

NONCOMMUTATIVE POLYNOMIALS NONNEGATIVE ON A VARIETY INTERSECT A CONVEX SET

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ABSTRACT. By a result of Helton and McCullough [HM12], open bounded convex free semialgebraic sets are exactly open (matricial) solution sets \mathcal{D}_L° of a linear matrix inequality (LMI) $L(X) \succ 0$. This paper gives a precise algebraic certificate for a polynomial being nonnegative on a convex semialgebraic set intersect a variety, a so-called “Perfect” Positivstellensatz.

For example, given a generic convex free semialgebraic set \mathcal{D}_L° we determine all “(strong sense) defining polynomials” p for \mathcal{D}_L° . Such polynomials must have the form

$$p = L \left(\sum_i^{\text{finite}} q_i^* q_i \right) L + \sum_j^{\text{finite}} (r_j L + C_j)^* L (r_j L + C_j),$$

where q_i, r_j are matrices of polynomials, and C_j are real matrices satisfying $C_j L = L C_j$.

This follows from our general result for a given linear pencil L and a finite set I of rows of polynomials. A matrix polynomial p is positive where L is positive and all $\iota \in I$ vanish if and only if

$$(P) \quad p = \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j + \sum_k^{\text{finite}} (r_k^* \iota_k + \iota_k^* r_k),$$

where each p_i, q_j and r_k are matrices of polynomials of appropriate dimension, and each ι_k is an element of the “ L -real radical” of I . In this representation, we can restrict p_i, q_i, ι_k and r_k to be elements of a low-dimensional subspace of matrices of polynomials, and in particular, their degrees depend in a very tame way only on the degree of p and the degrees of the elements of I . Further, this paper gives an efficient algorithm for computing the L -real radical of I .

This Positivstellensatz extends and unifies two different lines of results:

- (1) the free real Nullstellensatz of [CHMN, Nel] which gives an algebraic certificate corresponding to one polynomial being zero on the free variety where others are zero; this is (P) with $L = 1$;
- (2) the convex Positivstellensatz of [HKM12, KS11] which is (P) without I ; i.e., $I = \{0\}$.

The representation (P) has a number of additional consequences which will be presented.

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1. INTRODUCTION

In this section we introduce the main concepts which will be used throughout this paper. Subsections 1.1, 1.2, 1.3, 1.4 give basic definitions and include Theorem 1.1 which characterizes “defining polynomials” of a convex free semialgebraic set. Subsequently §1.5, §1.6 give more definitions and then in §1.7 we state our main general result Theorem 1.9 followed by statements of corollaries. The paper is devoted to proving these things and giving algorithms; a guide to the presentation is §1.8.

1.1. **Notation.** Given positive integers ν and ℓ , let $\mathbb{R}^{\nu \times \ell}$ denote the space of $\nu \times \ell$ real matrices. We use $E_{ij} \in \mathbb{R}^{\nu \times \ell}$ to denote the matrix unit with a 1 as the ij^{th} entry and a 0 for all other entries. If $\nu = 1$, then $E_{1j} = e_j \in \mathbb{R}^{1 \times \ell}$ is the row vector with 1 as the j^{th} entry and a 0 as all other entries. Let $\text{Id}_\nu \in \mathbb{R}^{\nu \times \nu}$ denote the $\nu \times \nu$ identity matrix. The transpose of a matrix $A \in \mathbb{R}^{\nu \times \ell}$ will be denoted by $A^* \in \mathbb{R}^{\ell \times \nu}$, and $\mathbb{S}^k \subseteq \mathbb{R}^{k \times k}$ is the space of real symmetric $k \times k$ matrices.

1.2. **Non-Commutative Polynomials.** Let $\langle x, x^* \rangle$ be the monoid freely generated by $x = (x_1, \dots, x_g)$ and $x^* = (x_1^*, \dots, x_g^*)$ —that is, $\langle x, x^* \rangle$ consists of words in the $2g$ free letters $x_1, \dots, x_g, x_1^*, \dots, x_g^*$, including the empty word \emptyset , which plays the role of the identity 1. Let $\mathbb{R}\langle x, x^* \rangle$ denote the \mathbb{R} -algebra freely generated by $\langle x, x^* \rangle$, i.e., the elements of $\mathbb{R}\langle x, x^* \rangle$ are polynomials in the non-commuting variables $\langle x, x^* \rangle$ with coefficients in \mathbb{R} . Call elements of $\mathbb{R}\langle x, x^* \rangle$ **non-commutative** or **NC** polynomials.

The **involution** on $\mathbb{R}\langle x, x^* \rangle$ is defined linearly so that $(x_i^*)^* = x_i$ for each variable x_i and $(pq)^* = q^*p^*$ for each $p, q \in \mathbb{R}\langle x, x^* \rangle$. For example,

$$(x_1x_2x_3 + 2x_3^*x_1 - x_3)^* = x_3^*x_2^*x_1^* + 2x_1^*x_3 - x_3^*$$

1.2.1. *Evaluation of NC Polynomials.* NC polynomials can be evaluated at a tuple of matrices in a natural way. Let $X = (X_1, \dots, X_g)$ be a tuple of real $n \times n$ matrices, that is $X \in (\mathbb{R}^{n \times n})^g$. Given $p \in \mathbb{R}\langle x, x^* \rangle$, let $p(X)$ denote the matrix defined by replacing each x_i in p with X_i , each x_i^* in p with X_i^* , and replacing the empty word with Id_n . Observe that $p^*(X) = p(X)^*$ for all $p \in \mathbb{R}\langle x, x^* \rangle$.

For example, if

$$p(x) = x_1^2 - 2x_1x_2^* - 3, \quad X_1 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

then

$$\begin{aligned}
p(X) &= X_1^2 - 2X_1X_2^* - 3\text{Id}_2 \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 12 \\ 18 & 21 \end{pmatrix}
\end{aligned}$$

1.2.2. *Matrices of NC Polynomials.* The space of $\nu \times \ell$ matrices with entries in $\mathbb{R}\langle x, x^* \rangle$ will be denoted as $\mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$. Each $p \in \mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$ can be expressed as

$$p = \sum_{w \in \langle x, x^* \rangle} A_w \otimes w \in \mathbb{R}^{\nu \times \ell} \otimes \mathbb{R}\langle x, x^* \rangle.$$

Given a tuple X of real $n \times n$ matrices, let $p(X)$ denote

$$p(X) = \sum_{w \in \langle x, x^* \rangle} A_w \otimes w(X) \in \mathbb{R}^{\nu n \times \ell n},$$

where \otimes denotes the Kronecker tensor product. The involution on $\mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$ is given by

$$p^* = \left(\sum_{w \in \langle x, x^* \rangle} A_w \otimes w \right)^* = \sum_{w \in \langle x, x^* \rangle} A_w^* \otimes w^* \in \mathbb{R}^{\ell \times \nu}\langle x, x^* \rangle.$$

Note that if X is a tuple of matrices, then $p^*(X) = p(X)^*$. If $p \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$, we say p is **symmetric** if $p = p^*$.

1.2.3. *Degree of NC Polynomials.* Let $|w|$ denote the **length** of a word $w \in \langle x, x^* \rangle$. A **monomial** in $\mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$ is a polynomial of the form $E_{ij} \otimes m$, where $m \in \langle x, x^* \rangle$. The **length** or **degree** of a monomial $E_{ij} \otimes m$ is $|E_{ij} \otimes m| := |m|$. The set of all monomials in $\mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$ is a vector space basis for $\mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$.

If p is a NC polynomial, define the degree of p , denoted $\deg(p)$, to be the largest degree of any monomial appearing in p . A NC polynomial p is **homogeneous of degree** d if every monomial appearing in p has degree d . If W is a subspace of $\mathbb{R}^{\nu \times \ell}\langle x \rangle$, define W_d to be the space spanned by all elements of W with degree at most d , and define W_d^{hom} to be the space spanned by all elements of W which are homogeneous of degree d .

1.2.4. *Operations on Sets.* If $A, B \subseteq \mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$, then define $A + B$ to be the set of polynomials of the form $a + b$, with $a \in A$, $b \in B$. In the case that $A \cap B = \{0\}$, we also denote $A + B$ as $A \oplus B$ —the expression $A \oplus B$ always asserts that $A \cap B = \{0\}$. If $A \subseteq \mathbb{R}^{\nu \times \ell}\langle x \rangle$ and $B \subseteq \mathbb{R}^{\ell \times \rho}\langle x \rangle$, let $AB \subseteq \mathbb{R}^{\nu \times \rho}\langle x, x^* \rangle$ denote the span of all polynomials of the form ab , with $a \in A$, $b \in B$. If $A \subseteq \mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$, then $A^* = \{a^* \mid a \in A\} \subseteq \mathbb{R}^{\ell \times \nu}\langle x, x^* \rangle$. If $A \subseteq \mathbb{R}^{\nu \times \ell}$

and $B \subseteq \mathbb{R}\langle x, x^* \rangle$, then $A \otimes B$ is defined to be the span of all simple tensors $a \otimes b$, where $a \in A$ and $b \in B$.

If $p \in \mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$, then expressions of the form $p + A$, pB , Cp , $D \otimes p$, where A , B , C , and D , are sets, denote $\{p\} + A$, $\{p\}B$, $C\{p\}$, and $D \otimes \{p\}$ respectively.

1.2.5. *Positivity sets.* Given a symmetric matrix of NC polynomials p , define its **positivity domain** \mathcal{D}_p by

$$\mathcal{D}_p(n) := \{X \in (\mathbb{R}^{n \times n})^g : p(X) \succeq 0\} \subseteq (\mathbb{R}^{n \times n})^g \quad \mathcal{D}_p := \bigcup_n \mathcal{D}_p(n).$$

If $p(0) \succ 0$ we also introduce

$$\mathcal{D}_p(n)^\circ := \text{principal component of } \{X \in (\mathbb{R}^{n \times n})^g : p(X) \succ 0\} \quad \mathcal{D}_p^\circ := \bigcup_n \mathcal{D}_p(n)^\circ$$

and its **(detailed) boundary** $\widehat{\partial \mathcal{D}_p}^\circ$ defined by

$$\widehat{\partial \mathcal{D}_p}^\circ := \{(X, v) : X \in \overline{\mathcal{D}_p}^\circ, p(X)v = 0\}$$

1.3. **Linear Pencils.** A **linear pencil** is a symmetric polynomial $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$, for some $\nu \in \mathbb{N}$, with $\deg(L) \leq 1$. Every $\nu \times \nu$ linear pencil can be expressed as

$$L = A_0 + A_1 \otimes x_1 + \cdots + A_g \otimes x_g + A_1^* \otimes x_1^* + \cdots + A_g^* \otimes x_g^*,$$

where each $A_i \in \mathbb{R}^{\nu \times \nu}$ and A_0 is symmetric. A linear pencil is **monic** if $A_0 = L(0) = \text{Id}_\nu$. For the purposes of this paper, we still call L a linear pencil even if $A_0 \neq 0$.

A **linear matrix inequality** or **(LMI)** is an expression of the form $L(x) \succeq 0$, where L is a linear pencil and x is a tuple of real scalar variables. When x is a tuple of real scalar variables, the set $\mathcal{D}_L(1)$ is the **positivity set of L** or the **spectrahedron** defined by L . Optimization of linear objective functions over spectrahedra is called semidefinite programming (SDP) [BV96, Tod01, WSV00], and is an important subfield of convex optimization.

One problem which arises in SDP is dealing with spectrahedra with empty interior. Every convex set with empty interior is contained in an affine hyperplane; we call these **thin convex sets**. A spectrahedron which is not thin will be called **thick**. Hence a spectrahedron is thick if it is the closure of its interior. Correspondingly we refer to thin and thick linear pencils L as those for which $\mathcal{D}_L(1)$ is thin, or respectively thick. A paper of Klep and Schweighofer gives an iterative process for finding a set of linear polynomials in $\mathbb{R}[x]$ whose zero set defines the affine subspace in which a spectrahedron lies [KS13, §3].

A **matricial relaxation of an LMI** is an expression of the form $L(X) \succeq 0$, where X is a tuple of square matrix variables. Over $\mathbb{R}\langle x, x^* \rangle$, the matricial relaxation of an LMI is important because every convex bounded noncommutative basic open semialgebraic set \mathcal{D}_p° is the positivity set \mathcal{D}_L° of a some linear pencil L ; see [HM12]. Sets of the form \mathcal{D}_L° are called

free open spectrahedra, while \mathcal{D}_L are **free spectrahedra**. Further, one can use results on the matricial relaxation of an LMI to prove new results about the original, scalar LMI.

1.4. Behavior of Polynomials on Real Zero Sets. One of our main themes is taking into account behavior of zero sets. For the free algebra $\mathbb{R}\langle x, x^* \rangle$, there is a ‘‘Real Nullstellensatz’’. Let $p_1, \dots, p_k, q \in \mathbb{R}\langle x, x^* \rangle$. If $q(X)v = 0$ for every $(X, v) \in \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{n \times n})^g \times \mathbb{R}^n$ such that $p_1(X)v = \dots = p_k(X)v = 0$, then q is an element of the ‘‘real radical’’ of the left ideal generated by p_1, \dots, p_k , see [CHMN]. In [Nel] this result was generalized to $\mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$. For sake of completeness we mention the free analog of Hilbert’s (complex) Nullstellensatz is given in [HMP07].

Now we lay out noncommutative analogs of several notions of classical (commutative) real algebraic geometry.

1.4.1. Left $\mathbb{R}\langle x, x^* \rangle$ -Modules. For matrices of NC polynomials we need to adapt the notion of left ideal and real left ideal. The space $\mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$ is a left $\mathbb{R}\langle x, x^* \rangle$ -module with the following action: if $q \in \mathbb{R}\langle x, x^* \rangle$, $A \in \mathbb{R}^{\nu \times \ell}$ and $r \in \mathbb{R}\langle x, x^* \rangle$, then

$$q \cdot (A \otimes r) := (\text{Id}_\nu \otimes q)(A \otimes r) = A \otimes qr,$$

where $\text{Id}_\nu \in \mathbb{R}^{\nu \times \nu}$ denotes the $\nu \times \nu$ identity matrix. In the sequel, we will simplify notation by identifying q with $\text{Id}_\nu \otimes q$ and simply writing $q(A \otimes r)$ when we mean $q \cdot (A \otimes r)$. We will also simplify our terminology by referring to left $\mathbb{R}\langle x, x^* \rangle$ -submodules $I \subseteq \mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$ as **left modules**.

1.4.2. Zero Sets of Left $\mathbb{R}\langle x, x^* \rangle$ -Modules. For $S \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$, for each $n \in \mathbb{N}$, define $V(S)^{(n)}$ to be

$$V(S)^{(n)} := \{(X, v) \in (\mathbb{R}^{n \times n})^g \times \mathbb{R}^{\ell n} \mid p(X)v = 0 \text{ for every } p \in S\},$$

and let $V(S)$ be

$$V(S) := \bigcup_{n \in \mathbb{N}} V(S)^{(n)}.$$

If $W \subseteq \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{n \times n})^g \times \mathbb{R}^{\ell n}$, define $\mathcal{I}(W)$ to be

$$\mathcal{I}(W) := \{p \in \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle \mid p(X)v = 0 \text{ for every } (X, v) \in W\}.$$

The set $\mathcal{I}(W) \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ is clearly a left module. If $I \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ is a left module, define the (vanishing) **radical** of I to be

$$\sqrt{I} := \mathcal{I}(V(I)).$$

Finally, we define the **free Zariski closure**, $\mathcal{Z}(W)$, of W to be

$$\mathcal{Z}(W) := V(\mathcal{I}(W)).$$

Before launching into full generality we give an appealing corollary of our main results; our concern here is the nature of defining polynomials for \mathcal{D}_L ; namely, a polynomial p which is nonnegative on \mathcal{D}_L with $\mathcal{D}_p^\circ = \mathcal{D}_L^\circ$. The following theorem applies to monic pencils L with the rather natural **zero determining property**:

$$(1.1) \quad \mathcal{Z}(\widehat{\partial\mathcal{D}}_L^\circ) = V(L) = \{(X, v) : L(X)v = 0\}.$$

Theorem 1.1 (Randstellensatz). *Let $L \in \mathbb{R}^{\ell \times \ell}\langle x, x^* \rangle$ be a monic linear pencil with the zero determining property. Let $p \in \mathbb{R}^{\ell \times \ell}\langle x, x^* \rangle$. Then*

$$\mathcal{D}_L \subseteq \mathcal{D}_p \quad \text{and} \quad \widehat{\partial\mathcal{D}}_L^\circ \subseteq \widehat{\partial\mathcal{D}}_p^\circ$$

if and only if

$$p = L \left(\sum_i^{\text{finite}} q_i^* q_i \right) L + \sum_j^{\text{finite}} (r_j L + C_j)^* L (r_j L + C_j)$$

where each $q_i \in \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$, each $r_j \in \mathbb{R}^{\ell \times \ell}\langle x, x^* \rangle$, and each $C_j \in \mathbb{R}^{\ell \times \ell}$ satisfies $C_j L = L C_j$.

Proof. See §5. ■

This describes all p which are defining polynomials of \mathcal{D}_L with boundary containment happening in a strong sense. It is a slight superset of this class, since if $q_j L$ and $r_j L + C_j$ all vanish simultaneously on a big enough set, then p might define a smaller set than \mathcal{D}_L .

The zero determining property holds for an $\ell \times \ell$ pencil L provided that

- (a) $\deg(\det L) = \ell$; and
- (b) $\det L$ is the smallest degree polynomial vanishing on $\partial\mathcal{D}_L(1)$;

see Corollary 5.5(2). These properties are easy to check with computer algebra, and *they hold generically* (see Corollary 5.6).

We now move towards the presentation of our main theorem. Its generality forces a number of definitions.

1.4.3. *Real Left Modules.* In classical real algebraic geometry [BCR98, Las10, Lau09, Mar08, PD01, Put93, Sce09] at the core of the real Nullstellensatz are real ideals and the real radical of an ideal. These correspond to vanishing ideals of a variety. Now we shall study a variety intersect a positivity domain \mathcal{D}_L . The appropriate notion in free algebras is what we call L -real left modules and L -real radicals. We now introduce them.

Let $I \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a left module, and $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$. We say that I is L -**real** if whenever

$$\sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in \mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell}$$

for some $p_i \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ and $q_j \in \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$, then each $p_i \in I$ and each $Lq_j \in \mathbb{R}^{\nu \times 1} I$. Note that $\mathbb{R}^{\ell \times 1} I$ is the subspace of $\ell \times \ell$ matrices whose rows are elements of I , and $(\mathbb{R}^{\ell \times 1} I)^* = I^* \mathbb{R}^{1 \times \ell}$ is the subspace of $\ell \times \ell$ matrices whose columns are elements of I^* . We call I **real** if it is L -real for $L = 1$.

The following result shows that no generality is lost by defining a real left module in terms of only rows of matrices.

Proposition 1.2. *A left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is L -real if and only if whenever*

$$(1.2) \quad \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* Lq_j \in \mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell},$$

for some $p_i \in \mathbb{R}^{\nu_i \times \ell} \langle x, x^* \rangle$ and $q_j \in \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$, then each $p_i \in \mathbb{R}^{\nu_i \times 1} I$ and each $Lq_j \in \mathbb{R}^{\nu \times 1} I$.

Proof. One direction is clear. For the converse, suppose I is L -real, and suppose that (1.2) holds. For polynomials $p_i \in \mathbb{R}^{\nu_i \times \ell} \langle x, x^* \rangle$ we have

$$p_i^* p_i = p_i^* \text{Id}_{\nu_i} p_i = \sum_{j=1}^{\nu_i} p_i^* E_{jj} p_i = \sum_{j=1}^{\nu_i} (e_j^* p_i)^* (e_j^* p_i),$$

so that

$$\sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* Lq_j = \sum_i^{\text{finite}} \sum_{j=1}^{\nu_i} (e_j^* p_i)^* (e_j^* p_i) + \sum_j^{\text{finite}} q_j^* Lq_j \in \mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell}.$$

Since I is L -real, each $e_j^* p_i \in I$ and each $Lq_j \in \mathbb{R}^{\nu \times 1} I$. Therefore, for each i ,

$$p_i = \text{Id}_{\nu_i} p_i = \sum_{j=1}^{\nu_i} e_j e_j^* p_i \in \mathbb{R}^{\nu_i \times 1} I. \quad \blacksquare$$

Corollary 1.3. *A left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is real if and only if whenever*

$$(1.3) \quad \sum_i^{\text{finite}} p_i^* p_i \in \mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell},$$

for some $p_i \in \mathbb{R}^{\nu_i \times \ell} \langle x, x^* \rangle$, then each $p_i \in \mathbb{R}^{\nu_i \times 1} I$.

