} \to -\frac{1}{49}$, $(Z_{112221})_{11} \to \frac{7}{40}$, $(Z_{112221})_{12} \to -\frac{1}{158}$, $(Z_{112221})_{21} \to -\frac{7}{109}$, $(Z_{112221})_{22} \to \frac{27}{350}$, $(Z_{112222})_{11} \to \frac{21}{86}$, $(Z_{112222})_{12} \to \frac{17}{222}$, $(Z_{112222})_{21} \to \frac{1}{148}$, $(Z_{112222})_{22} \to \frac{8}{85}$, $(Z_{121112})_{11} \to \frac{2}{75}$, $(Z_{121112})_{12} \to \frac{11}{137}$, $(Z_{121112})_{21} \to \frac{1}{286}$, $(Z_{121112})_{22} \to \frac{1}{129}$, $(Z_{121121})_{11} \to \frac{19401}{113}$, $(Z_{121121})_{12} \to \frac{6}{43}$, $(Z_{121121})_{22} \to \frac{7672}{45}$, $(Z_{121122})_{11} \to \frac{26}{49}$, $(Z_{121122})_{12} \to -\frac{135}{106}$, $(Z_{121122})_{21} \to \frac{6}{91}$, $(Z_{121122})_{22} \to -\frac{3}{17}$, $(Z_{121212})_{11} \to \frac{7}{100}$, $(Z_{121212})_{12} \to \frac{5}{71}$, $(Z_{121212})_{21} \to 0$, $(Z_{121212})_{22} \to 0$, $(Z_{121221})_{11} \to -\frac{7}{102}$, $(Z_{121221})_{12} \to \frac{71}{75}$, $(Z_{121221})_{21} \to \frac{1}{281}$, $(Z_{121221})_{22} \to \frac{2}{23}$, $(Z_{121222})_{11} \to \frac{13}{121}$, $(Z_{121222})_{12} \to \frac{10}{127}$, $(Z_{121222})_{21} \to 0$, $(Z_{121222})_{22} \to \frac{1}{3350}$, $(Z_{122112})_{11} \to \frac{131}{91}$, $(Z_{122112})_{12} \to \frac{8}{89}$, $(Z_{122112})_{21} \to -\frac{27}{59}$, $(Z_{122112})_{22} \to -\frac{47}{92}$, $(Z_{122122})_{11} \to \frac{37}{91}$, $(Z_{122122})_{12} \to -\frac{17}{67}$, $(Z_{122122})_{21} \to \frac{1}{754}$, $(Z_{122122})_{22} \to \frac{279}{106}$, $(Z_{122212})_{11} \to \frac{5}{64}$, $(Z_{122212})_{12} \to \frac{11}{177}$, $(Z_{122212})_{21} \to \frac{1}{436}$, $(Z_{122212})_{22} \to \frac{1}{286}$, $(Z_{122221})_{11} \to \frac{34973}{135}$, $(Z_{122221})_{12} \to -\frac{173}{63}$, $(Z_{122221})_{22} \to \frac{38353}{141}$, $(Z_{122222})_{11} \to \frac{25}{127}$, $(Z_{122222})_{12} \to \frac{16}{105}$, $(Z_{122222})_{21} \to \frac{1}{9811}$, $(Z_{122222})_{22} \to \frac{1}{3641}$, $(Z_{211112})_{11} \to \frac{22123}{85}$, $(Z_{211112})_{12} \to -\frac{217}{202}$, $(Z_{211112})_{22} \to \frac{28977}{112}$, $(Z_{211122})_{11} \to \frac{9}{115}$, $(Z_{211122})_{12} \to \frac{1}{43}$,

$(Z_{211122})_{21} \to \frac{1}{19}$, $(Z_{211122})_{22} \to \frac{5}{131}$, $(Z_{211212})_{11} \to \frac{53}{56}$, $(Z_{211212})_{12} \to \frac{87}{92}$, $(Z_{211212})_{21} \to -\frac{73}{75}$, $(Z_{211212})_{22} \to -\frac{108}{113}$, $(Z_{211222})_{11} \to \frac{50}{47}$, $(Z_{211222})_{12} \to -\frac{31}{23}$, $(Z_{211222})_{21} \to -\frac{131}{92}$, $(Z_{211222})_{22} \to \frac{237}{49}$, $(Z_{212122})_{11} \to \frac{13}{111}$, $(Z_{212122})_{12} \to \frac{1}{34}$, $(Z_{212122})_{21} \to \frac{19}{161}$, $(Z_{212122})_{22} \to \frac{4}{135}$, $(Z_{212212})_{11} \to \frac{18284}{107}$, $(Z_{212212})_{12} \to \frac{54}{167}$, $(Z_{212212})_{22} \to \frac{2561}{15}$, $(Z_{212222})_{11} \to \frac{13}{106}$, $(Z_{212222})_{12} \to \frac{1}{19}$, $(Z_{212222})_{21} \to \frac{13}{106}$, $(Z_{212222})_{22} \to \frac{3}{58}$, $(Z_{221122})_{11} \to \frac{41456}{243}$, $(Z_{221122})_{12} \to -\frac{38}{85}$, $(Z_{221122})_{22} \to \frac{33873}{197}$, $(Z_{221222})_{11} \to \frac{2}{13}$, $(Z_{221222})_{12} \to \frac{5}{44}$, $(Z_{221222})_{21} \to \frac{3}{71}$, $(Z_{221222})_{22} \to \frac{3}{98}$, $(Z_{222222})_{11} \to \frac{15}{68}$, $(Z_{222222})_{12} \to \frac{7}{88}$, $(Z_{222222})_{22} \to \frac{54}{73}$. ■

4.6. **Stopping criteria.** Finally, we present two criteria for the stopping of the Lasserre-Parrilo lifting hierarchy.

4.6.1. *Positivstellensatz inspired stopping criterion.* The first criterion mirrors the analogous statement ([HKM16, Theorem 6.8]) in the Γ-free context of matrix convex sets.