Here is a connection between vanishing sets and real left modules:

Proposition 1.4. *Let $V \subseteq \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{n \times n})^g \times \mathbb{R}^{\ell n}$. The space*

$$\mathcal{J}_L(V) := \{p \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle \mid p(X)v = 0 \text{ for all } (X, v) \in \mathcal{I}(V) \text{ satisfying } L(X) \succeq 0\}$$

is an L -real left module.

Proof. Suppose

$$\sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in \mathbb{R}^{\ell \times 1} \mathcal{J}_L(V) + \mathcal{J}_L(V)^* \mathbb{R}^{1 \times \ell},$$

where each $p_i \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ and each $q_j \in \mathbb{R}^{\nu \times \ell}$. For each $(X, v) \in V$ with $L(X) \succeq 0$, we have

$$\sum_i^{\text{finite}} v^* p_i(X)^* p_i(X) v + \sum_j^{\text{finite}} v^* q_j(X)^* L(X) q_j(X) v = 0.$$

Therefore $p_i(X)v = 0$ and $L(X)q_j(X)v = 0$, which implies that each $p_i \in \mathcal{J}_L(V)$, and each $Lq_j \in \mathbb{R}^{\nu \times 1} \mathcal{J}_L(V)$. \blacksquare

Corollary 1.5. *Let $V \subseteq \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{n \times n})^g \times \mathbb{R}^{\ell n}$. The space $\mathcal{I}(V) \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is a real left module.*

1.4.4. *The Real Radical.* We now introduce a generalization of the real radical of a left module for use in studying the positivity set of a linear pencil L . Just like the vanishing radical of I consists of polynomials vanishing on the variety $V(I)$, the L -real radical of I consists of polynomials vanishing on the intersection of $V(I)$ with the positivity set \mathcal{D}_L of L , see Proposition 1.6 below.

An intersection of L -real left modules is itself an L -real left module. Define the **L -real radical** of a left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ to be

$${}^{(L)}\sqrt{I} = \bigcap_{\substack{J \supseteq I \\ J \text{ } L\text{-real}}} J = \text{the smallest } L\text{-real left module containing } I.$$

The L -real radical of I with $L = 1$ is called the **real radical** of I and denoted by ${}^{\text{re}}\sqrt{I}$. As we explain later, the article [Nel] in §9.1 presents an algorithm for computing ${}^{\text{re}}\sqrt{I}$ for a finitely-generated left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$.

Proposition 1.4 implies that for each left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$,

$$I \subseteq {}^{\text{re}}\sqrt{I} \subseteq \sqrt{I}.$$

Much more difficult to prove is ${}^{\text{re}}\sqrt{I} = \sqrt{I}$ for finitely generated I , and this is [Nel, Theorem 1.3]. We also describe this here in §4.5 in the context of our more general theory.

That L -real radicals are closely related to vanishing-positivity is shown in the next proposition, where we show that ${}^{(L)}\sqrt{I}$ is the vanishing ideal (i.e., a “free Zariski closure”) of $V(I) \cap \mathcal{D}_L$. More precisely, ${}^{(L)}\sqrt{I} = \mathcal{I}(\mathcal{Z}(V(I) \cap \mathcal{D}_L))$.

Proposition 1.6. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finitely-generated left module, and let $p \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. Then $p(X)v = 0$ whenever $(X, v) \in V(I)$ and $L(X) \succeq 0$ if and only if $p \in {}^{(L)}\sqrt{I}$.*

The proof requires some of the heaviest results of this paper and is presented in §4.7.

1.5. Right Chip Spaces. We now introduce a natural class of polynomials needed for the proofs, chip spaces. Also we state our main theorems in terms of chip spaces since keeping track of the chip space where each polynomial lies adds significant generality, and leads to optimal degree and size bounds; cf. [KP10].

Consider $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. A monomial $e_i \otimes a$ **divides** another monomial $e_j \otimes b$ **on the right** if $i = j$ and $b = wa$ for some $w \in \langle x, x^* \rangle$ so that $w(e_i \otimes a) = e_j \otimes b$. If additionally $e_i \otimes a \neq e_j \otimes b$, then $e_i \otimes a$ **properly divides** $e_j \otimes b$ **on the right**. We call $e_i \otimes a$ a **(proper) right chip** of $e_j \otimes b$ if $e_i \otimes a$ (properly) divides it on the right.

A vector subspace $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is a **right chip space** if \mathfrak{C} is spanned by a set of monomials such that whenever $e_i \otimes w_1 w_2 w_3$, $e_i \otimes w_3 \in \mathfrak{C}$ for some $w_1, w_2, w_3 \in \langle x, x^* \rangle$, then $e_i \otimes w_2 w_3 \in \mathfrak{C}$. A right chip space \mathfrak{C} is **finite** if \mathfrak{C} is finite dimensional. A right chip space \mathfrak{C} is **full** if for each $e_i \otimes w \in \mathfrak{C}$, all right chips of $e_i \otimes w$ are in \mathfrak{C} as well.

Example 1.7. The space $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle_d$, the space of all $1 \times \ell$ matrix NC polynomials of degree bounded by d , is a full, finite right chip space.

For a finitely-generated left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ we can find full, finite right chip spaces $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ such that the generators of I are in the space $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$.

Example 1.8. Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be generated by some polynomials in the span of the monomials m_1, \dots, m_k . The space

$$\mathfrak{C} := \text{span}\{m \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle \mid m \text{ a proper right chip of some } m_i\}$$

is a full, finite right chip space such that $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ contains all the generators of I .

At first reading the main results of this paper, soon to be stated, the reader should just think of \mathfrak{C} as being $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle_d$, cf. Example 1.7.

An appeal of right chip spaces is they are easily computable.

1.6. (L, \mathfrak{C}) -Real Radical Modules. Chip spaces lead to an extension of the notion of an L -real radical of a left module.

Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module, let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$, and let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a right chip space. We say that I is **(L, \mathfrak{C}) -real** if whenever

$$(1.4) \quad \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in \mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell}$$

for some $p_i \in \mathfrak{C}$ and $q_j \in \mathbb{R}^{\nu \times 1} \mathfrak{C}$, then each $p_i \in I$ and each $Lq_j \in \mathbb{R}^{\nu \times 1} I$. We say I is **strongly** (L, \mathfrak{C}) -**real** if whenever (1.4) holds, then each $p_i \in I$ and each $q_j \in \mathbb{R}^{\nu \times 1} I$.

Define the (L, \mathfrak{C}) -**real radical** of I to be

$${}^{(L, \mathfrak{C})} \sqrt{I} = \bigcap_{\substack{J \supseteq I \\ J \text{ (L, } \mathfrak{C})\text{-real}}} J = \text{smallest } (L, \mathfrak{C})\text{-real left module containing } I.$$

Define the **strong** (L, \mathfrak{C}) -**real radical** of I to be

$${}^{(L, \mathfrak{C})+} \sqrt{I} = \bigcap_{\substack{J \supseteq I \\ J \text{ strongly} \\ (L, \mathfrak{C})\text{-real}}} J = \text{smallest strongly } (L, \mathfrak{C})\text{-real left module containing } I.$$

If $\mathfrak{C} = \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, we omit it and talk about L -real modules, and the (strongly) L -real radical of I . These definitions extend the notion of a real module as given in §1.4, e.g. a left module I is real if it is 1-real. As we will see in §7 all these real radicals are algorithmically computable.

1.7. Overview. The main results and some consequences. The main general result of this paper is the following theorem, proved in §4. At first reading the reader is advised to think of \mathfrak{C} as all of $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$.

Theorem 1.9. *Suppose $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ is a linear pencil. Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite chip space, let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left $\mathbb{R} \langle x, x^* \rangle$ -module generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, and let $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ be a symmetric polynomial.*

(1) $v^* p(X) v \geq 0$ whenever $(X, v) \in V(I)$ and $L(X) \succeq 0$ if and only if p is of the form

$$(1.5) \quad p = \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j + \sum_k^{\text{finite}} (r_k^* \iota_k + \iota_k^* r_k)$$

where each $p_i, r_k \in \mathfrak{C}$, each $q_j \in \mathbb{R}^{\ell \times 1} \mathfrak{C}$ and each $\iota_k \in {}^{(L, \mathfrak{C})} \sqrt{I} \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$.

(2) $v^* p(X) v \geq 0$ whenever $(X, v) \in V(I)$ and $L(X) \succ 0$, if and only if p is of the form (1.5) where each $p_i, r_k \in \mathfrak{C}$, each $q_j \in \mathbb{R}^{\nu \times 1} \mathfrak{C}$ and each $\iota_k \in {}^{(L, \mathfrak{C})+} \sqrt{I} \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$

Remark 1.10. Our machinery of chip spaces allows us to give additional information on the size of the testing matrices X in (1) and (2). Indeed, for certain cases we shall obtain provably optimal size and degree bounds; see §4.4 for details.

Theorem 1.9 is a general theorem from which we deduce several interesting corollaries.

1.7.1. *A Real Nullstellensatz for $\mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$.* One corollary is [Nel, Theorem 1.3], which is that paper's main theorem and is a generalization of the Real Nullstellensatz from [CHMN]. The heavy machinery developed in [Nel] which is used to prove this is also essential to many proofs in this paper.

Corollary 1.11 ([Nel, Theorem 1.3]). *Let p_1, \dots, p_k be such that each $p_i \in \mathbb{R}^{\nu_i \times \ell} \langle x, x^* \rangle$ for some $\nu_i \in \mathbb{N}$. Suppose $q \in \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$, with $\nu \in \mathbb{N}$, has the property that whenever $p_1(X)v, \dots, p_k(X)v = 0$, where $(X, v) \in \bigcup_{n \in \mathbb{N}} (\mathbb{R}^{n \times n})^g \times \mathbb{R}^{\ell n}$, then $q(X)v = 0$. Then q is an element of the space I defined by*

$$I := \mathbb{R}^{\nu \times 1} \operatorname{re} \sqrt{\sum_{i=1}^k \mathbb{R}^{1 \times \nu_i} \langle x, x^* \rangle p_i}.$$

Consequently, if the left module

$$(1.6) \quad \sum_{i=1}^k \mathbb{R}^{1 \times \nu_i} \langle x, x^* \rangle p_i$$

is real, and if $q(X)v = 0$ whenever $p_1(X)v, \dots, p_k(X)v = 0$, then q is of the form

$$q = r_1 p_1 + \dots + r_k p_k$$

for some $r_i \in \mathbb{R}^{\nu \times \nu_i} \langle x, x^* \rangle$.

Proof. See §4.5. ■

1.7.2. *Convex Positivstellensatz.* Applying Theorem 1.9 in the case $I = \{0\}$ gives an extension of the Convex Positivstellensatz of Helton, Klep, and McCullough [HKM12] to the case where the positivity set \mathcal{D}_L of a linear pencil L may have empty interior. This is given in Corollary 4.3. Note also that Corollary 4.3 gives a substantial refinement of the degree bounds obtained in [HKM12] by using right chip spaces.

1.7.3. *Thick Pencils.* Some basic properties of L -real radicals follow from Theorem 1.9.

Proposition 1.12. *Let $L \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ be a monic linear pencil and let $I_L = \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle L$. Then $I_L = \sqrt[\nu]{I_L} = \sqrt{I_L}$.*

This will be proved in §5.

We note L having the zero determining property (1.1) is equivalent to the statement $\sqrt[\ell]{I_L} = I_L$. This is a consequence of Propositions 1.12 and 1.6, and by Corollary 5.5 this holds for a generic and computationally checkable L . Of course for any L we have $\mathcal{Z}(\widehat{\partial \mathcal{D}}_L^\circ)$ is contained in $V(L)$, that is, $\sqrt[\ell]{I_L} \supseteq I_L$.

1.7.4. *Thin Pencils.* In §6 we will use the main theorem of this paper, Theorem 1.9, to prove results about LMIs with empty interior, i.e., thin spectrahedra. More precisely, if L is a linear pencil which defines a thin spectrahedron we will apply the main theorem to L with $I = \{0\}$ to prove results about thin spectrahedra. In §7.2.1 we give an efficient algorithm for computing the affine hull (i.e., the smallest affine subspace containing it) of the thin spectrahedron $\mathcal{D}_L(1)$.

There is an appealing connection between the space ${}^{(L)}\sqrt{\{0\}}$ and its corresponding ideal in $\mathbb{R}[x]$, the space of polynomials in commuting variables. We say that the **commutative collapse** of a polynomial $p \in \mathbb{R}\langle x, x^* \rangle$ to $\mathbb{R}[x]$ is the polynomial produced by letting the variables in p commute and setting $x = x^*$. The **commutative collapse** of a subset $S \subseteq \mathbb{R}\langle x, x^* \rangle$ to $\mathbb{R}[x]$ is the set of projections of all the elements of S to $\mathbb{R}[x]$. There is a natural decomposition of a thin linear pencil in terms of a thick one restricted to a special subspace as we now describe.

Theorem 1.13. *Let $L \in \mathbb{R}^{\nu \times \nu}[x]$ be a linear pencil, where $x = (x_1, \dots, x_g)$ is a tuple of commuting variables. Let $I \subseteq \mathbb{R}[x]$ be the commutative collapse of ${}^{(L)}\sqrt{\{0\}}$ onto $\mathbb{R}[x]$.*

- (1) $I \subseteq \mathbb{R}[x]$ is an ideal generated by linear polynomials.
- (2) There exists a linear pencil $\tilde{L} \in \mathbb{R}^{\nu' \times \nu'}[x]$, where $\nu' \leq \nu$, whose positivity set has nonempty interior such that

$$\{x \in \mathbb{R}^g \mid L(x) \succeq 0\} = \{x \in \mathbb{R}^g \mid \tilde{L}(x) \succeq 0 \text{ and } \iota(x) = 0 \text{ for each } \iota \in I\}.$$

The proof of Theorem 1.13 is based on taking $I = \{0\}$, $p = 1$ and will be given in §6.3.

Geometrically, Theorem 1.13 implies that given a linear pencil L which defines a spectrahedron with empty interior, either the spectrahedron $\mathcal{D}_L(1)$ is empty—that is, $L(x) \succeq 0$ is infeasible—or it can be viewed as a spectrahedron with non-empty interior lying inside a proper affine subspace of \mathbb{R}^g . In §7.2 we give an algorithm for computing the ideal $I \subseteq \mathbb{R}[x]$ and the linear pencil $\tilde{L} \in \mathbb{R}^{\nu' \times \nu'}[x]$ described in Theorem 1.13. In particular, we will see that the algorithm discussed in Theorem 1.14 is a generalization of the process of finding the affine subspace on which a spectrahedron lies, as given in [KS13].

1.7.5. *Algorithms.* Applying Theorem 1.9 requires one to compute the (L, \mathfrak{C}) -real radical of a left module I . In §7 we will present an algorithm for doing so. In addition, in §7 we also give more refined algorithms for the special cases of computing $\sqrt[\mathfrak{e}]{I}$ and ${}^{(L)}\sqrt{\{0\}}$. This generalizes the algorithm for the special case $L = 1$ found in [Nel]. Here is a theorem listing the algorithms' desirable properties. We emphasize this algorithm works even for polynomials $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$ which are not linear.

Theorem 1.14. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle_{\sigma}$ be a symmetric polynomial, let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite right chip space, and let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module. The L -Real Radical algorithm for ${}^{(L, \mathfrak{C})}\sqrt{I}$ in §7.5 has the following properties.*

- (1) *The algorithm terminates in a finite number of steps.*
- (2) *If I is generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$, then the algorithm involves computations on polynomials in $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$.*
- (3) *The algorithm outputs a left Gröbner basis for ${}^{(L, \mathfrak{C})}\sqrt{I}$.*

1.7.6. *Completely Positive Maps.* In §8 we apply our results on thin spectrahedra to give algebraic certificates for completely positive maps between (nonunital) subspaces of matrix algebras. We shall see that complete positivity of a map is equivalent to LMI domination between a pair of associated linear pencils.

1.8. **Context and Reader's Guide.** To give a broad perspective on the topic of this paper we point out that it fits in the area of Free Real Algebraic Geometry. This in turn lies within the booming area called Free Analysis the earliest and most developed branch of which is Free Probability, see [VDN92] for a survey. Also developing rapidly is Free Analytic Function Theory, see [Voi04, Voi10, KVV, MS11, Poe10, AM, BB07]. We refer the reader to [NCAalgebra](#) [HOSM] and [NCS0Stools](#) [CKP11] for computer algebra packages adapted to deal with free noncommuting variables.

While Free Null-Positivstellensätze as we develop in this paper date back less than a decade, already Free Positivstellensätze have found physical applications. For instance, applications to quantum physics are explained by Pironio, Navascués, Acín [PNA10] who also consider computational aspects related to noncommutative sum of squares. Doherty, Liang, Toner, Wehner [DLTW08] employ free positivity and the Positivstellensatz [HM04] to consider the quantum moment problem and multi-prover games.

Turning from the general to the very specific we describe the organization of the rest of this paper. §2 proves some basic results about L -real radicals and (L, \mathfrak{C}) -left modules. §3 describes how to construct positive linear functionals on spaces of square matrices of NC polynomials for use in the proof of the main theorem. §4 proves the main result, Theorem 1.9, and many of the corollaries of this paper. §5 proves Theorem 1.1 and Proposition 1.12, which pertain to thick spectrahedra. §6 characterizes the L -real radical of $\{0\}$, which pertains to thin spectrahedra. §7 describes algorithms for computing different real radicals appearing in our main results; many of these algorithms are improvements on previously known algorithms. §8 gives nonlinear algebraic certificates for complete positivity of maps between (nonunital) subspaces of matrix algebras. Finally, §9 gives direct analogs of the results of this paper in the case where all the variables x_j are symmetric.

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2. PROPERTIES OF (L, \mathfrak{C}) -REAL LEFT MODULES

In this section we prove some useful properties of (L, \mathfrak{C}) -real left modules $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. Here $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ is a matrix polynomial, and $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is a right chip space.

2.1. **L -Real Left Modules.** One class of L -real left modules which arise naturally are left modules $\mathcal{I}(\{(X, v)\})$, where X is a tuple of matrices with $L(X) \succeq 0$; cf. Proposition 1.4 in §1.4.3.

Proposition 2.1. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a symmetric polynomial, and let $X \in (\mathbb{R}^{n \times n})^g$ be such that $L(X) \succeq 0$. For each vector $v \in \mathbb{R}^{\ell n}$, the left module $\mathcal{I}(\{(X, v)\})$ is L -real. If also $L(X) \succ 0$, then $\mathcal{I}(\{(X, v)\})$ is strongly L -real for each v .*

Proof. Suppose

$$\sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in \mathbb{R}^{\ell \times 1} \mathcal{I}(\{(X, v)\}) + [\mathcal{I}(\{(X, v)\})]^* \mathbb{R}^{1 \times \ell}$$

for some polynomials $p_i \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ and $q_j \in \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$. For each $\iota \in \mathcal{I}(\{(X, v)\})$, we have

$$v^* \iota(X) v = 0 \quad \text{and} \quad v^* \iota(X)^* v = 0.$$

Therefore

$$v^* \left(\sum_i^{\text{finite}} p_i(X)^* p_i(X) + \sum_j^{\text{finite}} q_j(X)^* L(X) q_j(X) \right) v = 0.$$

For each i , and, since $L(X) \succeq 0$, for each j we have

$$v^* p_i(X)^* p_i(X) v \geq 0 \quad \text{and} \quad v^* q_j(X)^* L(X) q_j(X) v \geq 0.$$

Therefore, for each i ,

$$v^* p_i(X)^* p_i(X) v = \|p_i(X) v\|^2 = 0,$$

and for each j ,

$$v^* q_j(X)^* L(X) q_j(X) v = \|\sqrt{L(X)} q_j(X) v\|^2 = 0.$$

Hence each $p_i(X) v = 0$, or equivalently, $p_i \in \mathcal{I}(\{(X, v)\})$. Further, each $\sqrt{L(X)} q_j(X) v = 0$, so $L(X) q_j(X) v = 0$, which implies $L q_j \in \mathbb{R}^{\nu \times 1} \mathcal{I}(\{(X, v)\})$. If in addition $L(X) \succ 0$, then $L(X)$ is invertible, so $L(X) q_j(X) v = 0$ if and only if $q_j(X) v = 0$, which implies $q_j \in \mathbb{R}^{\nu \times 1} \mathcal{I}(\{(X, v)\})$. ■

2.2. Homogeneous Left Modules. We next consider $\sqrt[\ell]{I}$ for a homogeneous left module I . A left module $I \subseteq \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$ is **homogeneous** if it is generated by homogeneous polynomials.

Proposition 2.2. *Let $I \subseteq \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$ be a left module. The following are equivalent:*

- (i) I is homogeneous;
- (ii) $p \in I$ if and only if p is a sum of homogeneous polynomials in I ;
- (iii) $p \in \mathbb{R}^{\ell \times \nu} I + I^* \mathbb{R}^{\nu \times \ell}$ if and only if p is a sum of homogeneous elements of $\mathbb{R}^{\ell \times \nu} I + I^* \mathbb{R}^{\nu \times \ell}$;
- (iv) $p \in \mathbb{R}^{\ell \times \nu} I + I^* \mathbb{R}^{\nu \times \ell}$ if and only if p is a sum of homogeneous polynomials which are each in $\mathbb{R}^{\ell \times \nu} I$ or $I^* \mathbb{R}^{\nu \times \ell}$.

Proof. Straightforward. ■

Recall a linear pencil L is monic if its constant term is the identity matrix.

Proposition 2.3. *Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a homogeneous left module, and let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a monic linear pencil. The following are equivalent:*

- (i) I is real;
- (ii) I is L -real;
- (iii) I is strongly L -real.

Proof. By definition, (i) \Leftrightarrow (ii) \Leftrightarrow (iii). Therefore suppose that I is real. Let

$$(2.1) \quad \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in \mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell},$$

where each $p_i \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ and each $q_j \in \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$. Let δ be the minimum degree such that at least one of the p_i or q_j have terms of degree δ . Let \tilde{p}_i and \tilde{q}_j be the terms of p_i and q_j respectively with degree δ . The terms of (2.1) with degree 2δ are

$$(2.2) \quad \sum_i^{\text{finite}} \tilde{p}_i^* \tilde{p}_i + \sum_j^{\text{finite}} \tilde{q}_j^* \tilde{q}_j.$$

Since I is homogeneous, (2.2) must be in $\mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell}$. Since I is real, each $\tilde{p}_i \in I$ and $\tilde{q}_j \in \mathbb{R}^{\nu \times 1} I$. Therefore,

$$\sum_i^{\text{finite}} (p_i - \tilde{p}_i)^* (p_i - \tilde{p}_i) + \sum_j^{\text{finite}} (q_j - \tilde{q}_j)^* L (q_j - \tilde{q}_j) \in \mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell}.$$

We repeat this process to show that each homogeneous part of p_i is in I and each homogeneous part of q_j is in $\mathbb{R}^{\nu \times 1} I$. Hence I is strongly L -real. ■

A special example of a homogeneous left module is $\{0\}$. Proposition 2.3 implies that ${}^{(L)}\sqrt{\{0\}} = \{0\}$ if L is monic. In the non-monic case—and in particular, if $\mathcal{D}_L = \{X \mid L(X) \succeq 0\}$ has empty interior—there is more to say about ${}^{(L)}\sqrt{\{0\}}$, as we will see in §6.

2.3. (L, \mathfrak{C}) -Real Left Modules for Finite Right Chip Spaces. For a finite right chip space \mathfrak{C} and a left module I , the (L, \mathfrak{C}) -real radical ${}^{(L, \mathfrak{C})}\sqrt{I}$ is generated as a left module by polynomials in a restricted vector subspace, as shown in the following proposition.

Proposition 2.4. *Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite right chip space, let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module, and let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a symmetric polynomial of degree σ . The left module ${}^{(L, \mathfrak{C})}\sqrt{I}$ is generated by I together with some subset of polynomials in $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$, and ${}^{(L, \mathfrak{C})+}\sqrt{I}$ is generated by I together with some subset of polynomials in \mathfrak{C} .*

Proof. We will construct an increasing chain of left modules $I^{(a)}$ such that

$$I \subsetneq I^{(1)} \subsetneq \dots \subsetneq I^{(k)} = {}^{(L, \mathfrak{C})}\sqrt{I}.$$

Suppose inductively that $I^{(a)} \subseteq {}^{(L, \mathfrak{C})}\sqrt{I}$ is generated by I and by some polynomials in $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$. Consider a polynomial

$$(2.3) \quad \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in \mathbb{R}^{\ell \times 1} I^{(a)} + (I^{(a)})^* \mathbb{R}^{1 \times \ell}$$

for some $p_i \in \mathfrak{C}$ and $q_j \in \mathbb{R}^{\nu \times 1} \mathfrak{C}$. Since $I^{(a)} \subseteq {}^{(L, \mathfrak{C})}\sqrt{I}$, we have that each $p_i \in {}^{(L, \mathfrak{C})}\sqrt{I}$ and $L q_j \in \mathbb{R}^{\nu \times 1} {}^{(L, \mathfrak{C})}\sqrt{I}$, which implies that $e_k L q_j \in {}^{(L, \mathfrak{C})}\sqrt{I}$ for each standard unit vector $e_k \in \mathbb{R}^{1 \times \nu}$. If not all of the $p_i \in I^{(a)}$ and $L q_j \in \mathbb{R}^{\nu \times 1} I^{(a)}$, then let $I^{(a+1)}$ be generated by $I^{(a)}$ and by the p_i and $e_k L q_j$. In this case, $I^{(a)} \subsetneq I^{(a+1)} \subseteq {}^{(L, \mathfrak{C})}\sqrt{I}$. Furthermore, $I^{(a+1)}$ is generated by I and some polynomials in $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$.

This process must terminate since $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$ is finite dimensional. Therefore we arrive at a point where (2.3) holds if and only if $p_i \in I^{(a)}$ and $L q_j \in \mathbb{R}^{\nu \times 1} I^{(a)}$. At this point, $I^{(a)}$ is (L, \mathfrak{C}) -real and $I \subseteq I^{(a)} \subseteq {}^{(L, \mathfrak{C})}\sqrt{I}$. Hence $I^{(a)} = {}^{(L, \mathfrak{C})}\sqrt{I}$.

The ${}^{(L, \mathfrak{C})+}\sqrt{I}$ case is similar and its proof is omitted. ■

We will give algorithms for computing ${}^{(L, \mathfrak{C})}\sqrt{I}$ in §7.

3. POSITIVE LINEAR FUNCTIONALS ON $\mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$

This section contains fundamental properties of positive linear functionals on $\mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$.

A \mathbb{R} -linear functional λ on $W \subseteq \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ is **symmetric** if $\lambda(\omega^*) = \lambda(\omega)$ for each pair $\omega, \omega^* \in W$. A linear functional λ on a subspace $W \subseteq \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ is **positive** if it is symmetric and if $\lambda(\omega^* \omega) \geq 0$ for each $\omega^* \omega \in W$.

3.1. The GNS Construction. Proposition 3.1 below describes a variant of the well-known Gelfand-Naimark-Segal (GNS) construction.

Proposition 3.1. *Let λ be a positive linear functional on $\mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$, and let*

$$I = \{\vartheta \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle \mid L(\vartheta^* \vartheta) = 0\}.$$

There exists an inner product on the quotient space $\mathcal{H} := \mathbb{R}^{1 \times \ell} \langle x, x^ \rangle / I$, a tuple of operators X on \mathcal{H} , and a vector $v \in \mathcal{H}^n$ such that for each $p \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ we have*

$$\langle p(X)v, v \rangle = \lambda(p),$$

and $\mathcal{H} = \{q(X)v \mid q \in \mathbb{R}^{1 \times \ell} \langle x, x^ \rangle\}$.*

Proof. The proof follows the classical argument; alternately, see [Nel, Proposition 5.3] for a detailed proof. ■

We shall apply Proposition 3.1 in the next subsection to “flat” linear functionals, in which case the obtained quotient space \mathcal{H} is finite-dimensional, and X is thus simply a tuple of matrices. We refer to [Pop10, HKM12] for more on flat linear functionals in a free algebra.

3.2. Flat Extensions of Positive Linear Functionals. We next turn to flat extensions of positive linear functionals on $\mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$. The reader is referred to [CF96, CF98] for the classical theory of flatness on $\mathbb{R}[x]$. The content of this subsection comes from [Nel] and is summarized now for future use.

Let $W \subseteq \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ be a vector subspace and let λ be a positive linear functional on W . Suppose

$$\{\omega \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle \mid \omega^* \omega \in W\} = J \oplus T$$

where $J, T \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ are vector subspaces with

$$J := \{\vartheta \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle \mid \vartheta^* \vartheta \in W \text{ and } \lambda(\vartheta^* \vartheta) = 0\}.$$

An extension $\bar{\lambda}$ of λ to a space $U \supseteq W$ is a **flat extension** if $\bar{\lambda}$ is positive and if

$$\{u \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle \mid u^* u \in U\} = I \oplus T$$

where

$$I = \{\iota \in \mathbb{R}^{1 \times \ell} \mid \iota^* \iota \in U \text{ and } \bar{\lambda}(\iota^* \iota) = 0\}.$$

Proposition 3.2. *Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite right chip space, and let λ be a positive linear functional on $\mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$.*

- (1) *There exists a positive extension of λ to the space $\mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_2 \mathfrak{C}$ if and only if whenever $\vartheta \in \mathfrak{C}$ satisfies $\lambda(\vartheta^* \vartheta) = 0$, then $\lambda(b^* c \vartheta) = 0$ for each polynomial $b \in \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ and each $c \in \mathbb{R} \langle x, x^* \rangle$ satisfying $c \vartheta \in \mathfrak{C}$.*

(2) If there exists a positive extension of λ to the space $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_2\mathfrak{C}$, then there exists a unique flat extension $\bar{\lambda}$ of λ to $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle\mathfrak{C}$. In this case, the space

$$\{\theta \in \mathbb{R}\langle x, x^* \rangle\mathfrak{C} \mid \bar{\lambda}(\theta^*\theta) = 0\}$$

is generated as a left module by the set

$$(3.1) \quad \{\iota \in \mathbb{R}\langle x, x^* \rangle_1\mathfrak{C} \mid \lambda(b^*\iota) = 0 \text{ for every } b \in \mathfrak{C}\}.$$

(3) Given the existence of a flat extension $\bar{\lambda}$ of λ to $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle\mathfrak{C}$, there exists a flat extension of $\bar{\lambda}$ to all of $\mathbb{R}^{\ell \times \ell}\langle x, x^* \rangle$.

Proof. See [Nel, Proposition 5.2]. ■

Corollary 3.3. Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a full, finite right chip space. Let λ be a positive linear functional on $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$, and let J be the set

$$J := \{\vartheta \in \mathfrak{C} \mid \lambda(\vartheta^*\vartheta) = 0\}.$$

Suppose that if $\vartheta \in J$, then $\lambda(b^*c\vartheta) = 0$ for each polynomial $b \in \mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ and each $c \in \mathbb{R}\langle x, x^* \rangle$ such that $c\vartheta \in \mathfrak{C}$. Let $n := \dim(\mathfrak{C}) - \dim(J \cap \mathfrak{C})$, and suppose $n > 0$. Then there exists a g -tuple X of $n \times n$ real matrices, and a vector $v \in \mathbb{R}^{\ell n}$ such that for each $p \in \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ we have

$$v^*p(X)v = \lambda(p),$$

and $\mathbb{R}^{\ell n} = \{p(X)v \mid p \in \mathfrak{C}\}$.

Proof. By Proposition 3.2, there exists a flat extension $\bar{\lambda}$ of λ to all of $\mathbb{R}^{\ell \times \ell}\langle x, x^* \rangle$. Given this flat extension, apply Proposition 3.1 to produce the desired X and v . ■

3.3. Truncated Test Modules. Let $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$ for some $\nu \in \mathbb{N}$. Let $T \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ and $U \subseteq \mathbb{R}^{\nu \times \ell}\langle x, x^* \rangle$ be vector spaces. Define $M_{T,U}(L)$ as

$$(3.2) \quad M_{T,U}(L) := \left\{ \sum_i^{\text{finite}} t_i^* t_i + \sum_j^{\text{finite}} u_j^* L u_j \mid t_i \in T, u_j \in U \right\}.$$

We call $M_{T,U}(L)$ a **truncated (quadratic) module**.

Let $I \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a left module, $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$ a symmetric polynomial, and $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ a right chip space. Decompose \mathfrak{C} as

$$\mathfrak{C} = (I \cap \mathfrak{C}) \oplus T,$$

for some space $T \subseteq \mathfrak{C}$. Decompose $\mathbb{R}^{\nu \times 1}T$ as

$$\mathbb{R}^{\nu \times 1}T = J \oplus K,$$

where J is the subspace of $\mathbb{R}^{\nu \times 1}T$ defined by

$$J = \{\vartheta \in \mathbb{R}^{\nu \times 1}T \mid L\vartheta \in \mathbb{R}^{\nu \times 1}I\},$$

and $K \subseteq \mathfrak{C}$ is some complementary subspace. Since $J \cap K = \{0\}$, we have that $L\kappa \notin \mathbb{R}^{\nu \times 1}I$ for each $\kappa \in K \setminus \{0\}$. The following is a **truncated test module** for I , L and \mathfrak{C} :

$$(3.3) \quad M := M_{T,K}(L) = \left\{ \sum_i^{\text{finite}} \tau_i^* \tau_i + \sum_j^{\text{finite}} \kappa_j^* L \kappa_j \mid \tau_i \in T, \kappa_j \in K \right\}.$$

Lemma 3.4. *Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module, let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a symmetric polynomial, and let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite right chip space. Let $M = M_{T,K}(L)$ be a truncated test module for I , L and \mathfrak{C} , as in (3.3). If I is (L, \mathfrak{C}) -real, then*

$$(\mathbb{R}^{\ell \times 1}I + I^* \mathbb{R}^{1 \times \ell}) \cap M = \{0\}.$$

Proof. Suppose

$$\sum_i^{\text{finite}} \tau_i^* \tau_i + \sum_j^{\text{finite}} \kappa_j^* L \kappa_j \in \mathbb{R}^{\ell \times 1}I + I^* \mathbb{R}^{1 \times \ell},$$

where each $\tau_i \in T$ and each $\kappa_j \in K$. Since I is (L, \mathfrak{C}) -real, it must be that each $\tau_i \in I$ and each $L\kappa_j \in \mathbb{R}^{\nu \times 1}I$, which implies that each $\tau_i = 0$ and each $\kappa_j = 0$. \blacksquare

3.4. Building Positive Linear Functionals via Matrices. Recall that \mathbb{S}^k is the set of $k \times k$ symmetric matrices over \mathbb{R} . Define $\langle A, B \rangle := \text{Tr}(AB)$ to be the inner product on \mathbb{S}^k .

Lemma 3.5. *Let $\mathcal{B} \subseteq \mathbb{S}^k$ be a vector subspace. Then exactly one of the following holds:*

- (1) *There exists $B \in \mathcal{B}$ such that $B \succ 0$, and there exists no nonzero $A \in \mathcal{B}^\perp$ with $A \succeq 0$.*
- (2) *There exists $A \in \mathcal{B}^\perp$ such that $A \succ 0$, and there exists no nonzero $B \in \mathcal{B}$ with $B \succeq 0$.*
- (3) *There exist nonzero $B \in \mathcal{B}$ and $A \in \mathcal{B}^\perp$ with $A, B \succeq 0$, but there exist no $B \in \mathcal{B}$ nor $A \in \mathcal{B}^\perp$ with either $A \succ 0$ or $B \succ 0$.*

Proof. This is a consequence of the Bohnenblust [Bon48] dichotomy; see [Nel, Lemma 5.7] for a detailed proof. \blacksquare

Using Lemma 3.5 we can establish the existence of certain positive linear functionals with desirable properties.

Lemma 3.6. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil, $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite right chip space, $I \subseteq \mathbb{R} \langle x, x^* \rangle^{1 \times \ell}$ be a left module generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, and let $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ be a symmetric polynomial. Let $M = M_{T,K}(L)$ be a truncated test module for ${}^{(L, \mathfrak{C})} \sqrt{I}$, L , and \mathfrak{C} . If*

$$p \notin M + \mathbb{R}^{\ell \times 1} {}^{(L, \mathfrak{C})} \sqrt{I} + ({}^{(L, \mathfrak{C})} \sqrt{I})^* \mathbb{R}^{1 \times \ell},$$

then there exists a positive linear functional λ on $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ with the following properties:

- (1) $\lambda(a) > 0$ for each $a \in M \setminus \{0\}$;
- (2) $\lambda(\iota) = 0$ for each $\iota \in (\mathbb{R}^{\ell \times 1} \sqrt[L, \mathfrak{C}]{I} + [\sqrt[L, \mathfrak{C}]{I}]^* \mathbb{R}^{1 \times \ell}) \cap \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$;
- (3) $\lambda(p) < 0$.

Proof. Without loss of generality we may assume $I = \sqrt[L, \mathfrak{C}]{I}$ since by Proposition 2.4, $\sqrt[L, \mathfrak{C}]{I}$ is also generated by polynomials in $\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$.

First, $T \neq \{0\}$ since otherwise $I = \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. Further, the case where $K = \{0\}$ is similar to the case where $K \neq \{0\}$, so without loss of generality assume that $K \neq \{0\}$.

If

$$M \cap (\mathbb{R}p + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) = \{0\},$$

then let $W = \mathbb{R}p$. Otherwise, let $W = \{0\}$. In either case, by Lemma 3.4, we have

$$M \cap (W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) = \{0\}.$$

Let τ_1, \dots, τ_μ be a basis for T and let $\kappa_1, \dots, \kappa_\sigma$ be a basis for K . Define column vectors $\tau := (\tau_i)_{1 \leq i \leq \mu}$, $\kappa := (\kappa_j)_{1 \leq j \leq \sigma}$, and $L\kappa := (L\kappa_j)_{1 \leq j \leq \sigma}$. The set M is characterized as being the set of polynomials of the form $\tau^*A\tau + \kappa^*B(L\kappa)$, where A and B are positive-semidefinite matrices. By hypothesis, if $\tau^*A\tau + \kappa^*B(L\kappa) \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}$ and $A, B \succeq 0$, then $A, B = 0$.

Let $\mathcal{Z} \subseteq \mathbb{S}^\mu \times \mathbb{S}^\sigma$ be defined by

$$\mathcal{Z} := \{ (Z_\tau, Z_\kappa) \mid \tau^*Z_\tau\tau + \kappa^*Z_\kappa(L\kappa) \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell} \}.$$

By assumption, the space \mathcal{Z} contains no pairs (Z_τ, Z_κ) with $Z_\tau, Z_\kappa \succeq 0$ except $(0, 0)$. Therefore there is no nonzero positive-semidefinite matrix in the space $\widehat{\mathcal{Z}} \subseteq \mathbb{S}^{\mu+\sigma}$ defined by

$$\widehat{\mathcal{Z}} := \{ Z_\tau \oplus Z_\kappa \mid (Z_\tau, Z_\kappa) \in \mathcal{Z} \}.$$

By Lemma 3.5 there exists a positive definite matrix $C \in \widehat{\mathcal{Z}}^\perp$. Let $C_\tau \in \mathbb{S}^\mu$ and $C_\kappa \in \mathbb{S}^\sigma$ be such that C is of the form

$$C = \begin{pmatrix} C_\tau & \tilde{C} \\ \tilde{C}^* & C_\kappa \end{pmatrix}$$

for some block matrix \tilde{C} . Since $C \succ 0$, we have $C_\tau, C_\kappa \succ 0$, and if $(Z_\tau, Z_\kappa) \in \mathcal{Z}$, then since $C \in \widehat{\mathcal{Z}}^\perp$,

$$\langle C_\tau, Z_\tau \rangle + \langle C_\kappa, Z_\kappa \rangle = \langle C, Z_\tau \oplus Z_\kappa \rangle = 0.$$

Decompose $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ as

$$\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C} = (M + W + [(\mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) \cap \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}]) \oplus S,$$

for some space $S \subseteq \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$. Define $\tilde{\lambda}$ on $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ as follows:

$$(3.4) \quad \tilde{\lambda}(\tau^*A\tau + \kappa^*B(L\kappa) + \iota + s) = \text{Tr}(AC_\tau) + \text{Tr}(BC_\kappa),$$

where $A \in \mathbb{R}^{\mu \times \mu}$, $B \in \mathbb{R}^{\sigma \times \sigma}$, $\iota \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}$ and $s \in S$. We now verify that $\tilde{\lambda}$ is well defined.