Given $\alpha, \beta, \nu \in \mathbb{N}$, and an $\ell \times \ell$ symmetric matrix-valued noncommutative polynomial p, set

$$\mathrm{QM}^{\nu}_{\alpha,\beta}(p) := \Sigma^{\nu}_{\alpha} + \Big\{ \sum_{i}^{\text{finite}} f_i^* p f_i : \ f_i \in \mathbb{C}^{\ell\times\nu}\langle x\rangle_{\beta} \Big\} \ \subseteq \ \mathbb{C}^{\nu\times\nu}\langle x\rangle_{\max\{2\alpha,2\beta+a\}}, \tag{4.33}$$

where $a = \deg(p)$ and Σ^{ν}_{α} denotes all $\nu\times\nu$ sums of squares of degree at most 2α. Clearly, if $f \in \mathrm{QM}^{\nu}_{\alpha,\beta}(p)$ then $f|_{\mathcal{D}_p} \succeq 0$. We call $\mathrm{QM}^{\nu}_{\alpha,\beta}(p)$ the **truncated quadratic module** defined by p. For notational convenience, we write $\mathrm{QM}^{\nu}_{k} \subseteq \mathbb{C}^{\nu\times\nu}\langle x\rangle_{2k}$ for $\mathrm{QM}^{\nu}_{k,\lfloor\frac{2k-a}{2}\rfloor}$. We also introduce

$$\mathrm{QM}^{\nu}(p) := \bigcup_{\alpha,\beta} \mathrm{QM}^{\nu}_{\alpha,\beta}(p),$$

the **quadratic module** defined by p. If $\nu = 1$ we shall often omit the superscript ν. Observe that p is Archimedean if the convex cone $\mathrm{QM}^{\nu}(p)$ has an order unit, i.e., for all symmetric $\nu\times\nu$ matrix-valued polynomials f there is $N \in \mathbb{N}$ with $N - f \in \mathrm{QM}^{\nu}(p)$. (This notion is easily seen to be independent of ν, cf. [HKM13, §6].)

Definition 4.15. We say that p has the **Γ-positivity certificate property (Γ-PCP)**, if for some $N \in \mathbb{N}$, every $\nu \in \mathbb{N}$ and every Γ-pencil L^{Γ} of size $\nu\times\nu$, we have

$$L^{\Gamma}|_{\mathcal{D}_p} \succeq 0 \quad \Rightarrow \quad L^{\Gamma} \in M^{\nu}_{N}(p).$$ ■

We refer the reader to [Scw04, NiS07, LPR20, SL23] for the classical commutative study of degree bounds needed in Positivstellensatz certificates. See also [HN09, HN10] for an application of these bounds to convexity and Lasserre-Parrilo lifts in the commutative.

The next theorem says if Γ-PCP holds, then one of the truncated Lasserre–Parrilo lifts gives exactly the Γ-convex hull of $\mathcal{D}^{\infty}_{p}$.

Proposition 4.16. *Suppose $\mathcal{D}_p$ is bounded, $p(0) \succeq 0$, and $\Gamma(0) = 0$. If p has the Γ-PCP, then*

$$\Gamma\text{- conv}(\mathcal{D}_p^\infty) = \hat{\mathfrak{L}}_p^\Gamma\Big(\cdot, \left\lceil \frac{N}{2} \right\rceil\Big).$$

Proof. Let $\eta = \lceil \frac{N}{2} \rceil$. Since $\mathcal{D}_p$ is bounded and p has the Γ-PCP, p is Archimedean. By Theorem 4.7, $\Gamma\text{- conv}(\mathcal{D}_p^\infty) \subseteq \hat{\mathfrak{L}}_p^\Gamma(\cdot, \eta)$.

Now let $\nu \in \mathbb{N}$ and $Y \in \hat{\mathfrak{L}}_p^\Gamma(\nu, \eta) \setminus \Gamma\text{- conv}(\mathcal{D}_p^\infty)$ be given. Choose $W \in \mathfrak{L}_p^\Gamma(\nu, \eta)$ satisfying $\hat{W} = Y$. In particular, $H_\eta(W) \succeq 0$ and $H_{p,\eta}^{\Uparrow}(W) \succeq 0$. By the $\Gamma-$Hahn-Banach Theorem 1.9 (cf. Remark 1.10), there is a Γ-pencil L^Γ (of size ν) with $L^\Gamma|_{\Gamma\text{- conv}(\mathcal{D}_p^\infty)} \succeq 0$ and $L^\Gamma(Y) \not\succeq 0$. By the Γ-PCP property for p, we have that $L^\Gamma \in \mathrm{QM}_N^\nu(p)$, i.e.,

$$L^\Gamma = \sum_k h_k^* h_k + \sum_{i=1}^r f_i^* p f_i, \tag{4.34}$$

where $\deg(h_k) \leq \lfloor \frac{N}{2} \rfloor$ and $2\deg(f_i) + \deg(p) \leq N$ for $i = 1, \dots, r$. Now apply the Riesz moment map

$$\Phi_W^\nu : \mathbb{C}^{\nu\times\nu}\langle x\rangle_{\leq N} \to M_{\mu\,\nu}, \quad \sum_{\alpha\in\langle x\rangle} B_\alpha \alpha \mapsto \sum_{\alpha\in\langle x\rangle} B_\alpha \otimes W_\alpha,$$

to (4.34):

$$\Phi_W^\nu(L^\Gamma) = \sum_k \Phi_W^\nu(h_k^* h_k) + \sum_{i=1}^r \Phi_W^\nu(f_i^* p f_i). \tag{4.35}$$

Since $H_\eta(W) \succeq 0$ and $H_{p,\eta}^{\Uparrow}(W) \succeq 0$, the right hand side of (4.35) is positive semidefinite. On the other hand, by linearity in the γ_j built into the definition of a Γ-moment sequence, equation (4.7) implies

$$\Phi_W^\nu(L^\Gamma) = L^\Gamma(\hat{W}) = L^\Gamma(Y) \not\succeq 0,$$

a contradiction. ■

4.6.2. *Moment problem inspired stopping criterion.* Given $d, \eta \in \mathbb{N}$ with $d \leq \eta$, and a truncated moment sequence $Z = (Z_\alpha)_{|\alpha|\leq 2\eta}$, we say that the moment sequence Z or its truncated Hankel matrix $H_\eta(Z)$ is d**-flat** if $\operatorname{rank} H_{\eta-d}(Z) = \operatorname{rank} H_\eta(Z)$. Versions of this flatness condition are important in the classical commutative theory of moments since they imply the moment problem on Z has a solution [CF08, Lau09].