First, if $\tau^*A\tau + \kappa^*B(L\kappa) \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}$ for some $A \in \mathbb{R}^{\mu \times \mu}$ and $B \in \mathbb{R}^{\sigma \times \sigma}$, not necessarily symmetric, then $\tau^*A^*\tau + \kappa^*B^*(L\kappa) \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}$, which implies that

$$\tau^*(A + A^*)\tau + \kappa^*(B + B^*)(L\kappa) \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}.$$

Therefore

$$\text{Tr}(AC_\tau) + \text{Tr}(BC_\kappa) = \frac{1}{2}(\text{Tr}[(A + A^*)C_\tau] + \text{Tr}[(B + B^*)C_\kappa]) = 0.$$

Next, suppose $\tau^*A_1\tau + \kappa^*B_1(L\kappa) + \iota_1 + s_1 = \tau^*A_2\tau + \kappa^*B_2(L\kappa) + \iota_2 + s_2$, where $A_1, A_2 \in \mathbb{R}^{\mu \times \mu}$, $B_1, B_2 \in \mathbb{R}^{\sigma \times \sigma}$, $\iota_1, \iota_2 \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}$, and $s_1, s_2 \in S$. Then

$$\tau^*(A_1 - A_2)\tau + \kappa^*(B_1 - B_2)(L\kappa) + (\iota_1 - \iota_2) + (s_1 - s_2) = 0.$$

By construction, we must have $s_1 - s_2 = 0$. Therefore

$$\tau^*(A_1 - A_2)\tau + \kappa^*(B_1 - B_2)(L\kappa) \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}.$$

Hence

$$\begin{aligned} \tilde{\lambda}(\tau^*A_1\tau + \kappa^*B_1(L\kappa) + \iota_1 + s_1) &= \text{Tr}(A_1C_\tau) + \text{Tr}(B_1C_\kappa) \\ &= \text{Tr}(A_2C_\tau) + \text{Tr}(B_2C_\kappa) \\ &\quad + \text{Tr}([A_1 - A_2]C_\tau) + \text{Tr}([B_1 - B_2]C_\kappa) \\ &= \tilde{\lambda}(\tau^*A_2\tau + \kappa^*B_2(L\kappa) + \iota_2 + s_2). \end{aligned}$$

Therefore $\tilde{\lambda}$ is well defined.

Next, if $W = \{0\}$, let $\lambda = \tilde{\lambda}$. If $W = \mathbb{R}p$, we define λ as follows. Choose a symmetric functional ξ on $\mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ such that $\xi(p) < 0$ and $\xi(\mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) = \{0\}$, which exists by the Hahn-Banach Theorem. Let $\tilde{C}_\tau = (\xi[\tau_i^*\tau_j])_{1 \leq i, j \leq \mu}$ and $\tilde{C}_\kappa = (\xi[\kappa_i^*L\kappa_j])_{1 \leq i, j \leq \sigma}$. Choose $\epsilon > 0$ such that $C_\tau + \epsilon\tilde{C}_\tau \succ 0$ and $C_\kappa + \epsilon\tilde{C}_\kappa \succ 0$. Define $\lambda = \tilde{\lambda} + \epsilon\xi$.

It follows immediately, by definition of $\tilde{\lambda}$ that $\tilde{\lambda}(\iota) = 0$ for each $\iota \in W + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}$. If $W = \{0\}$, then $\lambda(\mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) = 0$. If $W = \mathbb{R}p$, since $\xi(\mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) = \{0\}$, we also have $\lambda(\mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) = 0$. It is also clear that $\lambda(b^*) = \lambda(b)$ for each $b \in \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ in both cases.

Each nonzero element of M is of the form $\tau^*A\tau + \kappa^*B(L\kappa)$, where $A, B \succeq 0$, and at least one of A and B is nonzero. If $W = \{0\}$, then

$$\lambda(\tau^*A\tau + \kappa^*B(L\kappa)) = \text{Tr}(AC_\tau) + \text{Tr}(BC_\kappa) > 0,$$

since $C_\tau, C_\kappa \succ 0$. If $W = \mathbb{R}p$, and if $A = (A_{ij})_{1 \leq i, j \leq \mu}$, and $B = (B_{ij})_{1 \leq i, j \leq \sigma}$, then

$$\begin{aligned} \lambda(\tau^*A\tau + \kappa^*B(L\kappa)) &= \text{Tr}(AC_\tau) + \text{Tr}(BC_\kappa) \\ &+ \epsilon \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} a_{ij} \xi(\tau_i^* \tau_j) + \epsilon \sum_{i=1}^{\sigma} \sum_{j=1}^{\sigma} b_{ij} \xi(\kappa_i^* L \kappa_j) \\ &= \text{Tr}(A[C_\tau + \epsilon \tilde{C}_\tau]) + \text{Tr}(B[C_\kappa + \epsilon \tilde{C}_\kappa]) > 0, \end{aligned}$$

since $C_\tau + \epsilon \tilde{C}_\tau, C_\kappa + \epsilon \tilde{C}_\kappa \succ 0$. Further, if $q \in \mathfrak{C}$, then $q = \iota + \tau$ for some $\iota \in I$ and $\tau \in T$. We see that

$$\lambda(q^*q) = \lambda(\tau^*\tau) + \lambda(\iota^*\tau) + \lambda(\tau^*\iota) + \lambda(\iota^*\iota) = \lambda(\tau^*\tau) \geq 0,$$

since $\iota \in I$ and $\tau^*\tau \in M$. Therefore λ is positive.

Finally, consider $\lambda(p)$. If $W = \mathbb{R}p$, then $\tilde{\lambda}(p) = 0$. Hence $\lambda(p) = \epsilon \xi(p) < 0$. If $W = \{0\}$, then

$$\mathbb{R}p \cap (M + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}) \neq \{0\}.$$

Therefore $\alpha p + \iota = m$ for some $\alpha \in \mathbb{R}$ and some $m \in M \setminus \{0\}$. We cannot have $\alpha > 0$ since this would imply that $p \in M + \mathbb{R}^{\ell \times 1}I + I^*\mathbb{R}^{1 \times \ell}$. Similarly, $\alpha \neq 0$ since otherwise $\iota \in M \setminus \{0\}$. Hence $\alpha < 0$, so that $\lambda(p) = \alpha \lambda(m) < 0$. \blacksquare

Lemma 3.7. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil, $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a full finite right chip space, $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, and $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ be a symmetric polynomial. Let $M = M_{T, K}(L)$ be a truncated test module for I, L and \mathfrak{C} . Set $n = \dim(\mathfrak{C}) - \dim({}^{(L, \mathfrak{C})} \sqrt{I} \cap \mathfrak{C})$.*

- (1) *If $p \notin M + \mathbb{R}^{\ell \times 1} {}^{(L, \mathfrak{C})} \sqrt{I} + ({}^{(L, \mathfrak{C})} \sqrt{I})^* \mathbb{R}^{1 \times \ell}$ then there exists $(X, v) \in V(I)^{(n)}$ such that $v^* p(X) v < 0$ but $L(X) \succeq 0$.*
- (2) *If $p \notin M + \mathbb{R}^{\ell \times 1} ({}^{(L, \mathfrak{C})} \sqrt{I})^+ + ({}^{(L, \mathfrak{C})} \sqrt{I})^* \mathbb{R}^{1 \times \ell}$, then there exists $(X, v) \in V(I)^{(n)}$ such that $v^* p(X) v < 0$ and $L(X) \succ 0$.*

Proof. Without loss of generality, let I be (L, \mathfrak{C}) -real since, by Proposition 2.4, the (L, \mathfrak{C}) -real radical of I is also generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$. Also, $p \notin \mathbb{R}^{\ell \times 1} I$ implies that $I \neq \mathbb{R} \langle x, x^* \rangle \mathfrak{C}$. In particular, this implies that $n = \dim(\mathfrak{C}) - \dim(I \cap \mathfrak{C}) > 0$. Let λ be a linear functional with the properties described by Lemma 3.6. By Corollary 3.3, we produce a tuple of $n \times n$ matrices X , together with a vector $v \in \mathbb{R}^{\ell n}$ such that

$$v^* a(X) v = \lambda(a)$$

for each $a \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, and such that $\mathbb{R}^{\ell n} = \{q(X)v \mid q \in \mathfrak{C}\}$.

If $\iota \in I \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, then for each $q \in \mathfrak{C}$, we have

$$(q(X)v)^* \iota(X)v = \lambda(q^* \iota) = 0$$

since $\lambda([\mathbb{R}^{\ell \times 1} I + I^* \mathbb{R}^{1 \times \ell}] \cap \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}) = \{0\}$. Therefore $\iota(X)v = 0$. Since I is generated by $I \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, this implies that $(X, v) \in V(I)^{(n)}$.

Next, consider $L(X)$. Let $q \in \mathbb{R}^{\nu \times 1} \mathfrak{C}$ be decomposed as $q = \varphi + \kappa$, where $L\varphi \in \mathbb{R}^{\nu \times 1} \langle x, x^* \rangle_1 \mathfrak{C} \cap I$, and $\kappa \in K$. We then see that

$$(q(X)v)^* L(X)(q(X)v) = v^* \kappa(X)^* L(X) \kappa(X) v = \lambda(\kappa^* L \kappa) \geq 0.$$

Hence $L(X) \succeq 0$. If I is strongly (L, \mathfrak{C}) -real, then $\varphi \in \mathbb{R}^{\nu \times 1} I$. Therefore $q(X)v \neq 0$ if and only if $\kappa \neq 0$. In this case,

$$(q(X)v)^* L(X)(q(X)v) = \lambda(\kappa^* L \kappa) > 0$$

since $\kappa^* L \kappa \in M \setminus \{0\}$. Hence $L(X) \succ 0$.

Finally, $v^* p(X)v = \lambda(p) < 0$. ■

As a consequence of Lemma 3.7, we can describe ${}^{(L)}\sqrt{I} \cap \mathfrak{C}$ for linear pencils L .

Corollary 3.8. *Suppose $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ is a linear pencil. Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a full finite right chip space, and let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$. Then,*

$${}^{(L)}\sqrt{I} \cap \mathfrak{C} = {}^{(L, \mathfrak{C})}\sqrt{I} \cap \mathfrak{C} \quad \text{and} \quad {}^{(L)\dagger}\sqrt{I} \cap \mathfrak{C} = {}^{(L, \mathfrak{C})\dagger}\sqrt{I} \cap \mathfrak{C}.$$

Proof. If a left module is L -real, then by definition it is also (L, \mathfrak{C}) -real, so ${}^{(L, \mathfrak{C})}\sqrt{I} \subseteq {}^{(L)}\sqrt{I}$. Conversely, assume there exists $p \in \left({}^{(L)}\sqrt{I} \cap \mathfrak{C} \right) \setminus \left({}^{(L, \mathfrak{C})}\sqrt{I} \cap \mathfrak{C} \right)$. Let M be a truncated test module for I , L and \mathfrak{C} . We claim that

$$-p^* p \notin M + \mathbb{R}^{\ell \times 1} {}^{(L, \mathfrak{C})}\sqrt{I} + ({}^{(L, \mathfrak{C})}\sqrt{I})^* \mathbb{R}^{1 \times \ell}.$$

Indeed, as otherwise there would exist $m \in M$ such that

$$p^* p + m \in \mathbb{R}^{\ell \times 1} {}^{(L, \mathfrak{C})}\sqrt{I} + ({}^{(L, \mathfrak{C})}\sqrt{I})^* \mathbb{R}^{1 \times \ell},$$

which would imply that $p \in {}^{(L, \mathfrak{C})}\sqrt{I}$. Now by Lemma 3.7 there exists $(X, v) \in V(I)$ such that

$$-v^* p(X)^* p(X) v < 0 \quad \text{and} \quad L(X) \succeq 0.$$

Since $\mathcal{I}(\{(X, v)\})$ is L -real by Proposition 2.1, we see that $p \notin \mathcal{I}((X, v)) \supseteq {}^{(L)}\sqrt{I}$, which is a contradiction.

The ${}^{(L)\dagger}\sqrt{I}$ case is similar. ■

4. MAIN RESULTS

In this section we use the results of §3 to prove several Positivstellensätze and Nullstellensätze. We first prove the main theorem of this paper, Theorem 1.9.

4.1. Proof of The Main Theorem 1.9. If p is of the form of (1.5), then Proposition 2.1 implies that $v^*p(X)v \geq 0$ if $(X, v) \in V(I)$ and $L(X) \succeq 0$.

Conversely, suppose p is not of the form of (1.5). By Proposition 2.4, ${}^{(L, \mathfrak{C})}\sqrt{I}$ is generated by polynomials in $\mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C}$. By [Nel, Lemma 4.2], the set of symmetric elements of the set

$$(\mathbb{R}^{\ell \times 1} {}^{(L, \mathfrak{C})}\sqrt{I} + {}^{(L, \mathfrak{C})}\sqrt{I} \mathbb{R}^{1 \times \ell}) \cap \mathfrak{C}^* \mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C}$$

is all elements of the form

$$\sum_k^{\text{finite}} (r_k^* \iota_k + \iota_k^* r_k),$$

with each $\iota_k \in {}^{(L, \mathfrak{C})}\sqrt{I} \cap \mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C}$ and each $r_k \in \mathfrak{C}$. Therefore,

$$p \notin M + \mathbb{R}^{\ell \times 1} {}^{(L, \mathfrak{C})}\sqrt{I} + ({}^{(L, \mathfrak{C})}\sqrt{I})^* \mathbb{R}^{1 \times \ell},$$

where M is some truncated test module for I , L and \mathfrak{C} . Now Lemma 3.7 implies that there exists $(X, v) \in V(I)$ such that $v^*p(X)v < 0$ and $L(X) \succeq 0$.

The strongly L -real case is similar, so its proof is omitted. ■

In the rest of this section we state and prove a few corollaries of Theorem 1.9. These contain several of those listed in the introduction.

4.2. Degree Bounds. Using the machinery of right chip spaces we deduce degree bounds on the terms appearing in the Positivstellensatz certificate (1.5).

Corollary 4.1. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module generated by polynomials with degree bounded by d , degree in each variable x_k bounded by d_k , and degree in each variable x_k^* bounded by d_{k+g} . Let $p \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ be a symmetric polynomial of degree δ , degree δ_k in each variable x_k , and degree δ_{k+g} in each variable x_k^* .*

(1) $v^*p(X)v \geq 0$ whenever $(X, v) \in V(I)$ and $L(X) \succeq 0$ if and only if p is of the form

$$(4.1) \quad p = \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j + \sum_k^{\text{finite}} (r_k^* \iota_k + \iota_k^* r_k)$$

where each $p_i, r_k \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, each $q_j \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ and $\iota_k \in {}^{(L)}\sqrt{I}$, with the following degree bounds:

- (a) each p_i, q_j , and r_k has degree bounded by $\max \{d - 1, \lceil \frac{\delta-1}{2} \rceil, 0\}$.
- (b) each ι_k has degree bounded by $\max \{d, \lceil \frac{\delta+1}{2} \rceil\}$.

- (c) each p_i, q_j, r_k and ι_k has degree in each variable x_k bounded by $\max\{d_k, \delta_k\}$,
(d) each p_i, q_j, r_k and ι_k has degree in each variable x_k bounded by $\max\{d_{k+g}, \delta_{k+g}\}$,
(2) $v^*p(X)v \geq 0$ whenever $(X, v) \in V(I)$ and $L(X) \succ 0$, if and only if p is of the form (4.1) where each $p_i, r_k \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, each $q_j \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ and $\iota_k \in {}^{(L)}\sqrt{I}$, and the same degree bounds as in (1) hold.

Proof. Let \mathfrak{C} be spanned by all monomials in $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ with degree bounded by $\max\{d - 1, \lceil \frac{\delta-1}{2}, 0 \rceil\}$, degree in each x_i bounded by $\max\{d_i, \delta_i\}$, and degree in each x_i^* bounded by $\max\{d_{i+g}, \delta_{i+g}\}$. Then \mathfrak{C} is a full finite right chip space, I is generated by some polynomials in $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, and $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$. Also note that ${}^{(L)}\sqrt{I} \cap \mathfrak{C} = {}^{(L, \mathfrak{C})}\sqrt{I} \cap \mathfrak{C}$ and ${}^{(L)}\sqrt{I} \cap \mathfrak{C} = {}^{(L, \mathfrak{C})}\sqrt{I} \cap \mathfrak{C}$ by Corollary 3.8. The result now follows directly from Theorem 1.9. ■

Remark 4.2. Given a finitely-generated left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ and a symmetric polynomial $p \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$, in general one can construct a right chip space \mathfrak{C} satisfying the conditions of Theorem 1.9 with dimension much smaller than the space of polynomials with degree bounds given in Corollary 4.1.

4.3. Convex Positivstellensatz for General Linear Pencils. Theorem 1.9 and Corollary 4.3 below are extensions of the Convex Positivstellensatz from [HKM12].

Corollary 4.3. *Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a full, finite right chip space. Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. Let $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ be a symmetric polynomial. Then $p(X) \succeq 0$ whenever $L(X) \succeq 0$ if and only if p is of the form*

$$p = \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j + \sum_k^{\text{finite}} (r_k^* \iota_k + \iota_k^* r_k)$$

where $p_i, r_k \in \mathfrak{C}$, $q_j \in \mathbb{R}^{\nu \times 1} \mathfrak{C}$ and $\iota_k \in {}^{(L, \mathfrak{C})}\sqrt{\{0\}} \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$.

Proof. Apply Theorem 1.9 with $I = \{0\}$. ■

Here is the restriction of Corollary 4.3 to the monic case (cf. [HKM12, Theorem 1.1 (2)]).

Corollary 4.4. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a monic linear pencil, let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite chip space, and suppose $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ is symmetric. Then $p(X) \succeq 0$ whenever $L(X) \succeq 0$ if and only if p is of the form*

$$(4.2) \quad p = \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j,$$

where each $p_i \in \mathfrak{C}$ and each $q_j \in \mathbb{R}^{\nu \times \ell} \mathfrak{C}$.

Proof. Since L is monic, by Proposition 2.3 we have ${}^{(L)}\sqrt{\{0\}} = \{0\}$. Now apply Corollary 4.3. \blacksquare

Our results on right chip spaces yield tighter bounds on the polynomials p_i and q_j in (4.2) than previous results in [HKM13, HKM12].

4.4. Size Bounds. In this section we present size bounds; that is, given a linear pencil L and a polynomial $p \in \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$, the positivity of p on \mathcal{D}_L only needs to be tested on $n \times n$ matrices $X \in \mathcal{D}_L$ for $n = \dim(\mathfrak{C})$. More precisely, we have

Corollary 4.5. *Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a right chip space. Let $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$ be a linear pencil and let $p \in \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$. Set*

$$n = \dim(\mathfrak{C}) - \dim({}^{(L, \mathfrak{C})}\sqrt{\{0\}} \cap \mathfrak{C}) \quad \text{and} \quad n_+ = \dim(\mathfrak{C}) - \dim({}^{(L, \mathfrak{C})+}\sqrt{\{0\}} \cap \mathfrak{C})$$

Then:

- (1) $p|_{\mathcal{D}_L} \succ 0$ if and only if $p|_{\mathcal{D}_L(n_+)} \succ 0$;
- (2) $p|_{\mathcal{D}_L} \succeq 0$ if and only if $p|_{\mathcal{D}_L(n)} \succeq 0$.

Proof. The proof of (1) essentially the same as the proof of (2), so we will only give the proof of (2).

First, the implication (\Rightarrow) is clear. Let $I = {}^{(L, \mathfrak{C})}\sqrt{\{0\}}$. If $p|_{\mathcal{D}_L} \not\succeq 0$, then p is not of the form (1.5), so Lemma 3.7 implies that there exists $(X, v) \in (\mathbb{R}^{n \times n})^g \times \mathbb{R}^n$ with $v^*p(X)v < 0$ and $L(X) \succeq 0$. \blacksquare

Remark 4.6. If $\deg(p) \leq 2k + 1$, then we have $p \in \mathfrak{C}^*\mathbb{R}\langle x, x^* \rangle_1\mathfrak{C}$ for $\mathfrak{C} = \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle_k$.

Corollary 4.7. *Let $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$ and $\hat{L} \in \mathbb{R}^{\ell \times \ell}\langle x, x^* \rangle$ be linear pencils. Let*

$$n = \ell - \dim({}^{(L, \mathbb{R}^{1 \times \ell})}\sqrt{\{0\}} \cap \mathbb{R}^{1 \times \ell}) \quad \text{and} \quad n_+ = \ell - \dim({}^{(L, \mathbb{R}^{1 \times \ell})+}\sqrt{\{0\}} \cap \mathbb{R}^{1 \times \ell})$$

Then:

- (1) $\hat{L}|_{\mathcal{D}_L} \succ 0$ if and only if $\hat{L}|_{\mathcal{D}_L(n_+)} \succ 0$;
- (2) $\hat{L}|_{\mathcal{D}_L} \succeq 0$ if and only if $\hat{L}|_{\mathcal{D}_L(n)} \succeq 0$.

Proof. Let $\mathfrak{C} = \mathbb{R}^{1 \times \ell}$ and apply Corollary 4.5. \blacksquare

Note that Corollaries 4.5 and 4.7 do not assume that \mathcal{D}_L is bounded nor do they assume that it has an interior point.

4.5. **The Left Nullstellensatz.** In this section we prove Corollary 1.11, which is the main result of [Nel] and is a generalization of the Real Nullstellensatz from [CHMN].

We begin with the following corollary of Theorem 1.9:

Corollary 4.8. *If $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is a finitely-generated left module, then $\sqrt[\text{re}]{I} = \sqrt{I}$.*

Proof. Let $L = 1$. By definition, $\sqrt[\text{re}]{I} = {}^{(L)}\sqrt{I}$. By Proposition 1.4, we have $\sqrt[\text{re}]{I} \subseteq \sqrt{I}$. Suppose $r \in \sqrt{I}$. It follows that $-v^*r(X)^*r(X)v = 0$ for each $(X, v) \in V(I)$. Since $L(X) \succeq 0$ for each X , Theorem 1.9 implies that $-r^*r$ is of the form

$$-r^*r = \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* q_j + \iota + \iota^*,$$

where $\iota \in \mathbb{R}^{\ell \times 1} \sqrt[\text{re}]{I}$. Therefore

$$r^*r + \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* q_j \in \mathbb{R}^{\ell \times 1} \sqrt[\text{re}]{I} + \left(\sqrt[\text{re}]{I} \right)^* \mathbb{R}^{1 \times \ell},$$

which implies that $r \in \sqrt[\text{re}]{I}$. ■

We now prove Corollary 1.11.

Proof of Corollary 1.11. Note that $p_i(X)v = 0$ means each row of $p_i(X)v$ is 0, i.e. $e_k^* p_i(X)v = 0$ for each $e_k \in \mathbb{R}^{1 \times \nu_i}$. Therefore

$$V(I) = V \left(\sum_{i=1}^k \mathbb{R}^{1 \times \nu_i} \langle x, x^* \rangle p_i \right).$$

The first part of the result now follows from Corollary 4.8.

Next, if q is an element of the left module (1.6), then

$$q = \sum_i^{\text{finite}} \sum_{j=1}^k a_{ij} b_{ij} p_j$$

for some $a_{ij} \in \mathbb{R}^{\nu \times 1}$ and $b_{ij} \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. Therefore,

$$q = \sum_{j=1}^k \left(\sum_i^{\text{finite}} a_{ij} b_{ij} \right) p_j. \quad \blacksquare$$

4.6. Positivity on a Left Module. We can characterize polynomials p which are positive on the variety of a left module as follows:

Corollary 4.9. *Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a full, finite right chip space. Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$, and let $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ be a symmetric polynomial. Then $v^* p(X) v \geq 0$ for each $(X, v) \in V(I)$ if and only if p is of the form*

$$p = \sum_i^{\text{finite}} q_i^* q_i + \sum_j^{\text{finite}} (r_j^* \iota_j + \iota_j^* r_j),$$

where each $q_i, r_j \in \mathfrak{C}$ and each $\iota_j \in \sqrt[\nu]{I} \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$.

Proof. If $L = 1$, then $\sqrt[\nu]{I} = \sqrt[L]{I}$ by definition. We see $L(X) \succ 0$ for all tuples of matrices X , and we see for any $q \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ that $q^* L q = q^* q$. Therefore Theorem 1.9 gives the result. \blacksquare

4.7. Zero on the Intersection of the Variety of a Left Module and the Positivity Set of a Linear Pencil. We return to polynomials p which vanish on the intersection of the variety of a left module with a spectrahedron. We next prove Proposition 1.6 and its strongly L -real radical analog:

Corollary 4.10. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finitely-generated left module, and let $p \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$.*

- (1) $p(X)v = 0$ whenever $(X, v) \in V(I)$ and $L(X) \succeq 0$ if and only if $p \in \sqrt[L]{I}$;
- (2) $p(X)v = 0$ whenever $(X, v) \in V(I)$ and $L(X) \succ 0$ if and only if $p \in \sqrt[L]{I}^+$.

Proof. Let $(X, v) \in V(I)$ be such that $L(X) \succeq 0$. Proposition 2.1 implies that $\mathcal{I}(\{(X, v)\})$ is an L -real left module containing I . Therefore, $\sqrt[L]{I} \subseteq \mathcal{I}(\{(X, v)\})$.

Conversely, suppose $p(X)v = 0$ whenever $(X, v) \in V(I)$ and $L(X) \succeq 0$. Then

$$v^* (-p(X)^* p(X)) v \geq 0$$

whenever $(X, v) \in V(I)$ and $L(X) \succeq 0$. Theorem 1.9 implies that

$$-p^* p = \sum_j^{\text{finite}} q_j^* q_j + \sum_k^{\text{finite}} r_k^* L r_k + \iota + \iota^*$$

for some $q_j \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, $r_k \in \mathbb{R}^{\nu \times \ell} \langle x, x^* \rangle$, and $\iota \in \mathbb{R}^{\ell \times 1} \sqrt[L]{I}$. Therefore

$$p^* p + \sum_j^{\text{finite}} q_j^* q_j + \sum_k^{\text{finite}} r_k^* L r_k \in \mathbb{R}^{\ell \times 1} \sqrt[L]{I} + \left[\sqrt[L]{I} \right]^* \mathbb{R}^{1 \times \ell},$$

which implies that $p \in \sqrt[L]{I}$.

The strongly L -real case is similar. ■

5. THICK SPECTRAHEDRA AND THICK LINEAR PENCILS

This section proves a “Randstellensatz” for \mathcal{D}_L and properties of L -real radicals for monic linear pencils L satisfying the zero determining property (ZDP). These are Theorem 1.1 and Proposition 1.12 stated in the introduction. Then in Subsection 5.2 we exhibit big classes of linear pencils having ZDP.

5.1. Randstellensatz.

Definition 5.1. If $L \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ is a linear pencil, let $I_L = \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle L \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be the left module generated by the rows of L .

Proposition 5.2. *Let $L \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ be a monic linear pencil. Then $\sqrt[\mathbb{R}]{I_L} = I_L$.*

Proof. Let L be

$$(5.1) \quad L = \text{Id}_\ell - \Lambda \quad \text{where} \quad \Lambda = \sum_{i=1}^{\text{finite}} (A_i \otimes x_i + A_i^* \otimes x_i^*)$$

and each $A_i \in \mathbb{R}^{\ell \times \ell}$. Consider $\sqrt[\mathbb{R}]{I_L}$. By Proposition 2.4, $\sqrt[\mathbb{R}]{I_L} = \sqrt[\mathbb{C}]{I_L}$ is generated by I_L together with possibly some constant polynomials. Let $c \in \sqrt[\mathbb{R}]{I_L}$ be constant. To show that $\sqrt[\mathbb{R}]{I_L} = I_L$ it suffices to show that $c \in I_L$.

By Corollary 4.8, if $(X, v) \in (\mathbb{R}^{n \times n})^g \times \mathbb{R}^{\ell n}$, then $L(X)v = 0$ implies that $cv = 0$. Using the embedding $\mathbb{C} \rightarrow \mathbb{R}^{2 \times 2}$ given by

$$a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

we can consider evaluating L at tuples of complex numbers. Fix a variable x_i and let $x_i = a_i + ib_i$, where a_i and b_i are real variables. If v is an eigenvector of $A_i + A_i^*$ with nonzero eigenvalue λ , then λ must be real and

$$L\left(0, \dots, 0, \frac{1}{\lambda}, 0, \dots, 0\right)v = \text{Id}_\ell v - \frac{1}{\lambda}(A_i + A_i^*)v = 0.$$

Hence $cv = 0$. Since $A_i + A_i^*$ is symmetric, there exists an orthonormal basis for \mathbb{R}^ℓ consisting of eigenvectors of $A_i + A_i^*$. Therefore, since c^* is orthogonal to all eigenvectors with nonzero eigenvalues, c^* must be an eigenvector of $A_i + A_i^*$ with eigenvalue 0.

Similarly, consider $\imath A_i - \imath A_i^*$. If v is an eigenvector of $\imath A_i - \imath A_i^*$ with nonzero eigenvalue λ , then λ must be real and

$$L\left(0, \dots, 0, \frac{\imath}{\lambda}, 0, \dots, 0\right)v = \text{Id}_\ell v - \frac{1}{\lambda}(\imath A_i - \imath A_i^*)v = 0.$$

Since $\iota A_i - \iota A_i^*$ is Hermitian, there exists an orthonormal basis for \mathbb{C}^ℓ consisting of eigenvectors of $\iota A_i - \iota A_i^*$. Therefore, since c^* is orthogonal to all eigenvectors with nonzero eigenvalues, c^* must be an eigenvector of $\iota A_i - \iota A_i^*$ with eigenvalue 0. This implies that $\Lambda c^* = 0$.

After a change of basis, if $c = e_1$, then $\Lambda e_1 = 0$, which implies

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & L_{22} & \cdots & L_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & L_{\ell 2} & \vdots & L_{\ell\ell} \end{pmatrix}$$

Therefore $c \in I_L$. ■

Proof of Proposition 1.12. By Proposition 5.2, $\sqrt[\mathbb{R}]{I_L} = I_L$. The equality $\sqrt{I_L} = \sqrt[\mathbb{R}]{I_L}$ is [Nel, Theorem 1.3]. ■

Proposition 5.3. *Let $L \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ be a monic linear pencil satisfying the hypotheses of Theorem 1.1. Suppose $p \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ is of the form*

$$(5.2) \quad p = \sum_i^{\text{finite}} q_i^* q_i + \sum_j^{\text{finite}} r_j^* L r_j \in \mathbb{R}^{\ell \times 1} I_L + I_L^* \mathbb{R}^{1 \times \ell}.$$

Then each $q_i \in I_L$ and for each r_j ,

$$r_j \in \mathbb{R}^{\ell \times 1} I_L + \{C \in \mathbb{R}^{\ell \times \ell} \mid LC = CL\}.$$

Proof. From (5.2) it follows that q_i and $L r_j$ vanish on $\widehat{\partial \mathcal{D}}_L^\circ$, so by (1.1) they vanish on $V(L)$, i.e., $q_i \in I_L$ and $L r_j \in \sqrt{I_L} = I_L$ by Proposition 1.12. So consider some $L r \in I_L$. Let \mathcal{N} be the vector subspace

$$\mathcal{N} = \left(\bigcap_{i=1}^g \text{Null}(A_i) \right) \cap \left(\bigcap_{i=1}^g \text{Null}(A_i^*) \right).$$

We consider two cases.

Case 1: Suppose $\mathcal{N} = \{0\}$. Let W be the set of all monomials which are not the leading monomial of an element of I_L . Decompose r as $\theta + \tilde{r}$, where $\theta \in \mathbb{R}^{\ell \times 1} I_L$ and

$$\tilde{r} = \sum_{\omega \in W} R_\omega \otimes \omega.$$

We see that $L \tilde{r} = L r - L \theta \in \mathbb{R}^{\ell \times 1} I_L$. If $\deg(\tilde{r}) > 0$, then the leading degree terms of $L \tilde{r}$ are

$$\sum_{i=1}^g \sum_{|\omega| = \deg(\tilde{r})} (A_i R_\omega \otimes x_i \omega + A_i^* R_\omega \otimes x_i^* \omega),$$

which must be nonzero since $\mathcal{N} = \{0\}$. Since I_L is generated by polynomials of degree at most 1, there exists a left Gröbner basis for I_L consisting of polynomials with degree bounded by 1. We see, however, that the leading degree terms of $L\tilde{r}$ are not divisible on the right by the leading terms of the polynomials in the left Gröbner basis for I_L , since their rightmost degree $\deg(\tilde{r})$ piece is in W . This is a contradiction. Hence \tilde{r} is constant.

Suppose $L\tilde{r} = qL$ for some matrix polynomial q . If q is of the form

$$q = \sum_{m \in \langle x \rangle} Q_m \otimes m$$

then the leading degree terms of qL are

$$\sum_{i=1}^g \sum_{|m|=\deg(q)} Q_m (A_i \otimes mx_i + A_i^* \otimes mx_i^*),$$

which are nonzero since $\mathcal{N} = \{0\}$. Because

$$\deg(qL) = \deg(L\tilde{r}) \leq 1,$$

we see that q is constant. Therefore

$$L\tilde{r} = \tilde{r} - \sum_{i=1}^g (A_i \tilde{r} \otimes x_i + A_i^* \tilde{r} \otimes x_i^*) = qL = q - \sum_{i=1}^g (qA_i \otimes x_i + qA_i^* \otimes x_i^*).$$

Matching up terms shows $q = \tilde{r}$ and thus $L\tilde{r} = \tilde{r}L$.

Case 2: Suppose $\mathcal{N} \neq \{0\}$. After applying an orthonormal change of basis to L we may assume that L is of the form

$$L = \begin{pmatrix} \tilde{L} & 0 \\ 0 & \text{Id}_\nu \end{pmatrix} \quad \text{and} \quad \tilde{L} = \text{Id}_{\ell-\nu} - \sum_{i=1}^g (\tilde{A}_i \otimes x_i + \tilde{A}_i^* \otimes x_i^*)$$

for some ν , where

$$\left(\bigcap_{i=1}^g \text{Null}(\tilde{A}_i) \right) \cap \left(\bigcap_{i=1}^g \text{Null}(\tilde{A}_i^*) \right) = \{0\}.$$

Next, express r as

$$r = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

where r_{12} and r_{22} have column dimension ν , then $r_{12}, r_{22} \in \mathbb{R}^{\ell \times 1} I_L$. Further, there exists a $q \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ such that

$$\begin{pmatrix} \tilde{L} & 0 \\ 0 & \text{Id}_\nu \end{pmatrix} \begin{pmatrix} r_{11} & 0 \\ r_{21} & 0 \end{pmatrix} = \begin{pmatrix} \tilde{L}r_{11} & 0 \\ r_{21} & 0 \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} \tilde{L} & 0 \\ 0 & \text{Id}_\nu \end{pmatrix} = \begin{pmatrix} q_{11}\tilde{L} & q_{12} \\ q_{21}\tilde{L} & q_{22} \end{pmatrix}$$

which shows $r_{21}, \tilde{L}r_{11} \in \mathbb{R}^{\ell \times 1} I_L$. By Case 1, r_{11} may be decomposed as $r_{11} = s\tilde{L} + C$, where $s \in \mathbb{R}^{\ell \times \ell} \langle x \rangle$ and C is a constant matrix satisfying $\tilde{L}C = C\tilde{L}$. Then

$$\begin{pmatrix} r_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} L + \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} L = L \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}. \quad \blacksquare$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, $L(X) \succeq 0$ implies that $p(X) \succeq 0$ is equivalent to p being of the form (5.2) by [HKM12, Theorem 1.1 (2)]. Further, $L(X)v = 0$ implies that $p(X)v = 0$ is equivalent to $p \in \mathbb{R}^{\ell \times 1} \sqrt[\ell]{I_L} = \mathbb{R}^{\ell \times 1} I_L$ by Corollary 4.8. Therefore Proposition 5.3 gives the result. \blacksquare

5.2. The zero determining property, ZDP. In this subsection we shall describe a rich class of pencils L with the ZDP. We do not know of any examples of minimal pencils which fail to satisfy it.

Let p be a classical commutative polynomial with $p(0) > 0$. The closed set \mathcal{C}_p is defined to be the closure of the connected component of 0 of

$$\{x \in \mathbb{R}^g : p(x) > 0\}.$$

We call \tilde{p} a **minimum degree defining polynomial** for \mathcal{C}_p if \tilde{p} is the lowest degree polynomial for which $\mathcal{C}_p = \mathcal{C}_{\tilde{p}}$. Recall from [HV07] (see Lemma 5.9 for details) there is only one minimum degree defining polynomial for \mathcal{C}_p , i.e., \tilde{p} is unique up to multiplication by a positive scalar. Denote by $\deg(\mathcal{C}_p)$ the degree of such a minimal \tilde{p} . Observe that this definition also applies to spectrahedra $\mathcal{D}_L(n) \subseteq (\mathbb{R}^{n \times n})^g$ associated to a monic linear pencil L .

Given a linear pencil L , let

$$(5.3) \quad \delta_n(X) := \det(L(X)) \quad \text{for } X \in (\mathbb{R}^{n \times n})^g.$$

For example, consider the **free ball**

$$L(x) = \begin{pmatrix} 1 & x_1^* & x_2^* & \cdots & x_g^* \\ x_1 & 1 & 0 & \cdots & 0 \\ x_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ x_g & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{D}_L(n) = \{X \in (\mathbb{R}^{n \times n})^g \mid \|X\| \leq 1\}$, and

$$\delta_n(X) = \det \left(I - \sum_{j=1}^g X_j^* X_j \right)$$

by way of Schur complements. Note δ_n is a degree $2n$ polynomial in the entries of the X_j s.

Theorem 5.4. *Suppose L is a monic linear pencil.*

- (1) $\mathcal{Z}(\widehat{\partial\mathcal{D}}_L^\circ) \subseteq V(L)$.
- (2) Suppose δ_n is a minimal degree defining polynomial for $\mathcal{D}_L(n)$ for every n , then $\mathcal{Z}(\widehat{\partial\mathcal{D}}_L^\circ) = V(L)$.
- (3) The conclusion of (2) holds even if $\delta_n = \mu_n^m$ with μ_n the minimal degree defining polynomial for $\mathcal{D}_L(n)$.

Corollary 5.5. *Suppose L is an $\ell \times \ell$ monic pencil.*

- (1) $\deg \delta_n \geq \deg(\mathcal{D}_L(n)) \geq n \deg \delta_1$.
- (2) Suppose δ_1 is a minimum degree defining polynomial for $\mathcal{D}_L(1)$. If $\deg \delta_1 = \ell$, then $\deg(\mathcal{D}_L(n)) = n \deg \delta_1$ and δ_n is a minimal degree defining polynomial for $\mathcal{D}_L(n)$ for every n , so $\mathcal{Z}(\widehat{\partial\mathcal{D}}_L^\circ) = V(L)$.
- (3) If \mathcal{D}_L is the free ball, then $\deg(\mathcal{D}_L(n)) = n \deg \delta_1$ and δ_n is a minimal degree defining polynomial for $\mathcal{D}_L(n)$ for every n , so $\mathcal{Z}(\widehat{\partial\mathcal{D}}_L^\circ) = V(L)$.