Proposition 4.17. *Suppose p is a symmetric matrix-valued noncommutative polynomial and let $\delta = \deg \Gamma$, $a = \deg p$ and $2\eta \geq \delta + 2a$. If each $Y \in \hat{\mathfrak{L}}_p^\Gamma(\cdot, \eta)$ admits a lift $W \in \mathfrak{L}_p^\Gamma(\cdot, \eta)$ that is a-flat, then*

$$\hat{\mathfrak{L}}_p^\Gamma(\cdot, \eta) = \Gamma\text{- conv}(\mathcal{D}_p) = \Gamma\text{- conv}(\mathcal{D}_p^\infty).$$

We split off the key step in establishing the proposition into a lemma that may be of independent interest.

Lemma 4.18. *Let $\delta = \deg \Gamma$, $a = \deg p$ and $2\eta \geq \delta + 2a$. If $Y \in \hat{\mathfrak{L}}_p^{\Gamma}(\cdot, \eta)$ admits a lift $W \in \mathfrak{L}_p^{\Gamma}(\cdot, \eta)$ that is a-flat, then $Y \in \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p)$.*

Proof. The proof goes along the same lines as that of Theorem 4.7, so we only give a sketch of the proof, leaving some of the details to the reader.

Define a sesquilinear form $[\cdot, \cdot]$ on $\mathcal{V} = \mathbb{C}\langle x \rangle_\eta \otimes \mathbb{C}^n$ by

$$[s, t] = \sum_{|\alpha|, |\beta| \leq \eta} \langle Y_{\beta^* \alpha} s_\alpha, t_\beta \rangle \tag{4.36}$$

where $s = \sum \alpha \otimes s_\alpha$ and $t = \sum \beta \otimes t_\beta$. The assumption $H_\eta(W) \succeq 0$ implies $[s, s] \geq 0$ for all $s \in \mathcal{V}$, so that this sesquilinear form is positive semidefinite. As before,

$$\mathcal{N} = \{s : [s, s] = 0\}$$

is a subspace of $\mathcal{V}$. Modding out $\mathcal{N}$ produces a well-defined and positive semidefinite form

$$\langle s, t \rangle = [s + \mathcal{N}, t + \mathcal{N}]_Y$$

on the quotient $\mathcal{W}$ of $\mathcal{V}$ by $\mathcal{N}$, where, as is standard practice, $s \in \mathcal{V}$ is identified with its image $s + \mathcal{N}$ in the quotient. At this point, observe $\langle \cdot, \cdot \rangle$ is a positive definite sesquilinear form on the finite-dimensional vector space $\mathcal{W}$. Hence $\mathcal{W}$ is already a Hilbert space.

Since W is a-flat, $\operatorname{rank} H_{\eta-1}(W) = \operatorname{rank} H_\eta$. Hence $\mathcal{W}$ is spanned by $\{\alpha \otimes h : |\alpha| \leq \eta - 1,\, h \in \mathbb{C}^n\}$. If $s \in \mathcal{N}$ has degree at most $\eta - 1$, then $x_j s$ has degree at most η and $[x_j s, t] = 0$. It follows that x_j determines a (well defined) operator Z_j on $\mathcal{W}$. Since $\mathcal{W}$ is a finite-dimensional Hilbert space, Z_j is a bounded operator. If s as in equation (4.17) has degree at most $\eta - a$, then the computation of equation (4.17) gives $\langle p(Z)s, s \rangle \geq 0$. Since $\operatorname{rank} H_{\eta-a}(W) = \operatorname{rank} H_\eta$, polynomials of degree at most $\eta - a$ span $\mathcal{V}$. Thus $p(Z) \succeq 0$. Hence $Z \in \mathcal{D}_p$ and, letting V denote the usual isometry, (Z, V) is a Γ-pair and $V^* Z^\alpha V = W_\alpha$ for all $|\alpha| \leq \eta$. As a side remark, we can extend W to a full Γ-moment sequence via this formula. ∎

Proof of Proposition 4.17. Simple corollary of Lemma 4.18. Using the obvious inclusion of (4.22) of Theorem 4.10 (which does not require Archimedeanity), we can deduce

$$\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p) \subseteq \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty) \subseteq \bigcap_{d=0}^{\infty} \hat{\mathfrak{L}}_p^{\Gamma}(\cdot, d) \subseteq \hat{\mathfrak{L}}_p^{\Gamma}(\cdot, \eta) \subseteq \Gamma\text{-}\operatorname{conv}(\mathcal{D}_p), \tag{4.37}$$

whence we have equalities throughout. ∎

4.7. **The operator Γ-convex hulls.** Most of the results in this section have their counterpart in the context of operator Γ-convex sets as we now explain.

Let $p \in M_\mu(\mathbb{C}\langle x \rangle)$ be a symmetric matrix-valued noncommutative polynomial in $\mathtt{g}$ variables, and consider its free operator semialgebraic set $\mathcal{D}_p^\infty$, whose operator Γ-convex hull is denoted $\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty)$. Thus $\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty)$ is just $\Gamma\text{-}\operatorname{conv}(\mathcal{D}_p^\infty)$ together with a new infinite level $\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty)(\infty)$, where a tuple $X \in \mathbb{S}_n^{\mathtt{g}}$ lies in $\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty)(\infty)$ if there exists a tuple

$Y \in \mathcal{D}_p^\infty$ acting on some separable Hilbert space $\mathcal{H}$ and an isometry $V : \mathcal{H} \to \mathcal{H}$ such that (Y, V) is a Γ-pair and $X = V^*YV$. By construction, this operator Γ-convex hull is the smallest operator Γ-convex set containing $\mathcal{D}_p^\infty$.

Since every Γ-pencil L^Γ as in (4.2) is a matrix-valued polynomial, this definition (4.4) applies to L^Γ yielding the **operator Γ-spectrahedron** $\mathcal{D}_{L^\Gamma}^\infty$ and then the **operator Γ-spectrahedrop** $\operatorname{proj}_x(\mathcal{D}_{L^\Gamma}^\infty)^\infty$.

Further, the set of all Γ-moment sequences $\mathfrak{M}^\Gamma$ is extended by accepting moment sequences of operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$ to yield the infinite level $\mathfrak{M}^\Gamma(\infty)$. To each $Y \in \mathfrak{M}^\Gamma(\infty)$ we can assign, as in Definition 4.2, the Γ-Hankel "matrix" $H(Y)$ as in (4.8), the truncated Γ-Hankel "matrix" $H_d(Y)$ as in (4.9), and their localizing counterparts.

Finally, $\mathfrak{L}_p^\Gamma$ of Subsection 4.3 is extended to include the infinite level

$$\mathfrak{L}_p^\Gamma(\infty) = \{Y = (Y_\alpha)_\alpha \in \mathfrak{M}^\Gamma(\infty) \mid \ H(Y) \succeq 0, \ H_p^{\Uparrow}(Y) \succeq 0\}. \tag{4.38}$$

The obtained lift is denoted $\mathfrak{L}_p^{\Gamma,\infty}$.

Given $Y \in \mathfrak{L}_p^{\Gamma,\infty}(n)$ for $n \in \mathbb{N} \cup \{\infty\}$ let

$$\hat{Y} = (Y_{x_1}, Y_{x_2}, \ldots, Y_{x_{\mathrm{g}}}) \in \mathbb{S}_n^{\mathrm{g}} \tag{4.39}$$

and denote

$$\hat{\mathfrak{L}}_p^{\Gamma,\infty} = \{\hat{Y} \mid Y \in \mathfrak{L}_p^{\Gamma,\infty}\}.$$

Corollary 4.19. *If p is Archimedean, then*

$$\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty) = \hat{\mathfrak{L}}_p^{\Gamma,\infty}.$$

Proof. Proof is the same as that of Theorem 4.7, with obvious modifications. The main change is that in the GNS construction one now works with $\mathbb{C}\langle x\rangle \otimes \mathcal{H}$ instead of $\mathbb{C}\langle x\rangle \otimes \mathbb{C}^n$, where $\mathcal{H}$ denotes an infinite-dimensional Hilbert space. The routine details are left to the reader. ∎

Obvious modifications of the truncations (4.20) and projections (4.21) of Subsection 4.4 lead to the sets $\mathfrak{L}_p^{\Gamma,\infty}(\cdot, d)$ and $\hat{\mathfrak{L}}_p^{\Gamma,\infty}(\cdot, d)$, respectively. With this at hand, we can state our final corollary.

Corollary 4.20. *If p is an Archimedean symmetric matrix-valued noncommutative polynomial, then*

(a) *for every $n \in \mathbb{N} \cup \{\infty\}$,*

$$\bigcap_{d=0}^{\infty} \hat{\mathfrak{L}}_p^{\Gamma,\infty}(n, d) = \hat{\mathfrak{L}}_p^{\Gamma,\infty}(n) = \Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty)(n); \tag{4.40}$$

(b) *for every d there is a Γ-pencil L_d^Γ given by a tuple $A^{(d)}$ such that $\hat{\mathfrak{L}}_p^{\Gamma,\infty}(\cdot, d)$ is the projection of $\mathcal{D}^{\Gamma,\infty}_{A^{(d)}}$.*

Hence, the operator Γ-spectrahedrops $\hat{\mathfrak{L}}_p^{\Gamma,\infty}(\cdot, d)$ give increasingly finer outer approximations of the operator Γ-convex hull $\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty)$ of $\mathcal{D}_p^\infty$.

Proof. By Theorem 4.10 it suffices to treat the case $n = \infty$. To prove the nontrivial inclusion in the first equality of (4.40), let $Z \in \bigcap_d \hat{\mathfrak{L}}_p^{\Gamma,\infty}(\infty, d)$. For every d there is a (truncated) moment sequence $Y^{(d)} = (Y_\alpha^{(d)}) \in \hat{\mathfrak{L}}_p^{\Gamma,\infty}(\infty; d)$ such that

$$(Y_{x_1}^{(d)}, \dots, Y_{x_g}^{(d)}) = Z.$$

By Lemma 4.11 (more precisely, its operator counterpart that is established with the same proof), for a given word α, the sequence $(Y_\alpha^{(d)})_{|\alpha| \le 2d+\deg(p)}$ is bounded. Since there are countably many such sequences, there exists a moment sequence (Y_α) from $\mathbb{S}_\infty$ and a subsequence $(d_k)_k$ of indices such that $(Y_\alpha^{(d_k)})$ WOT-converges to $Y_\alpha \in \mathbb{S}_\infty$ for each word α.