Corollary 5.6. *A generic $\ell \times \ell$ monic linear pencil L in $g > 2$ variables has ZDP.*

Proof. It is clear that $\delta_1 = \det L(x)$ is of degree $\leq \ell$. Furthermore, the determinant of a generic symmetric matrix is irreducible (see e.g.. [Bôc07, §61, p. 177]). Then the zero set V of δ_1 is a generic linear section of this irreducible variety and δ_1 is thus irreducible by Bertini's theorem [Šaf99, Theorem II.3.§1.6, p. 249]. In particular, the Zariski closure of $\mathcal{D}_L(1)$ is V . Now if r is a minimum degree defining polynomial for $\mathcal{D}_L(1)$, then r vanishes on V and is thus a multiple of δ_1 by irreducibility. Hence δ_1 is a degree ℓ minimum degree defining polynomial for $\mathcal{D}_L(1)$ and the desired conclusion follows from Corollary 5.5. ■

Remark 5.7.

- (1) For a given L the minimality of δ_1 can be easily checked with computer algebra. It suffices to establish that the ideal in $\mathbb{R}[x]$ generated by $\det L$ is real radical. We refer the reader to [BN93, Neu98] for algorithmic aspects of real radicals in commutative polynomial rings.
- (2) From Corollary 5.6 we infer that there are numerous examples of ZDP pencils.
- (3) Also for perspective, any bivariate RZ polynomial $p(x_1, x_2)$ of degree ℓ , has a determinantal representation $p(x_1, x_2) = \det L(x_1, x_2)$ for some $\ell \times \ell$ monic linear pencil L , and $\mathcal{C}_p = \mathcal{D}_L(1)$ [HV07]. (However, in more than 2 variables p may only admit $\ell \times \ell$ determinantal representations for $\ell > \deg p$, cf. [Vin12].)

Example 5.8. Let

$$L(x_1, x_2) = \begin{pmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix}.$$

Then $\mathcal{D}_L(1) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 - x_1^2 - x_2^2 \geq 0\}$, so $\det L(x_1, x_2) = 1 - x_1^2 - x_2^2$ is the minimum degree defining polynomial for $\mathcal{C}_p = \mathcal{D}_L(1)$. Hence L has ZDP.

To prove the above theorem we need some lemmas and we set about to prepare them.

5.2.1. *Minimal degree defining polynomials.* Here we give some basic facts about minimal degree defining polynomials. We require background from the proof of Lemma 2.1 of [HV07], so the proof is reproduced in an online Appendix A.

Given a commutative polynomial p let $\text{Var}(p)$ denote its zero set. If $S \subseteq \mathbb{R}^m$, then $\text{Zar}(S) \subseteq \mathbb{R}^m$ denotes the Zariski closure of S , i.e., the set of common zeros of all polynomials vanishing on S .

Beware the polynomials in the lemma are commutative.

Lemma 5.9. *A minimum degree defining polynomial p for $\mathcal{C} = \mathcal{C}_p$ is unique up to a constant. Moreover,*

- (1) *any other polynomial q with $\mathcal{C}_q = \mathcal{C}_p$ is given by $q = ph$ where h is an arbitrary polynomial which is positive on a dense connected subset of \mathcal{C}_p .*
- (2) *any other polynomial q which vanishes on $\partial\mathcal{C}_p$ is given by $q = ph$ where h is an arbitrary polynomial.*
- (3) $\text{Zar}(\partial\mathcal{C}_p) = \text{Var}(p)$.

Proof. (1) This is Lemma A.1.

(2) Let V be the Zariski closure of $\partial\mathcal{C} \subseteq \mathbb{R}^m$, and let $V = V_1 \cup \cdots \cup V_k$ be the decomposition of V into irreducible components satisfying $\dim V_i = m - 1$ for each i established in the proof of Lemma A.1. Write $p = p_1 \cdots p_k$, where p_i is an irreducible polynomial vanishing on V_i .

Since q vanishes on $\partial\mathcal{C}$, it vanishes on each V and thus on each V_i . By the real Nullstellensatz for principal ideals [BCR98, Theorem 4.5.1], $q = p_1 r_1$ for some r_1 . Since p_1 does not identically vanish on V_2 , r_1 vanishes on V_2 . Thus there is r_2 with $r_1 = p_2 r_2$. Repeating this k times leads to $q = p_1 p_2 \cdots p_k h$ for some polynomial h .

- (3) This is basically a restatement of (2). ■

Let $\text{foot}(S)$ denote the **footprint**

$$\text{foot}(S) := \{X \mid (X, v) \in S\}$$

of the set S in $(\mathbb{R}^{n \times n})^g \times \mathbb{R}^n$.

Lemma 5.10. *Let L be a monic linear pencil, and suppose that δ_n is a minimal degree defining polynomial for $\mathcal{D}_L(n)$ for some n .*

(1) *Then $\partial^1 \mathcal{D}_L(1)$ defined by*

$$\partial^1 \mathcal{D}_L(n) = \{X \in (\mathbb{R}^{n \times n})^g \mid \dim \ker (L(X)) = 1 \text{ \& } L(X) \succeq 0\}$$

is nonempty and dense in $\partial \mathcal{D}_L(n)$.

(2) *$V^1(L)$ defined by*

$$V^1(L)(n) = \{X \in (\mathbb{R}^{n \times n})^g \mid \dim \ker (L(X)) = 1\}$$

is nonempty and dense in $\text{foot}[V(L)(n)]$.

Proof. (1) Consider Renegar's directional derivative δ'_n of δ_n . Like δ_n , it is an RZ polynomial, and the corresponding algebraic interior \mathcal{C}' contains \mathcal{C} , cf. [Ren06, §4] or [Vin12, §2]. By minimality of δ_n , $\mathcal{C}' \supsetneq \mathcal{C}$, i.e., there is $X \in \partial \mathcal{D}_L(n)$ with $\delta'_n(X) \neq 0$. That is, X is a simple root of δ_n and thus $\dim \ker L(X) = 1$.

Having established that $\partial^1 \mathcal{D}_L(n) \neq \emptyset$, the density follows from [Ren06, Theorem 6]. Note that in this case $\partial^1 \mathcal{D}_L(n)$ are exactly smooth points of $\partial \mathcal{D}_L(n)$ [Ren06, Lemma 7].

(2) We use the decomposition

$$\text{foot}[V(L)(n)] = V_1 \cup \cdots \cup V_k$$

into irreducible components described in the proof of Lemma 5.9. Each V_j has relatively open intersection with $\partial^1 \mathcal{D}_L(n)$, so there is an X^j in V_j for which $\dim \ker L(X^j) = 1$. Thus $\dim \ker L(X) = 1$ for X in an open dense subset of V_j . ■

Proof of Theorem 5.4. (1) is obvious.

(2) Since δ_n is the minimum degree defining polynomial for $\partial \mathcal{D}_L(n) = \text{foot}[\widehat{\partial \mathcal{D}}_L^\circ(n)]$, by Lemma 5.9(3) Zariski closures satisfy $\text{Zar}(\partial \mathcal{D}_L(n)) = \text{Var}(\delta_n)$ for each n . So

$$(5.4) \quad \text{foot}[\mathcal{Z}(\widehat{\partial \mathcal{D}}_L^\circ(n))] = \text{Zar}(\partial \mathcal{D}_L(n)) = \text{Var}(\delta_n) = \text{foot}[V(L)(n)]$$

Intuitively, this says the “footprint” of what we are trying to prove is as claimed. By Lemma 5.10, we get $1 = \dim \ker L(X)$ for an open dense set \mathcal{U} of X in $\text{foot}[V(L)(n)]$, so also for such $X \in \text{foot}[\mathcal{Z}(\widehat{\partial \mathcal{D}}_L^\circ(n))]$. This with (5.4) and (1) says $\ker L(X)(n) = \{u \in \mathbb{R}^n \mid (X, u) \in \mathcal{Z}(\widehat{\partial \mathcal{D}}_L^\circ(n))\}$, which proves (2).

(3) follows from the same arguments as (2). ■

Proof of Corollary 5.5. (1) To get a lower bound on the minimal degree required to define $\mathcal{D}_L(n)$ take $X = X^1 \oplus \cdots \oplus X^n$ diagonal matrices in $(\mathbb{R}^{n \times n})^g$, with $X^j \in \mathbb{R}^g$, and evaluate

$$(5.5) \quad \delta_n(X) = \det L(X) = \prod_{j=1}^n \det L(X^j) = \prod_{j=1}^n \delta_1(X^j).$$

Note it has (on commutative X) degree $n \deg \delta_1$. No lower degree polynomial will vanish on all of the diagonal $X \in \partial \mathcal{D}_L(n)$, since δ_1 is minimal, so $\deg(\mathcal{D}_L(n)) \geq n \deg \delta_1$. As δ_n is a defining polynomial for $\mathcal{D}_L(n)$, it follows that $\deg \delta_n \geq \deg(\mathcal{D}_L(n))$. This proves (1).

An implication of (1) is that when δ_1 is minimal, δ_n must be minimal for those L with $\deg \delta_n \leq n \deg \delta_1$. This forces

$$\deg(\mathcal{D}_L(n)) = \deg \delta_n = n \deg \delta_1.$$

From this we get (2) and (3) immediately:

(2) By the definition of the determinant we have $\deg \delta_n \leq nd$, where d is the size of L . By hypothesis $d = \deg \delta_1$, so $\deg \delta_n \leq n \deg \delta_1$, proving (2).

(3) Note for the ball that $n \deg \delta_1 = 2n$ and we already saw that δ_n has degree $2n$. ■

5.2.2. *Necessary side.* We present a necessary condition for a pencil to satisfy ZDP.

Lemma 5.11. *Suppose L is a monic linear pencil, and let μ denote the minimum degree defining polynomial for $\mathcal{D}_L(1)$. Necessary for the zero determining property of L is that*

$$\delta_1 = \mu^m \rho$$

for some m and polynomial ρ which has zeros only on $\text{Var}(\mu)$ but does not vanish everywhere on $\text{Var}(\mu)$. Moreover, by the real Nullstellensatz, the existence of such a ρ is equivalent to

$$f \delta_1 = \mu^m (\mu^{2s} + \text{SOS})$$

for some polynomial f and some s .

Proof. From Lemma 5.9(2) we get $\delta_1 = \mu^m \rho$ for some m and ρ which is not zero everywhere on $\text{Var}(\mu)$. If ρ is 0 at some $X \notin \text{Var}(\mu)$, then $X \notin \text{Zar}(\mathcal{D}_L)$ contradicting ZDP even at the “footprint level”.

We just proved $\rho = 0$ implies $\mu = 0$. The real Nullstellensatz says equivalent to this is

$$(5.6) \quad \mu^{2s} + \text{SOS} = f \rho$$

for some polynomial f . So $f \delta_1 = \mu^m f \rho = \mu^m (\mu^{2s} + \text{SOS})$. Conversely, if such f exists, then $\delta_1 = \mu^m \rho$ implies that ρ satisfies (5.6). ■

6. DECOMPOSITION OF THIN LINEAR PENCILS

The main concern of this section is a linear pencil L for which the spectrahedron $\mathcal{D}_L(1)$ has no interior, i.e., a thin linear pencil. The special case of Theorem 1.9 where $I = \{0\}$ and $p = 0$ gives a characterization of the space ${}^{(L)}\sqrt{\{0\}}$, which in turn gives a nice algebro-geometric interpretation of spectrahedra \mathcal{D}_L having no interior points.

6.1. Characterization of ${}^{(L)}\sqrt{\{0\}}$. Recall that a spectrahedron with empty interior lies on an affine hyperplane [Bar02]. We now give a matricial version of this result.

Proposition 6.1. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. The space ${}^{(L)}\sqrt{\{0\}} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is characterized by*

$${}^{(L)}\sqrt{\{0\}} = \{p \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle \mid p(X) = 0 \text{ whenever } L(X) \succeq 0\}.$$

Proof. Every pair (X, v) evaluated at 0 is 0. The result follows from Corollary 4.10. \blacksquare

Interestingly, there is a strong relation between a free spectrahedron \mathcal{D}_L and its scalar counterpart $\mathcal{D}_L(1)$.

Proposition 6.2. *Let L be a linear pencil, and let $\{0\} \subseteq \mathbb{R} \langle x, x^* \rangle$ be the trivial ideal. Then the following are equivalent:*

- (i) ${}^{(L)}\sqrt{\{0\}} = \mathbb{R} \langle x, x^* \rangle$.
- (ii) The linear matrix inequality $L(x) \succeq 0$ is infeasible over $x \in \mathbb{R}^g$, i.e., $\mathcal{D}_L(1) = \emptyset$.
- (iii) The linear matrix inequality $L(X) \succeq 0$ is infeasible over matrix tuples $X \in (\mathbb{R}^{n \times n})^g$ for each $n \in \mathbb{N}$, i.e., $\mathcal{D}_L = \emptyset$.

The equivalence between (ii) and (iii) of this proposition is easy—for example, see [KS11, Corollary 4.1.4]—but we present a new proof here using the real radical theory of this paper.

Proof. That (iii) implies (ii) is clear.

Next, assume (ii) holds. Lemma 3.7 implies that if $-1 \notin M + {}^{(L, \mathbb{R})}\sqrt{\{0\}} + ({}^{(L, \mathbb{R})}\sqrt{\{0\}})^*$, where M is a truncated test module for $\{0\}$, L and \mathbb{R} , then there exists a tuple X of $n \times n$ matrices such that $L(X) \succeq 0$, where $n = \dim(\mathbb{R}) - \dim(\{0\}) = 1$. Since $n = 1$, such an X is actually in \mathbb{R}^g , which is a contradiction. Therefore, $-1 \in M + {}^{(L, \mathbb{R})}\sqrt{\{0\}} + ({}^{(L, \mathbb{R})}\sqrt{\{0\}})^*$. Hence $1 + M \in {}^{(L, \mathbb{R})}\sqrt{\{0\}} + ({}^{(L, \mathbb{R})}\sqrt{\{0\}})^*$, which implies that $1 \in {}^{(L, \mathbb{R})}\sqrt{\{0\}}$, which implies (i).

Finally, suppose (i) holds. The condition $1(X) = 0$ is infeasible for all matrix tuples X . By Proposition 6.1, and since $1 \in {}^{(L)}\sqrt{\{0\}}$, it must be that $L(X) \succeq 0$ is infeasible over all matrix tuples, which gives (iii). \blacksquare

In $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, the L -real radical of $\{(0, \dots, 0)\}$ can be derived easily from ${}^{(L)}\sqrt{\{0\}} \subseteq \mathbb{R} \langle x, x^* \rangle$, as the following corollary shows.

Corollary 6.3. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. Let $\{0\}$ denote the ideal of $\mathbb{R} \langle x, x^* \rangle$ and let $\{(0, \dots, 0)\}$ denote the left $\mathbb{R} \langle x, x^* \rangle$ -module generated by $(0, \dots, 0) \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. Then ${}^{(L)}\sqrt{\{(0, \dots, 0)\}} = \mathbb{R}^{1 \times \ell} \otimes {}^{(L)}\sqrt{\{0\}}$.*

Proof. By Proposition 6.1, we have

$${}^{(L)}\sqrt{\{(0, \dots, 0)\}} = \{p \in \mathbb{R}^{1 \times \ell} \mid p(X) = 0 \text{ whenever } L(X) \succeq 0\}.$$

If $p = \sum_{i=1}^{\ell} e_i \otimes p_i \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, then $p(X) = 0$ means that each $p_i(X) = 0$. Therefore $p \in {}^{(L)}\sqrt{\{(0, \dots, 0)\}}$ if and only if each $p_i \in {}^{(L)}\sqrt{\{0\}}$. ■

Definition 6.4. A set $S \subseteq \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ is said to be ***-closed** if $S^* = S$. A ***-ideal** is a two-sided ideal $I \subseteq \mathbb{R} \langle x, x^* \rangle$ which is *-closed, that is, $I = I^*$. If $U \subseteq \mathbb{R} \langle x, x^* \rangle$, then the *-ideal generated by U is the two-sided ideal in $\mathbb{R} \langle x, x^* \rangle$ generated by $U + U^*$.

Corollary 6.5. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. Then ${}^{(L)}\sqrt{\{0\}} \subseteq \mathbb{R} \langle x, x^* \rangle$ is a *-closed real ideal.*

Proof. Let $p \in {}^{(L)}\sqrt{\{0\}}$. If $q \in \mathbb{R} \langle x, x^* \rangle$, then

$$p(X)q(X) = 0q(X) = q(X)0 = q(X)p(X) = 0.$$

Therefore $pq, qp \in {}^{(L)}\sqrt{\{0\}}$. Next, by definition, ${}^{(L)}\sqrt{\{0\}}$ is real. Further, if $p \in {}^{(L)}\sqrt{\{0\}}$, then $pp^* \in {}^{(L)}\sqrt{\{0\}}$, which implies, since ${}^{(L)}\sqrt{\{0\}}$ is real, that $p^* \in {}^{(L)}\sqrt{\{0\}}$. ■

Proposition 6.6. *If $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ is a linear pencil, then ${}^{(L)}\sqrt{\{0\}}$ is the *-ideal in $\mathbb{R} \langle x, x^* \rangle$ generated by $({}^{(L, \mathbb{R})}\sqrt{\{0\}})_1$. Further, if $L(0) \succeq 0$, then $({}^{(L, \mathbb{R})}\sqrt{\{0\}})_1$ is spanned by a set of homogeneous linear forms.*

Proof. First, if ${}^{(L)}\sqrt{\{0\}} = \mathbb{R} \langle x, x^* \rangle$, then Corollary 3.8 implies that $1 \in ({}^{(L, \mathbb{R})}\sqrt{\{0\}})$ so that $({}^{(L, \mathbb{R})}\sqrt{\{0\}})_1 = \mathbb{R} \langle x, x^* \rangle_1$, which gives the result. Therefore, by Corollary 6.2, the only case that remains is the case where $L(x) \succeq 0$ is feasible over \mathbb{R}^g . Without loss of generality, we can apply an affine change of variables to x so that $L(0) \succeq 0$. Further, if $\iota \in {}^{(L)}\sqrt{\{0\}}$, then Proposition 6.1 implies that $\iota(0) = 0$. In particular, this implies that $({}^{(L, \mathbb{R})}\sqrt{\{0\}})_1$ is spanned by linear forms.

Next, let $I \subseteq ({}^{(L)}\sqrt{\{0\}})$ be the *-ideal generated by $({}^{(L, \mathbb{R})}\sqrt{\{0\}})_1$. Suppose

$$(6.1) \quad \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in I.$$

Let \prec be a total order on the letters $x_1, \dots, x_g, x_1^*, \dots, x_g^*$. Since I is generated by linear forms, it is straightforward to reduce each p_i and q_j in (6.1) to have monomials with no letters which are the leading letter of an element of I . Further, we can express L as $L = \tilde{L} + L_I$, where $L_I \in \mathbb{R}^{\ell \times \ell} \otimes I$ and \tilde{L} has terms which contain no letters which are the leading letter of an element of I_1 . Under this condition

$$(6.2) \quad \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* \tilde{L} q_j = 0.$$

If at least some of the p_i or Lq_j in (6.2) are nonzero, let δ be the smallest degree such that at least some of the degree δ terms \tilde{p}_i of p_i or \tilde{q}_j of q_j satisfy $\tilde{p}_i \neq 0$ or $\tilde{L}\tilde{q}_j \neq 0$. For each $m \in \langle x, x^* \rangle_\delta$, let $A_{m,p_i} \in \mathbb{R}$ be the coefficient of p_i in m , and let $B_{m,q_j} \in \mathbb{R}^{\nu \times 1}$ be the coefficient of q_j in m . Then the coefficient of $m^* m$ in (6.2) is

$$(6.3) \quad \sum_i^{\text{finite}} A_{m,p_i}^* A_{m,p_i} + \sum_j^{\text{finite}} B_{m,q_j}^* L(0) B_{m,q_j} = 0.$$

Since $L(0) \succeq 0$, each $A_{m,p_i} = 0$ and each $L(0)B_{m,q_j} = 0$. Further, the terms of (6.2) in $m^* \mathbb{R} \langle x, x^* \rangle_1^{\text{hom}} m$ are

$$m^* \left(\sum_j^{\text{finite}} B_{m,q_j}^* (\tilde{L} - L(0)) B_{m,q_j} \right) m = 0.$$

Hence

$$\sum_j^{\text{finite}} B_{m,q_j}^* L B_{m,q_j} = \sum_j^{\text{finite}} B_{m,q_j}^* L_I B_{m,q_j} \in \mathbb{R}^{\ell \times 1} \sqrt[\langle L, \mathbb{R} \rangle]{\{0\}} + \left(\sqrt[\langle L, \mathbb{R} \rangle]{\{0\}} \right)^* \mathbb{R}^{1 \times \ell}.$$

This implies that $LB_{m,q_j} \in \mathbb{R}^{\nu \times 1} \sqrt[\langle L, \mathbb{R} \rangle]{\{0\}}_1$ for each j . Therefore, if (6.2) holds, then each $LB_{m,q_j} \in I$, which implies that $\tilde{L}B_{m,q_j} = 0$. Since m was arbitrary, this yields $\tilde{p}_i = 0$ and $\tilde{L}\tilde{q}_j = 0$, which is a contradiction. Hence each $p_i = 0$ and each $\tilde{L}q_j = 0$. Therefore, (6.1) holds if and only if each $p_i \in I$ and each $Lq_j \in \mathbb{R}^{\nu \times 1} I$, which implies that I is L -real. Since $I \subseteq \sqrt[\langle L \rangle]{\{0\}}$, we have $I = \sqrt[\langle L \rangle]{\{0\}}$. \blacksquare

6.2. Decomposition of Linear Pencils. In this subsection we express a thin spectrahedron as the intersection of a thick spectrahedron with an affine subspace.