For each k the sequence $Y^{(d_k)}$ satisfies (4.24) with $d = d_k$ and hence for all $d \le d_k$. Hence the limit Γ-moment sequence $Y = (Y_\alpha)_\alpha$ satisfies (4.24) for all indices d and thus Y satisfies (4.23) and hence belongs to $\mathfrak{L}_p^{\Gamma,\infty}(\infty)$. Thus, $Z \in \hat{\mathfrak{L}}_p^{\Gamma,\infty}(\infty)$ since

$$Z = (Y_{x_1}, \dots, Y_{x_g}) = \hat{Y}. \qquad \blacksquare$$

Remark 4.21. Finally, let us discuss the stopping criteria for the Lasserre-Parrilo lifts in the operator Γ-convex context. The Positivstellensatz inspired version, Proposition 4.16 extends to the operator Γ-convex case mutatis mutandis with the conclusion

$$\Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty) = \hat{\mathfrak{L}}_p^{\Gamma,\infty}\Big(\cdot, \Big\lceil \frac{N}{2} \Big\rceil\Big).$$

A version of the the moment inspired stopping criterion with a conclusion as in Proposition 4.17 requires the addition of an Archimedean hypothesis on the polynomial p and a reinterpretation of the flatness condition. Indeed, in this case, the vector space $\mathcal{W}$ with positive definite sesquilinear form $\langle \cdot, \cdot \rangle$ as in the proof of Lemma 4.18, will not (necessarily) be complete. It will be, in general, necessary to form the closure. Hence the Archimedean assumption is needed to guarantee that the operators Z_j extend to bounded operators T_j on the completion of $\mathcal{W}$.

The natural reformulation of the flatness condition of Subsection 4.6.2 in terms of a Schur complement extends readily to the operator (infinite-dimensional) setting. Borrowing from the presentation in [Dri04] (or see [FF90, Chapter XVI]), for a positive semidefinite block operator matrix

$$D = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

the Schur complement of A is the largest positive semidefinite operator X such that

$$\begin{pmatrix} A & B \\ B^* & C - X \end{pmatrix} \succeq 0.$$

Thus, if $0 \preceq Y$ and

$$\begin{pmatrix} A & B \\ B^* & C - Y \end{pmatrix} \succeq 0,$$

then $Y \preceq X$. Factorizing the positive operator matrix D leads to a contraction G such that $B = C^{1/2} G A^{1/2}$ and then $X = A^{1/2}(I - G^* G)A^{1/2}$ is the desired Schur complement.

There is another description of X that is well suited to the present application. Namely, again assuming $D \succeq 0$ and letting $\mathcal{E}$ denote the space that C acts on and $\mathcal{H}$ the space that A acts on,

$$\langle Xh, h\rangle = \inf \left\{ \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix}, \begin{pmatrix} h \\ g \end{pmatrix} \mid g \in \mathcal{E} \right\}, \quad h \in \mathcal{H}. \tag{4.41}$$

Consider the positive semidefinite sesquilinear form

$$[\begin{pmatrix} h \\ g \end{pmatrix}, \begin{pmatrix} h' \\ g' \end{pmatrix}] = \langle D \begin{pmatrix} h \\ g \end{pmatrix}, \begin{pmatrix} h' \\ g' \end{pmatrix}\rangle.$$

From the description (4.41) it is clear that if $X = 0$, then the equivalence classes of $h \in \mathcal{H} \subseteq \mathcal{H} \oplus \mathcal{E}$ in the quotient are dense in the Hilbert space obtained in the usual way by modding out null vectors and forming the completion.

Given $n, d, \eta \in \mathbb{N}$ with $d \leq \eta$, a truncated moment sequence $Z = (Z_\alpha)_{|\alpha| \leq 2\eta}$ with $Z \in \mathbb{S}_n^{\mathbf{g}}$, is d-flat if and only if, setting $D = H_\eta(Z)$ and $A = H_{\eta-d}(Z)$, the Schur complement of A (in D) is 0. Thus, we make this Schur complement condition the definition of d-flat when the Z_α are allowed to be operators on an infinite-dimensional Hilbert space. With these hypotheses, the conclusion of Proposition 4.17 in the operator Γ-convex setting is is

$$\hat{\mathfrak{L}}_p^{\Gamma,\infty}(\cdot, \eta) = \Gamma\text{-}\operatorname{opco}(\mathcal{D}_p) = \Gamma\text{-}\operatorname{opco}(\mathcal{D}_p^\infty). \qquad \blacksquare$$

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Appendix A. An operator convex set as an nc convex set

This appendix explains how a weak*-closed operator convex set can be viewed as an nc convex set. Let $\mathscr{R} = \mathscr{R}_{\mathtt{g}}$ denote $\mathtt{g}$-dimensional **row Hilbert space**. Thus $\mathscr{R} \subseteq \mathcal{B}(\mathbb{C}^{\mathtt{g}}) = M_{\mathtt{g}}$ is the operator space consisting of $\mathtt{g} \times \mathtt{g}$ matrices with zero entries except along the first row. It is common to view $\mathscr{R}$ as $M_{1,\mathtt{g}}$. Likewise, let $\mathscr{C} \subseteq \mathcal{B}(\mathbb{C}^{\mathtt{g}}) = M_{\mathtt{g}}$ denote $\mathtt{g}$-dimensional **column Hilbert space**. As an operator space, $\mathscr{R}$ is endowed with the following matrix norm structure it inherits from the space $M_n(\mathcal{B}(\mathbb{C}^{\mathtt{g}})) = \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^{\mathtt{g}})$. An element X in $M_n(\mathscr{R})$ is a $1 \times \mathtt{g}$ block matrix with entries from M_n,

$$X = \begin{pmatrix} X_1 & \cdots & X_{\mathtt{g}} \end{pmatrix},$$

and $\|X\| = \|X\|_r$, the norm of X, is the norm of X as an operator $\mathbb{C}^{\mathtt{g}} \otimes \mathbb{C}^n \to \mathbb{C}^{\mathtt{g}}$. In particular, $\|X\| \le 1$ if and only if $\sum X_j X_j^* \preceq I_n$. Similar considerations apply for the column space $\mathscr{C}$.

In general, the (standard) dual E^* of an operator space E is the operator space $\mathrm{CB}(E, \mathbb{C})$ of completely bounded maps from E to $\mathbb{C}$. The matrix norm structure on E^* is determined via the (isometric) identification of $M_n(E^*)$ with $\mathrm{CB}(E, M_n)$, the space of completely bounded maps from E into M_n; that is, for $F = (f_{j,k}) \in M_n(E^*)$,

$$\|F\| = \|(f_{j,k})\| = \sup\left\{ \| \left[f_{j,k}(Y(a,b)) \right]_{a,b} \| \,:\, Y = (Y(a,b)) \in M_m(E),\ \|Y\| = 1,\ m \in \mathbb{N} \right\}.$$

One finds (see [ER00, Section I.3.4] or [Pau02, Proposition 14.9]) that $\mathscr{C}^* = \mathscr{R}$, the operator space dual, completely isometrically via the canonical mapping $\gamma : \mathscr{R} \to \mathscr{C}^*$, where, for

$$x = \begin{pmatrix} x_1 & \dots & x_{\mathtt{g}} \end{pmatrix} \in \mathscr{R},$$

$\gamma[x] : \mathscr{C} \to \mathbb{C}$ is given by $\gamma[x](y) = \sum_j x_j\, y_j$. In particular, for $X \in M_n(\mathscr{R})$ and $Y \in M_m(\mathscr{C})$,

$$1_m \otimes (1_n \otimes \gamma)[X](Y) = \big(\gamma[X(j,k)](Y(a,b))\big) = \left(\sum_{\ell} X_\ell(j,k)\, Y_\ell(a,b) \right) = \sum X_\ell \otimes Y_\ell,$$

where $X(j,k) = \begin{pmatrix} (X_1)_{j,k} & \dots & (X_{\mathtt{g}})_{j,k} \end{pmatrix}$. Further,

$$\begin{aligned} \|X\| &= \| \sum X_\ell \otimes e_\ell^* \|_{M_{n,n\mathtt{g}}} = \sup\{ \| \sum X_\ell \otimes Y_\ell \| : m \in \mathbb{N},\ Y \in M_m(\mathscr{C}),\ \|Y\| = 1 \} \\ &= \|1_n \otimes \gamma[X]\|_{\mathrm{CB}(\mathscr{C}, M_n)}. \end{aligned} \tag{A.1}$$

To obtain the second equality in equation (A.1), observe that choosing $Y_\ell = e_1 e_\ell^*$ shows the supremum is at least $\|X\|$. For the reverse inequality, note given $Y \in M_m(\mathscr{C})$ with $\|Y\| = 1$, that

$$\widehat{X} = \sum_a X_a \otimes e_a^* \otimes I_m, \quad \widehat{Y} = \sum_b I_n \otimes e_b \otimes Y_b$$

have norm $\|X\|$ and $\|Y\| = 1$ respectively, since, for instance $\widehat{Y}^* \widehat{Y} = I_n \otimes \sum Y_j^* Y_j \preceq I \otimes \|Y\|^2$. It follows that

$$\|X\| = \|X\|\,\|Y\| \ge \|\widehat{X}\,\widehat{Y}\| = \| \sum X_\ell \otimes Y_\ell \|.$$

Thus the supremum is at most $\|X\|$. Hence $\mathscr{R}$ is a dual operator space with distinguished predual $\mathscr{C}$.

The convention in [DK+], specialized to our setting where $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}}$, is to endow $K_m \subseteq \mathbb{S}^{\mathrm{g}}_m \subseteq M_m(\mathscr{R})$ with the relative topology inherited from $M_m(\mathscr{R})$, where the latter is identified with $\mathrm{CB}(\mathscr{C}, M_m)$ given the **point-weak*** topology. The point-weak topology is the topology in which a net φ_λ converging to φ means $\varphi_\lambda(y)$ converges to $\varphi(y)$ weak* for each $y \in \mathscr{C}$. When $m \in \mathbb{N}$, both $\mathscr{C}$ and M_m are finite-dimensional, so this topological consideration is trivial. The case $n = \infty$ requires some explanation. By definition (see for instance [DK+, ER00]), M_∞ (resp. $M_\infty(\mathscr{R})$) consists of the infinite matrices $X = (x_{j,k})_{j,k=1}^\infty$ with $x_{j,k} \in \mathbb{C}$ (resp. $x_{j,k} \in \mathscr{R}$) for which there is a uniform bound on the norms of all finite submatrices. As is readily checked, $M_\infty = \mathcal{B}(\ell^2)$, and we identify $\mathcal{B}(\ell^2)$ with $\mathcal{B}(\mathcal{H})$ (recall the convention that $\mathcal{H}$ is a separable infinite-dimensional Hilbert space). Likewise,

$$M_\infty(\mathscr{R}) = \mathscr{R} \otimes \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}^{\mathrm{g}}, \mathcal{H}).^{6}$$

The topology on $M_\infty(\mathscr{R})$ is then the **point-weak*** topology on $\mathrm{CB}(\mathscr{C}, M_\infty) = \mathrm{CB}(\mathscr{C}, \mathcal{B}(\mathcal{H}))$. Thus, a net φ_λ from $\mathrm{CB}(\mathscr{C}, \mathcal{B}(\mathcal{H}))$ converges in this topology if and only if for each $y \in \mathscr{C}$, every $(Y_{\lambda,j})_\lambda$ converges weak* to some $Y_j \in \mathcal{B}(\mathcal{H})$ (with $\mathcal{B}(\mathcal{H})$ dual to trace class), where

$$\begin{pmatrix} Y_{\lambda,1} & \dots & Y_{\lambda,\mathrm{g}} \end{pmatrix} = \begin{pmatrix} \varphi_\lambda(y)[e_1] & \dots \varphi_\lambda(y)[e_{\mathrm{g}}] \end{pmatrix} = \varphi_\lambda(y).$$

Thus, the point-weak* topology on $M_\infty(\mathscr{R}) = CB(\mathscr{C}, \mathcal{B}(\mathcal{H}))$, is the same as the weak* topology on $\mathcal{B}(\mathcal{H}^{\mathrm{g}}, \mathcal{H})$ defined via trace class duality, which we note is uniquely determined since $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.