Proposition 6.7. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil such that the linear matrix inequality $L(X) \succeq 0$ is feasible. Decompose $\mathbb{R} \langle x, x^* \rangle_1$ as $(\sqrt[\langle L \rangle]{\{0\}})_1 \oplus T$ for some space T . Let L_T be the projection of L onto $\mathbb{R}^{\nu \times \nu} \otimes T$, so that $L - L_T \in \mathbb{R}^{\nu \times \nu} \otimes (\sqrt[\langle L \rangle]{\{0\}})_1$. Let*

$$\mathcal{N} = \{n \in \mathbb{R}^\nu \mid L_T n = 0\}$$

and suppose $\dim(\mathcal{N}) < \nu$. Let $\tilde{L} = C^ L_T C$, where the columns of C form a basis for \mathcal{N}^\perp .*

- (1) Given a tuple of matrices X , $L(X) \succeq 0$ if and only if $\tilde{L}(X) \succeq 0$ and $\iota(X) = 0$ for each $\iota \in \sqrt[\iota]{\{0\}}$.
- (2) There exists $x \in \mathbb{R}^g$ such that $\tilde{L}(x) \succ 0$.

Proof. (1) Let $X \in (\mathbb{R}^{n \times n})^g$ for some $n \in \mathbb{N}$.

First, suppose $L(X) \succeq 0$. Proposition 6.1 implies that $\iota(X) = 0$ for each $\iota \in \sqrt[\iota]{\{0\}}$. Since $L - L_T \in \mathbb{R}^{\nu \times \nu} \otimes \sqrt[\iota]{\{0\}}$, this implies that

$$C^*L(X)C = C^*([L - L_T](X) + L_T(X))C = \tilde{L}(X) \succeq 0.$$

Conversely, suppose $\tilde{L}(X) \succeq 0$ and $\iota(X) = 0$ for each $\iota \in \sqrt[\iota]{\{0\}}$. We have that $L - L_T \in \mathbb{R}^{\nu \times \nu} \otimes \sqrt[\iota]{\{0\}}$, hence $L(X) = L_T(X)$. Each $v \in \mathbb{R}^\nu$ can be expressed as $n_v + Cc_v$, where $n_v \in \mathcal{N}$ and $Cc_v \in \mathcal{N}^\perp$. Therefore,

$$v^*L(X)v = (Cc_v)^*L_T(X)Cc_v = c_v^*\tilde{L}(X)c_v \geq 0.$$

Hence $L(X) \succeq 0$.

- (2) Let $\nu' = \dim(\mathcal{N}^\perp)$. Suppose

$$\sum_i^{\text{finite}} p_i^*p_i + \sum_j^{\text{finite}} q_j^*\tilde{L}q_j = 0$$

for some $p_i \in \mathbb{R}$ and $q_j \in \mathbb{R}^{\nu' \times 1}$. Then

$$\sum_i^{\text{finite}} p_i^*p_i + \sum_j^{\text{finite}} q_j^*C^*LCq_j \in \mathbb{R}^{\nu' \times 1} \sqrt[\iota]{\{0\}} + (\sqrt[\iota]{\{0\}})^*\mathbb{R}^{1 \times \nu'}.$$

Therefore $LCq_j \in \mathbb{R}^{\nu' \times 1} \sqrt[\iota]{\{0\}}$ for each j . This implies that each $\tilde{L}q_j \in \mathbb{R}^{\nu' \times 1} \sqrt[\iota]{\{0\}}$. Since $\tilde{L} \in \mathbb{R}^{\nu' \times \nu'} \otimes T$, and $T \cap \sqrt[\iota]{\{0\}} = \{0\}$, we have $\tilde{L}q_j = 0$ for all j . By construction, however, this implies that each $q_j = 0$, which also implies that each $p_i = 0$. So $\{0\}$ is strongly (\tilde{L}, \mathbb{R}) -real.

Let M be a truncated test module for $\{0\}$, \tilde{L} , and \mathbb{R} . We see that $-1 \notin M$ since otherwise $1 + m = 0$ for some $m \in M$, which would imply that $1 \in \{0\} = \sqrt[\tilde{L}, \mathbb{R}]{\{0\}}$. By Lemma 3.7, there exists $(X, v) \in V(\{0\})^n$ such that $\tilde{L}(X) \succ 0$ and $-v^*1v < 0$, where $n = \dim(\mathbb{R}\langle x, x^* \rangle_0) - \dim(\{0\}) = 1$. Therefore $X \in \mathbb{R}^g$ and $\tilde{L}(X) \succ 0$. ■

6.3. Geometric Interpretation of $\sqrt[\iota]{\{0\}}$. Given a linear pencil $L \in \mathbb{R}^{\nu \times \nu}[x]$ of the form

$$L = A_0 + A_1x_1 + \cdots + A_gx_g$$

we can easily construct a linear pencil $L_0 \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ such that $L_0(x) = L(x)$ for each $x \in \mathbb{R}^g$, namely

$$L = A_0 + \frac{1}{2}A_1x_1 + \cdots + \frac{1}{2}A_gx_g + \frac{1}{2}A_1x_1^* + \cdots + \frac{1}{2}A_gx_g^*.$$

Using this, we now can prove Theorem 1.13.

Proof of Theorem 1.13. Let L_0 be such that $L_0(x) = L(x)$ for each $x \in \mathbb{R}^g$. Let \tilde{L} be the pencil obtained from L_0 using Proposition 6.7 Then Proposition 6.7 (1) implies that

$$\{x \in \mathbb{R}^g \mid L(x) \succeq 0\} = \{x \in \mathbb{R}^g \mid \tilde{L}(x) \succeq 0 \text{ and } \iota(x) = 0 \text{ for each } \iota \in I\},$$

since $x \in \mathbb{R}^g$ can be viewed as a tuple of 1×1 matrix variables. Further, Proposition 6.7 (2) implies that the spectrahedron $\mathcal{D}_{\tilde{L}}(1)$ has nonempty interior. ■

Here is the geometric interpretation of Theorem 1.13: if L is a linear pencil which defines a spectrahedron with empty interior, then either the spectrahedron is empty or it can be viewed as a spectrahedron inside a proper affine linear subspace of \mathbb{R}^g , with the new spectrahedron having nonempty interior. The affine linear subspace is defined by

$$\{x \in \mathbb{R}^g \mid \iota(x) = 0 \text{ for each } \iota \in \sqrt[\ell]{\{0\}}\}$$

and the new spectrahedron is defined by $\tilde{L}(x) \succeq 0$. Hence the commutative collapse of $\sqrt[\ell]{\{0\}}$ defines the affine subspace containing the thin spectrahedron $\mathcal{D}_L(1)$ found in [KS13].

7. COMPUTATION OF REAL RADICALS OF LEFT MODULES

Given a linear pencil $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$, a left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, and a right chip space $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, this section describes algorithms for computing the real radicals $\sqrt[\ell]{I}$, $\sqrt[\ell]{\{0\}}$ and $\sqrt[\ell, \mathfrak{C}]{I}$. Computing these radicals helps one verify whether or not a polynomial $p \in \mathbb{R}^{\ell \times \ell} \langle x, x^* \rangle$ is positive where L is positive and each $\iota \in I$ vanishes, i.e., whether (1.5) holds for p , which we describe in detail in §7.6.

7.1. Left Gröbner Bases. Left monomial orders on $\langle x, x^* \rangle$ are used to compute left Gröbner bases for left ideals $I \subseteq \mathbb{R} \langle x, x^* \rangle$. There is a general theory of one-sided Gröbner bases for one-sided modules with coherent bases over algebras with ordered multiplicative basis [Gre00]. In [Nel] there is a version of this theory specific to our case. Left Gröbner bases are easily computable and are used to algorithmically determine membership in a left module. In this section we recall the highlights of the left Gröbner basis theory found in [Nel].

Given a total order \prec on $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, we say the **leading monomial** of a polynomial p is the highest monomial, according to \prec , appearing in p . We denote this leading monomial

as $\text{Tip}(p)$. Given a subset $S \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, let $\text{Tip}(S)$ denote the set of leading monomials of elements of S .

A **left admissible order** \prec on the monomials in $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is a well-order such that $a \prec b$ for some monomials $a, b \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ implies that for each $c \in \langle x, x^* \rangle$ we have $ca \prec cb$. Given a left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, a subset $\mathcal{G} \subseteq I$ is a **left Gröbner basis of I with respect to \prec** if the left module generated by $\text{Tip}(\mathcal{G})$ equals the left module generated by $\text{Tip}(I)$. We say a polynomial p is **monic** if the coefficient of $\text{Tip}(p)$ in p is 1. We say a left Gröbner basis \mathcal{G} is **reduced** if the following hold:

- (1) Every element of \mathcal{G} is monic.
- (2) If $\iota_1, \iota_2 \in \mathcal{G}$, then $\text{Tip}(\iota_1)$ does not divide any of the terms of ι_2 on the right.

Proposition 7.1. *Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module and let \prec be a left admissible order. Then*

- (1) *There is a left Gröbner basis for I with respect to \prec .*
- (2) *There is a unique reduced left Gröbner basis for I with respect to \prec .*
- (3) *If \mathcal{G} is a left Gröbner basis for I with respect to \prec , then \mathcal{G} generates I as a left module.*
- (4) $\mathbb{R}^{1 \times \ell} = I \oplus \text{Span}(\text{NonTip}(I))$.

Proof. See [Gre00, Propositions 4.2, 4.4]. ■

Proposition 7.2 ([Nel, Lemma 8.2]). *Let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module and let $\{\iota_i\}_{i \in \alpha}$ be a left Gröbner basis for I . Every element $p \in I$ can be expressed uniquely as*

$$(7.1) \quad p = \sum_i^{\text{finite}} q_i \iota_i,$$

for some $q_i \in \mathbb{R} \langle x, x^ \rangle$. In particular, the leading monomial of p is divisible on the right by the leading monomial of one of the left Gröbner basis elements ι_i .*

7.1.1. Algorithm for Computing Reduced Left Gröbner Bases. Let \prec be a left monomial order on $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. Let I be the left module generated by polynomials $\iota_1, \dots, \iota_\mu \in \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. It is easy to show that inputting $\iota_1, \dots, \iota_\mu$ into the following algorithm computes a reduced left Gröbner basis for I .

- (1) Given: $\mathcal{G} = \{\iota_1, \dots, \iota_\mu\}$.
- (2) If $0 \in \mathcal{G}$, remove it. Further, perform scalar multiplication so that each element of \mathcal{G} is monic.
- (3) For each $\iota_i, \iota_j \in \mathcal{G}$, compare $\text{Tip}(\iota_i)$ with the terms of ι_j .
 - (a) If $\text{Tip}(\iota_i)$ divides a term of ι_j on the right, let $q \in \langle x, x^* \rangle$ and $\xi \in \mathbb{R}$ be such that $\xi q \text{Tip}(\iota_i)$ is a term in ι_j . Replace ι_j with $\iota_j - \xi q \iota_i$. Return to (2).

- (b) If $\text{Tip}(\iota_i)$ does not divide any terms of any ι_j on the right for any $i \neq j$, stop and output \mathcal{G} .

7.2. The L -Real Radical Algorithm. We now turn our attention to computing ${}^{(L,\mathfrak{C})}\sqrt{I}$. A special case of ${}^{(L,\mathfrak{C})}\sqrt{I}$ is ${}^{(L,\mathbb{R})}\sqrt{\{0\}}$. Proposition 6.6 implies that ${}^{(L,\mathbb{R})}\sqrt{\{0\}}$ is a generating set for ${}^{(L)}\sqrt{\{0\}}$. For each linear pencil L , each left module $I \subseteq \mathbb{R}\langle x, x^* \rangle$, and each full right chip space $\mathfrak{C} \subseteq \mathbb{R}\langle x, x^* \rangle$, we always have ${}^{(L)}\sqrt{\{0\}} \subseteq {}^{(L,\mathfrak{C})}\sqrt{I}$ since $0 \in I$ and $\mathbb{R} \subseteq \mathfrak{C}$. Further, Corollary 6.3 implies that computing ${}^{(L)}\sqrt{\{0\}}$ automatically gives ${}^{(L)}\sqrt{\{(0, \dots, 0)\}} \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$. As previously noted, the commutative collapse of ${}^{(L)}\sqrt{\{0\}}$ is generated by a set of linear polynomials, which is given by [KS13].

When a linear pencil $L \in \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle$ is inputted into the following algorithm, the algorithm outputs a generating set of linear polynomials for ${}^{(L)}\sqrt{\{0\}}$.

7.2.1. *The L -Real Radical Algorithm for ${}^{(L)}\sqrt{\{0\}}$.*

- (1) Let $I^{(0)} = \{0\}$, $T^{(0)} = \{1, x_1, \dots, x_g, x_1^*, \dots, x_g^*\}$. Fix a total order \prec on the letters $x_1, \dots, x_g, x_1^*, \dots, x_g^*$.
- (2) Compute the space $\mathcal{N}^{(0)} \subseteq \mathbb{R}^{\nu^{(i)}}$ defined as

$$\mathcal{N}^{(0)} = \{n \in \mathbb{R}^{\nu} \mid Ln = 0\}.$$

Define $\nu^{(0)}$ to be the dimension of $(\mathcal{N}^{(0)})^\perp$. If $\nu^{(0)} = 0$, then stop and output $I^{(0)} = \{0\}$ and $\tilde{L} = 1 \in \mathbb{R}^{1 \times 1}\langle x, x^* \rangle$. Otherwise, let $\{\xi_1^{(0)}, \dots, \xi_{\nu^{(0)}}^{(0)}\} \subseteq \mathbb{R}^{\nu}$ be an orthonormal basis for \mathcal{N}^\perp , and compress L onto $([\mathcal{N}^{(0)}]^\perp)^*([\mathcal{N}^{(0)}]^\perp)$ as

$$L^{(0)} := \begin{pmatrix} \xi_1^{(0)} & \dots & \xi_{\nu^{(0)}}^{(0)} \end{pmatrix}^* L \begin{pmatrix} \xi_1^{(0)} & \dots & \xi_{\nu^{(0)}}^{(0)} \end{pmatrix}.$$

- (3) Let $i = 0$.
- (4) Consider the problem

$$(7.2) \quad \text{Tr}(L^{(i)}A^{(i)}) + c^{(i)} = 0 \quad A^{(i)} \succeq 0, \quad c^{(i)} > 0.$$

- (a) If (7.2) has a solution with $c^{(i)} > 0$, output $\{1\}$, $\tilde{L} = 1 \in \mathbb{R}^{1 \times 1}\langle x, x^* \rangle$, and stop.
- (b) If (7.2) has a solution $0 \neq A^{(i)} \succeq 0$ and $c^{(i)} = 0$, then let $\iota_{(1,1)}^{(i)}, \dots, \iota_{(\nu^{(i)}, \nu^{(i)})}^{(i)}$ be defined by

$$(7.3) \quad \begin{pmatrix} \iota_{(1,1)}^{(i)} & \iota_{(1,2)}^{(i)} & \dots & \iota_{(1,\nu^{(i)})}^{(i)} \\ \iota_{(2,1)}^{(i)} & \iota_{(2,2)}^{(i)} & \dots & \iota_{(2,\nu^{(i)})}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \iota_{(\nu^{(i)},1)}^{(i)} & \iota_{(\nu^{(i)},2)}^{(i)} & \dots & \iota_{(\nu^{(i)},\nu^{(i)})}^{(i)} \end{pmatrix} := L^{(i)}\sqrt{A^{(i)}}.$$

(i) Define $I^{(i+1)}$ to be a reduced left Gröbner basis for the set

$$(7.4) \quad I^{(i)} \cup \left\{ \nu_{(j,k)}^{(i)} \right\}_{j,k=1}^{\nu^{(i)}}.$$

(ii) If $I^{(i+1)} = \{1\}$, stop and output $I^{(i+1)}$ and $\tilde{L} = 1$.

(iii) Let $T^{(i+1)}$ be the set containing 1 and all letters x_i and x_i^* such that neither x_i nor x_i^* is a leading letter of an element of $I^{(i+1)}$ or $(I^{(i+1)})^*$.

(iv) Perform division in order of \prec on the entries of $L^{(i)}$ using $I^{(i+1)}$ and $(I^{(i+1)})^*$ to get

$$L^{(i)} = L_I^{(i)} + L_T^{(i)},$$

where the entries of $L_I^{(i)}$ are spanned by $I^{(i+1)} + (I^{(i+1)})^*$ and the entries of $L_T^{(i)}$ are in $T^{(i+1)}$.

(v) Compute the space $\mathcal{N}^{(i+1)} \subseteq \mathbb{R}^{\nu^{(i)}}$ defined as

$$\mathcal{N}^{(i+1)} = \{n \in \mathbb{R}^{\nu^{(i)}} \mid L_T^{(i)} n = 0\}.$$

Define $\nu^{(i+1)}$ to be the dimension of $(\mathcal{N}^{(i+1)})^\perp$.

(vi) If $\nu^{(i+1)} = 0$ then stop and output $I^{(i+1)}$ and $\tilde{L} = 1 \in \mathbb{R}^{1 \times 1} \langle x, x^* \rangle$.

(vii) Otherwise, let $\{\xi_1^{(i+1)}, \dots, \xi_{\nu^{(i+1)}}^{(i+1)}\} \subseteq \mathbb{R}^{\nu^{(i)}}$ be a basis for $(\mathcal{N}^{(i+1)})^\perp$. Let $L^{(i+1)}$ be defined by

$$L^{(i+1)} := \begin{pmatrix} \xi_1^{(i+1)} & \dots & \xi_{\nu^{(i+1)}}^{(i+1)} \end{pmatrix}^* L_T^{(i)} \begin{pmatrix} \xi_1^{(i+1)} & \dots & \xi_{\nu^{(i+1)}}^{(i+1)} \end{pmatrix}.$$

(viii) Let $i := i + 1$ and go to (4).

(c) If (7.2) has no nonzero solution $A^{(i)} \succeq 0$ and $c^{(i)} \geq 0$, then stop and output $I^{(i)}$ and $L^{(i)}$.

7.2.2. Properties of the L-Real Radical Algorithm for ${}^{(L)}\sqrt{\{0\}}$.

Proposition 7.3. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$ be a linear pencil. The L-Real Radical algorithm for ${}^{(L)}\sqrt{\{0\}}$ in §7.2.1 has the following properties.*

- (1) *The algorithm terminates in a finite number of steps.*
- (2) *The only polynomials involved in the algorithm have degree ≤ 1 .*
- (3) *The algorithm outputs a space of linear polynomials which generate ${}^{(L, \mathbb{R})}\sqrt{\{0\}}$ and, sequentially, ${}^{(L)}\sqrt{\{0\}}$.*
- (4) *The algorithm also outputs a linear pencil \tilde{L} such that $L(X) \succeq 0$ if and only if $\tilde{L}(X) \succeq 0$ and $\iota(X) = 0$ for each $\iota \in {}^{(L)}\sqrt{\{0\}}$. Further, there exists a real scalar solution $x \in \mathbb{R}^g$ to the linear matrix equality $\tilde{L}(x) \succ 0$.*

Proof. First, it is clear that the algorithm only involves linear polynomials. Next, the algorithm stops at step (2) if and only if $L = 0$. In this case, it is trivial to see that ${}^{(0, \mathbb{R})}\sqrt{\{0\}} = \{0\}$, and so the algorithm outputs $\{0\}$ which generates ${}^{(0, \mathbb{R})}\sqrt{\{0\}}$. Also note that $\tilde{L} = 1$ is always positive definite.