The definition of a compact **nc convex** set with cardinal upper bound $\aleph_0$ from [DK+, Defintion 2.2.1] reads as follows. Given an operator space E, a graded set $\boldsymbol{K} = (K_n)_{1 \le n \le \infty}$ with $K_n \subseteq M_n(E)$ is nc convex if it is closed under countable direct sums of norm bounded families and isometric compressions. More concretely, given $1 \le m, n \le \infty$, if $X_j \in \boldsymbol{K}$ for $j \in \mathbb{N}$, then $\oplus X_j \in \boldsymbol{K}$; and if $X \in K_n$ and $V : \mathcal{H}_m \to \mathcal{H}_n$ is an isometry, then $V^* X V \in K_m$. Finally, assuming E is a dual operator space with distinguished predual E_*, the nc convex set $\boldsymbol{K}$ is compact if each K_n is compact (in the point-weak* topology). Here, $\mathrm{CB}(E_*, M_n)$ is endowed with the point-weak* topology, and $K_n \subseteq M_n(E)$ is endowed with the relative topology after identifying $M_n(E)$ with $\mathrm{CB}(E_*, M_n)$. Thus, a weak*-closed and (norm) bounded operator convex set $\boldsymbol{K} \subseteq \mathbb{S}^{\mathrm{g}} \subseteq M(\mathscr{R})$ is a compact nc convex set over the operator space $\mathscr{R}$. See also [Dav25, Example 16.4(3)] for a closely related example.

[6] Note that an $A \in \mathscr{R} \otimes \mathcal{B}(\mathcal{H})$ is identified with the row operator $A = \begin{pmatrix} A_1 & \dots & A_{\mathrm{g}} \end{pmatrix}$, where $A_j \in \mathcal{B}(\mathcal{H})$. By contrast a $B \in \mathscr{C} \otimes \mathcal{B}(\mathcal{H})$ is identified with a column operator with entries from $\mathcal{B}(\mathcal{H})$ so that $\mathscr{C} \otimes \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H}^{\mathrm{g}})$.

Appendix B. Proof of Pascoe's SOT-BW-MUST

As the title of this section suggests and for the readers convenience, this section contains a streamlined version of the proof given in [Man20] of Theorem 3.1 tailored to the version of that result presented here. Accordingly, suppose $\mathcal{H}$ is a separable infinite-dimensional Hilbert space and $(X^{(j)})_j$ is a norm bounded sequence from $\mathcal{B}(\mathcal{H})^{\mathsf{g}}$. The first objective is to prove that there exists a subsequence $(Y^{(n)})_n$ of $(X^{(j)})_j$ and a sequence of unitary operators (U_n) such that $(U_n^*)_n$ and $(U_n^* Y^{(n)} U_n)_n$ both SOT converge.

Let $\{e_1, e_2, \dots\}$ denote an orthonormal basis for $\mathcal{H}$. Let $\mathcal{H}_k = \operatorname{span}\{e_1, \dots, e_k\}$. Thus $\mathcal{H}_0 = \cup_{k=1}^{\infty} \mathcal{H}_k$ is dense in $\mathcal{H}$. Recall the evaluations Y^α of a word α in g noncommuting variables at a $Y \in B(\mathcal{H})^{\mathsf{g}}$ described in Subsection 1.1. For each $k, j \in \mathbb{N}$, let

$$\mathcal{H}_k(Y) = \operatorname{span}\{Y^\alpha e_n : 1 \le n \le k,\ 0 \le |\alpha| \le k\},$$

where $|\alpha|$ is the length of the word α. For instance,

$$\mathcal{H}_1(Y) = \operatorname{span}\{e_1, Y_1 e_1, \dots Y_{\mathsf{g}} e_1\}.$$

By construction, $\mathcal{H}_k \subseteq \mathcal{H}_k(Y)$ and

$$Y_\ell \mathcal{H}_k(Y) \subseteq \mathcal{H}_{k+1}(Y)$$

for all $k \in \mathbb{N}$ and $1 \le \ell \le \mathsf{g}$. Moreover,

$$\dim \mathcal{H}_{k-1}(Y) < \dim \mathcal{H}_k(Y) \le d_k := k \sum_{j=0}^{k} \mathsf{g}^j$$

and

$$\dim \mathcal{H}_k(Y) - \dim \mathcal{H}_{k-1}(Y) \le k\mathsf{g}^k + \sum_{j=0}^{k-1} \mathsf{g}^j = d_k - d_{k-1},$$

independent of Y.

Let $\mathcal{H}_Y = \cup_{k=1}^{\infty} \mathcal{H}_k(Y)$. From the observations in the previous paragraph, we can define, inductively, an isometry $W : \mathcal{H}_Y \to \mathcal{H}_0$ satisfying

$$\mathcal{H}_k \subseteq W\, \mathcal{H}_k(Y) \subseteq \mathcal{H}_{d_k}, \tag{B.1}$$

which then extends to an isometry W from $\mathcal{H}$ onto $\mathcal{H}$. Indeed, define W on $\mathcal{H}_1(Y)$ by $We_1 = e_1$ and then extend W to an isometry from $\mathcal{H}_1(Y)$ to $\mathcal{H}_{d_1} = \mathcal{H}_{\mathsf{g}+1}$ by choosing any isometry from $\mathcal{H}_1(Y) \ominus \mathcal{H}_1$ to $\mathcal{H}_{\mathsf{g}+1} \ominus \mathcal{H}_1$, which is possible since $\dim \mathcal{H}_1(Y) - \dim \mathcal{H}_1 \le \mathsf{g} = \dim \mathcal{H}_{\mathsf{g}+1} - \dim \mathcal{H}_1$. Now suppose $W : \mathcal{H}_k(Y) \to \mathcal{H}_{d_k}$ has been defined satisfying the inclusions in equation (B.1). If e_{k+1} is in $W\mathcal{H}_k(Y)$, then extend W to $\mathcal{H}_{k+1}(Y)$ by choosing any isometry from the at most $d_{k+1} - d_k$ dimensional space $\mathcal{H}_{k+1}(Y) \ominus \mathcal{H}_k(Y)$ to the at least $d_{k+1} - d_k$ dimensional space $\mathcal{H}_{d_{k+1}} \ominus W\mathcal{H}_k(Y)$. Otherwise, let P denote the projection onto $W\mathcal{H}_k(Y) \subseteq \mathcal{H}_{d_k}$ and let e denote the unit vector in the direction of $e_{k+1} - Pe_{k+1}$. Thus $e \in [W\mathcal{H}_k(Y)]^\perp$. On the other hand, since e_{k+1}, $Pe_{k+1} \in \mathcal{H}_{d_{k+1}}$ it follows that e is in

$\mathcal{H}_{d_{k+1}}$ and thus $e \in \mathcal{H}_{d_{k+1}} \ominus W\mathcal{H}_k(Y)$. Because $\dim \mathcal{H}_k(Y) < \dim \mathcal{H}_{k+1}(Y)$, there exists a unit vector $f \in \mathcal{H}_{k+1}(Y) \ominus \mathcal{H}_k(Y)$. Now choose any isometry from $\mathcal{H}_{k+1}(Y) \ominus \mathcal{H}_k(Y)$ into $\mathcal{H}_{d_{k+1}} \ominus W\mathcal{H}_k(Y)$ that sends f to e to extend W to an isometry $W : \mathcal{H}_{k+1}(Y) \to \mathcal{H}_{d_{k+1}}$. To complete the induction argument, note that both $Wf = e$ and Pe_{k+1} are in $W\mathcal{H}_{k+1}(Y)$ and hence $e_{k+1} \in W\mathcal{H}_{k+1}(Y)$ so that the inclusion $\mathcal{H}_{k+1} \subseteq \mathcal{H}_{k+1}(Y)$ also holds.

Letting V denote the unitary W^*, from equation (B.1) we obtain,

$$\mathcal{H}_k \subseteq V^* \mathcal{H}_k(Y) \subseteq \mathcal{H}_{d_k}, \tag{B.2}$$

for all $k \in \mathbb{N}$. It follows that

$$V\mathcal{H}_k \subseteq \mathcal{H}_k(Y) \subseteq V\mathcal{H}_{d_k}, \tag{B.3}$$

and also

$$V^*\mathcal{H}_k \subseteq V^*\mathcal{H}_k(Y) \subseteq \mathcal{H}_{d_k}, \tag{B.4}$$

since V is unitary and $\mathcal{H}_k \subseteq \mathcal{H}_k(Y)$. Finally,

$$V^*YV\mathcal{H}_k \subseteq V^*Y\mathcal{H}_k(Y) \subseteq V^*\mathcal{H}_{k+1}(Y) \subseteq \mathcal{H}_{d_{k+1}}. \tag{B.5}$$

Returning to the sequence $(X^{(j)})$, for each j there exists a unitary $V_j \in \mathcal{B}(\mathcal{H})$ such that equations (B.2) through (B.5) hold with $X^{(j)}$ and V_j in place of Y and V. Let $S^{(j)} = V_j^* X^{(j)} V_j$ and let $J_k : \mathcal{H}_k \to \mathcal{H}_{d_{k+1}}$ denote the inclusion. The sequence $(S^{(j)}J_k, V_j^* J_k)_j$ is a sequence of $(\mathrm{g}+1)$-tuples of operators between the finite-dimensional Hilbert spaces $\mathcal{H}_k$ and $\mathcal{H}_{d_{k+1}}$ and thus has a (norm) convergent subsequence. By a standard diagonalization argument with respect to the parameter k, there exists a subsequence $(T^{(n)}, U_n^*)_n$ of $(S^{(j)}, V_j^*)_j$ such that $(U_n^* h)_n$ and $(T_\ell^{(n)} h)$ converges in $\mathcal{H}$ for each $h \in \cup_k \mathcal{H}_k = \mathcal{H}_0$ and $1 \leq \ell \leq \mathrm{g}$. Since $(S^{(j)}, V_j^*)_j$ is a bounded sequence, so is $(T^{(n)}, U_n^*)_n$. Thus for each ℓ, the sequence $(T_\ell^{(n)})_n$ is norm bounded sequence that is pointwise Cauchy on the dense subset $\mathcal{H}_0$ of $\mathcal{H}$, and similarly for $(U_n^*)_n$. Hence $(T_\ell^{n)})_n$ converge SOT to some bounded operator $T \in \mathcal{B}(\mathcal{H})$ and U_n^* converges SOT to some bounded operator U^*. Since $((U_n^*)^* = U_n)_n$ converges WOT to U, it follows that $(U_n U_n^*)$ converges WOT to UU^* and also to the identity. Thus U^* is an isometry. Finally, $(Y_n = U_n T^{(n)} U_n^*)_n$, is a subsequence of $(X^{(j)})_j$ and $(U_n^* Y^{(n)} U_n)$ converges SOT to T, establishing the first part of the result.

To prove the last statement of the proposition, suppose $S \subseteq \mathcal{B}(\mathcal{H})^{\mathrm{g}}$ is SOT-closed and bounded and closed under isometric conjugation. Since S is assumed bounded and $\mathcal{H}$ is separable, to prove S is WOT compact, it suffices to prove that S is WOT sequentially compact. Given a sequence $(X^{(j)})_j$ from S there exists a subsequence $(Y^n)_n$ of $(X^{(j)})_j$ and a sequence $(U_n)_n$ of unitary operators on $\mathcal{H}$ such that $(U_n^*)_n$ converges SOT to an isometry V and $(U_n^* Y^{(n)} U_n)$ converges SOT to some operator T. Since S is closed under isometric conjugation, each $U_n^* Y^{(n)} U_n$ is in S and therefore, since S is SOT-closed, T and V^*TV are in S. Moreover, $(U_n \left[U_n^* Y^{(n)} U_n\right] U_n^*)_n$ converges WOT to $V^*TV \in S$. Thus $(Y^{(n)})_n$ converges WOT to $V^*TV \in S$ and the proof is complete. ∎

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