Given an index i , assume inductively that $I^{(i)} \subseteq ({}^{(L, \mathbb{R})}\sqrt{\{0\}})_1$, that

$$(7.5) \quad \mathbb{R}\langle x, x^* \rangle_1 = \text{span}(I^{(i)} + [I^{(i)}]^*) \oplus \text{span } T^{(i)},$$

and that for $i > 0$, the set $I^{(i)}$ is a reduced left Gröbner basis for ${}^{(L, \mathbb{R})}\sqrt{\{0\}}$. Also assume $L^{(i)}$ is of the form

$$(7.6) \quad L^{(i)} = (C^{(i)})(L - L_I^{(i)})(C^{(i)})^*,$$

where $L_I^{(i)} \in \mathbb{R}^{\nu \times \nu} \otimes (I^{(i)} + [I^{(i)}]^*)$ so that $L - L_I^{(i)} \in \mathbb{R}^{\nu \times \nu} \otimes T^{(i)}$, and $C^{(i)}$ is a matrix whose columns are a basis for the space $(\mathcal{Z}^{(i)})^\perp$ defined by

$$\mathcal{Z}^{(i)} := \{n \in \mathbb{R}^\nu \mid (L - L_I^{(i)})n = 0\}.$$

Suppose there is a nonzero solution to (7.2). Decompose $A^{(i)}$ as $A^{(i)} = (U^{(i)})^* \Lambda (U^{(i)})$, where Λ is a diagonal matrix with entries $\psi_1^{(i)}, \dots, \psi_{\nu^{(i)}}^{(i)}$ and $U^{(i)} \in \mathbb{R}^{\nu^{(i)} \times \nu^{(i)}}$ is an orthogonal matrix. Then

$$A^{(i)} = \sum_{j=1}^{\nu^{(i)}} \left(a_j^{(i)} \right) \left(a_j^{(i)} \right)^*,$$

where $a_j^{(i)} = \sqrt{\xi_j^{(i)}} U^* e_j^* \in \mathbb{R}^{\nu^{(i)}}$. Hence (7.2) implies that

$$c^{(i)} + \sum_{j=1}^{\nu^{(i)}} \left(a_j^{(i)} \right)^* L^{(i)} \left(a_j^{(i)} \right) = 0.$$

Therefore,

$$c^{(i)} + \sum_{j=1}^{\nu^{(i)}} \left([C^{(i)}]^* a_j^{(i)} \right)^* L \left([C^{(i)}]^* a_j^{(i)} \right) \in ({}^{(L, \mathbb{R})}\sqrt{\{0\}} + ({}^{(L, \mathbb{R})}\sqrt{\{0\}})^*,$$

since $I^{(i)} \subseteq ({}^{(L, \mathbb{R})}\sqrt{\{0\}})$. This implies that each $L^{(i)} [C^{(i)}]^* a_j^{(i)} \in ({}^{(L, \mathbb{R})}\sqrt{\{0\}})$ and $\sqrt{c^{(i)}} \in ({}^{(L, \mathbb{R})}\sqrt{\{0\}})$. If $c^{(i)} > 0$, then this implies that $({}^{(L, \mathbb{R})}\sqrt{\{0\}}) = \mathbb{R}\langle x, x^* \rangle$ so that the algorithm outputs $\{1\}$, a generating set for $({}^{(L, \mathbb{R})}\sqrt{\{0\}})$, and $\tilde{L} = 1$, and the condition that $\iota(X) = 0$ for each $\iota \in ({}^{(L, \mathbb{R})}\sqrt{\{0\}})$ is infeasible. If $c^{(i)} = 0$ but $A^{(i)} \neq 0$, then since not all of the $a_j^{(i)}$ are 0, it follows that each nonzero $L^{(i)} a_j^{(i)} \in ({}^{(L, \mathbb{R})}\sqrt{\{0\}})$. Therefore,

$$L^{(i)} \sqrt{A^{(i)}} = L^{(i)} \begin{pmatrix} a_1^{(i)} & \dots & a_{\nu^{(i)}}^{(i)} \end{pmatrix} U$$

has entries in $({}^{(L, \mathbb{R})}\sqrt{\{0\}}) \setminus I^{(i)}$.

Given $I^{(i+1)}$, we find $T^{(i+1)}$ which satisfies the necessary assumptions given above. Further, if $L_T^{(i)} = 0$, then we see that if $\iota(X) = 0$ for each $\iota \in I^{(i+1)}$, then $L(X) = 0$. In this case, $L(X) \succeq 0$ if and only if $\iota(X) = 0$ for each $\iota \in I^{(i+1)}$. Therefore ${}^{(L, \mathbb{R})}\sqrt{\{0\}}$ is generated by $I^{(i+1)}$, which we output, and we choose $\tilde{L} = 1$, which is always positive definite. If $L_T^{(i)} \neq 0$, then we have $L^{(i+1)}$ of the form $(C^{(i+1)})^*(L - L_I^{(i+1)})(C^{(i+1)})$, which satisfies the assumptions given above.

This algorithm must terminate in a finite number of iterations since at each iteration we add some linear polynomials to $I^{(i)}$ to get $\mathbb{R}\langle x, x^* \rangle I^{(i)} \subsetneq \mathbb{R}\langle x, x^* \rangle I^{(i+1)} \subseteq {}^{(L, \mathbb{R})}\sqrt{\{0\}}$. At the end, there is no nonzero solution to (7.2). Therefore, if

$$\sum_j^{\text{finite}} p_j^* p_j + \sum_k^{\text{finite}} q_k^* L q_k \in (\mathbb{R}\langle x, x^* \rangle I^{(i)}) + (\mathbb{R}\langle x, x^* \rangle I^{(i)})^*,$$

since the entries of $L^{(i)}$ are in $T^{(i)}$,

$$\sum_j^{\text{finite}} p_j^* p_j + \sum_k^{\text{finite}} (C^{(i)} q_k)^* L^{(i)} (C^{(i)} q_k) = 0.$$

Hence

$$\text{Tr} \left(L^{(i)} \left[\sum_k^{\text{finite}} (C^{(i)} q_k)(C^{(i)} q_k)^* \right] \right) + \left(\sum_j^{\text{finite}} p_j^* p_j \right) = 0,$$

which implies, since there is no nonzero solution to (7.2), that each $C^{(i)} q_k = 0$ and each $p_j = 0$. Therefore, each $L q_k = L^{(i)} q_k + L_I^{(i)} C^{(i)} q_k \in \mathbb{R}^{\nu^{(i)} \times 1} I^{(i)}$. This implies that $I^{(i)}$ is (L, \mathbb{R}) -real. Since $I^{(i)} \subseteq {}^{(L, \mathbb{R})}\sqrt{\{0\}}$, this implies that $I^{(i)} = {}^{(L, \mathbb{R})}\sqrt{\{0\}}$.

Finally, Proposition 6.7 implies, given the construction of $L^{(i)}$, that the outputted $\tilde{L} = L^{(i)}$ satisfies all the properties given in (4). \blacksquare

7.3. Examples. Here are some examples of linear pencils L and their real radicals ${}^{(L, \mathbb{R})}\sqrt{\{0\}}$.

Example 7.4. Let L be the pencil

$$L = \begin{pmatrix} 1 & x_1 \\ x_1^* & 1 \end{pmatrix}.$$

Note that

$$\mathcal{D}_L = \{X_1 \mid \|X_1\| \leq 1\},$$

and $L(X) \succ 0$ iff $\|X_1\| < 1$. Proposition 6.1 implies that ${}^{(L)}\sqrt{\{0\}} = \{0\}$. Proposition 2.3 also implies that ${}^{(L)}\sqrt{\{0\}} = \{0\}$. Therefore we expect the L -Real Radical algorithm to output $\{0\}$ and L .

We now run the algorithm. First, we see that $Ln \neq 0$ for any $n \in \mathbb{R}^2 \setminus \{0\}$ since $L(0) \succ 0$, whence $L^{(0)} = L$. Next, we see that if $A^{(0)} = (a_{jk})_{1 \leq j, k \leq 2}$, then

$$\text{Tr}(L^{(0)}A^{(0)}) = a_{11} + a_{22} + a_{12}x_1 + a_{21}x_1^*.$$

For this to be constant, we need $a_{12} = a_{21} = 0$. Next, if $A^{(0)} \succeq 0$, then $a_{11}, a_{22} \geq 0$. Hence there is no nonzero solution to (7.2). We therefore stop and output $\{0\}$ and $\tilde{L} = L$.

Example 7.5. Let L be the pencil

$$L = \begin{pmatrix} 1 & x_1 \\ x_1^* & 0 \end{pmatrix}.$$

Note that since there is a 0 on the diagonal, we have

$$\mathcal{D}_L = \{X_1 \mid X_1 = 0\}.$$

For the algorithm, we first see that $Ln = 0$ has no nonzero solution in \mathbb{R}^2 . Therefore $L^{(0)} = L$. Next, we see that if $A^{(0)} = E_{22} \succeq 0$, then

$$\text{Tr}(L^{(0)}A^{(0)}) = 0.$$

Since $\sqrt{A^{(0)}} = E_{22}$, we see

$$L^{(0)}\sqrt{A^{(0)}} = \begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix},$$

so $I^{(1)} = \mathbb{R}x_1$. We decompose $\mathbb{R}\langle x, x^* \rangle_1$ as

$$\mathbb{R}\langle x, x^* \rangle_1 = (I^{(1)} + [I^{(1)}]^*) \oplus T^{(1)} \quad \text{with} \quad T^{(1)} := \mathbb{R}.$$

When we project L onto $\mathbb{R}^{2 \times 2} \otimes T^{(1)}$ we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E_{11}.$$

The nullspace of E_{11} is $\mathcal{N}^{(1)} = \mathbb{R}e_2$. The compression of E_{11} onto the space $([\mathcal{N}^{(1)}]^\perp)^*([\mathcal{N}^{(1)}]^\perp)$ is $L^{(1)} = (1) \in \mathbb{R}^{1 \times 1}\langle x, x^* \rangle$.

There is no nonzero solution to

$$\text{Tr}(L^{(1)}A^{(1)}) + c = 0,$$

so we output ${}^{(L, \mathbb{R})}\sqrt{\{0\}} = \{x_1\}$ and $\tilde{L} = (1)$.

Example 7.6. Let L be the pencil

$$L = \begin{pmatrix} x_1 + x_1^* & 1 \\ 1 & 0 \end{pmatrix}.$$

This pencil L is clearly infeasible, i.e., $\mathcal{D}_L = \emptyset$.

Applying the algorithm, we get $L^{(0)} = L$, and we see that

$$\mathrm{Tr}(L^{(0)}E_{22}) = 0.$$

Next,

$$L^{(0)}\sqrt{E_{22}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so we add 1 to $I^{(0)}$ to get $I^{(1)} = \mathbb{R}\langle x, x^* \rangle_1$. Output now $I^{(1)}$ and $\tilde{L} = 1$.

Example 7.7. This is a version of [KS13, Example 4.6.3] presented in free non-symmetric variables. Let L be the pencil

$$L = \begin{pmatrix} 0 & x_1 & 0 \\ x_1^* & x_2 + x_2^* & 1 \\ 0 & 1 & x_1 + x_1^* \end{pmatrix}$$

Applying the algorithm, we get $L^{(0)} = L$ and thus

$$\mathrm{Tr}(L^{(0)}E_{11}) = 0.$$

Next,

$$L^{(0)}\sqrt{E_{11}} = \begin{pmatrix} 0 & 0 & 0 \\ x_1^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore we add x_1^* to $I^{(0)}$ to get $I^{(1)} = \mathbb{R}\langle x, x^* \rangle_{x_1^*}$. This leads to

$$L^{(1)} = \begin{pmatrix} x_2 + x_2^* & 1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$\mathrm{Tr}(L^{(1)}E_{22}) = 0,$$

so we see

$$L^{(1)}\sqrt{E_{22}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We thus add 1 to $I^{(1)}$ to obtain $I^{(2)} = \mathbb{R}\langle x, x^* \rangle$. Hence L is infeasible.

Example 7.8. Let L be the pencil

$$L = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ x_1^* & 1 & 0 & 0 \\ x_2^* & 0 & 1 & 0 \\ x_3^* & 0 & 0 & 0 \end{pmatrix}.$$

The pencil L has no nullspace, so $L^{(0)} = L$. We see that $\text{Tr}(L^{(0)}E_{44}) = 0$ and

$$L^{(0)}\sqrt{E_{44}} = \begin{pmatrix} 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore $I^{(1)} = \mathbb{R}x_3$. We then get

$$\mathbb{R}\langle x, x^* \rangle_1 = \left(I^{(1)} + [I^{(1)}]^* \right) \oplus T^{(1)} \quad \text{with} \quad T^{(1)} := \mathbb{R} + \sum_{j=1}^2 \mathbb{R}x_j + \sum_{j=1}^2 \mathbb{R}x_j^*.$$

When we project $L^{(0)}$ onto $T^{(1)}$ we get

$$\begin{pmatrix} 1 & x_1 & x_2 & 0 \\ x_1^* & 1 & 0 & 0 \\ x_2^* & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix has a null space $\mathcal{N}^{(1)} = \mathbb{R}e_4$, so we obtain

$$L^{(1)} = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1^* & 1 & 0 \\ x_2^* & 0 & 1 \end{pmatrix}.$$

This pencil has non-empty interior, so $\tilde{L} = L^{(1)}$. Geometrically, we see that the set $L(x) \succeq 0$ is the two-dimensional spectrahedron defined by \tilde{L} , which is the closed ball

$$\{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1\}.$$

Example 7.9. As our final example we present a classical example of a spectrahedron used in mathematical optimization to construct a semidefinite program (SDP) with nonzero duality gap, cf. [KS13, Example 4.6.4]. Let L be the pencil

$$L = \begin{pmatrix} \alpha + x_2 + x_2^* & 0 & 0 \\ 0 & x_1 + x_1^* & x_2 \\ 0 & x_2^* & 0 \end{pmatrix}$$

for some $\alpha > 0$.

Applying the algorithm, we get $L^{(0)} = L$ and thus

$$\text{Tr}(L^{(0)}E_{33}) = 0.$$

Next,

$$L^{(0)}\sqrt{E_{33}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore we add x_2 to $I^{(0)}$ to get $I^{(1)} = \mathbb{R}\langle x, x^* \rangle x_2$. We decompose $\mathbb{R}\langle x, x^* \rangle_1$ as

$$\mathbb{R}\langle x, x^* \rangle_1 = (I^{(1)} + [I^{(1)}]^*) \oplus T^{(1)} \quad \text{with} \quad T^{(1)} := \mathbb{R} + \mathbb{R}x_1 + \mathbb{R}x_1^*.$$

Projecting L onto $\mathbb{R}^{2 \times 2} \otimes T^{(1)}$ yields

$$L^{(1)} = \begin{pmatrix} \alpha & 0 \\ 0 & x_1 + x_1^* \end{pmatrix}.$$

There is no nonzero solution to (7.2). We therefore stop and output $I^{(1)}$ and $\tilde{L} = L^{(1)}$.

7.4. \mathfrak{C} -Bases. For right chip spaces $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$, we would ideally like to find an order on $\mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ satisfying $a \prec b$ whenever $a \in \mathfrak{C}$ and $b \notin \mathfrak{C}$. However, as it turns out right chip space \mathfrak{C} rarely admit left admissible orders (as defined previously in §7.1) with this property. Therefore, we discuss \mathfrak{C} -orders, which were introduced in [Nel].

Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a right chip space. Let \prec_0 be a degree order on $\langle x, x^* \rangle$, that is, \prec_0 is a total order on $\langle x, x^* \rangle$ satisfying $a \prec_0 b$ whenever $|a| < |b|$. We say that $\prec_{\mathfrak{C}}$ is a **\mathfrak{C} -order (induced by \prec_0)** if $\prec_{\mathfrak{C}}$ is a total order on $\mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ such that if $a, b \in \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$, then $a \prec_{\mathfrak{C}} b$ if one of the following hold:

- (1) $a \in \mathfrak{C}$ and $b \notin \mathfrak{C}$;
- (2) $a \in \mathbb{R}\langle x, x^* \rangle \mathfrak{C}$ and $b \notin \mathbb{R}\langle x, x^* \rangle \mathfrak{C}$;
- (3) $a = a_1 a_2$, $b = b_1 b_2$, where $a_2, b_2 \in \mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C} \setminus \mathfrak{C}$, $a_1, b_1 \in \langle x, x^* \rangle$, and $a_1 \prec_0 b_1$;
- (4) $a = w a_2$, $b = w b_2$, where $a_2, b_2 \in \mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C} \setminus \mathfrak{C}$, $w \in \langle x, x^* \rangle$, and $a_2 \prec_{\mathfrak{C}} b_2$.

The above conditions in and of themselves only define a partial order. By definition, a \mathfrak{C} order $\prec_{\mathfrak{C}}$ is defined in some way among the elements of \mathfrak{C} , $\mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C} \setminus \mathfrak{C}$, and $\mathbb{R}^{1 \times \ell} \setminus \mathbb{R}\langle x, x^* \rangle \mathfrak{C}$ respectively to make it a total order.

Further, let $I \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a left module generated by polynomials in $\mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C}$. We say that a pair of sets $(\{\iota_i\}_{i \in A}, \{\vartheta_j\}_{j \in B})$ is a **\mathfrak{C} -basis** for I if $\{\iota_i\}_{i \in A}$ is a maximal set of monic polynomials in $I \cap (\mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C} \setminus \mathfrak{C})$ with distinct leading monomials and if $\{\vartheta_j\}_{j \in B}$ is a maximal (possibly empty) set of monic polynomials in $I \cap \mathfrak{C}$ with distinct leading monomials.

Here is a useful property of \mathfrak{C} -bases.

Lemma 7.10 ([Nel, Lemma 3.4]). *Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a finite right chip space and let $\prec_{\mathfrak{C}}$ be a \mathfrak{C} -order induced by some degree order. Let $I \subseteq \mathbb{R}^{1 \times \ell}\langle x, x^* \rangle$ be a left module generated by*

polynomials in $\mathbb{R}\langle x, x^* \rangle_1 \mathfrak{C}$, and let $(\{\iota_i\}_{i=1}^\mu, \{\vartheta_j\}_{j=1}^\sigma)$ be a \mathfrak{C} -basis for I . Then each element of I can be represented uniquely as

$$(7.7) \quad \sum_{i=1}^{\mu} p_i \iota_i + \sum_{j=1}^{\sigma} \alpha_j \vartheta_j,$$

where each $p_i \in \mathbb{R}\langle x, x^* \rangle$ and $\alpha_j \in \mathbb{R}$.

Conversely, any pair of sets of monic polynomials $(\{\iota_i\}_{i=1}^\mu, \{\vartheta_j\}_{j=1}^\sigma)$ with distinct leading monomials such that any element of I can be expressed in the form (7.7) is a \mathfrak{C} -basis for I .

7.5. The L -Real Radical Algorithm for ${}^{(L, \mathfrak{C})}\sqrt{I}$. Here is an algorithm for the more general real radical ${}^{(L, \mathfrak{C})}\sqrt{I}$, where $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle_\sigma$ is any symmetric polynomial, $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is some finite right chip space, and $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ is a left module. When a generating set $\{p_1, \dots, p_\mu\}$ for I is inputted into the following algorithm, it outputs a \mathfrak{C} -basis for ${}^{(L, \mathfrak{C})}\sqrt{I}$.

- (1) Fix some \mathfrak{C} -order on $\mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$. Compute a \mathfrak{C} -basis for I , and let $I^{(0)}$ be the outputted pair of sets.
- (2) Let $i = 0$.
- (3) Let $T^{(i)} \subseteq \mathfrak{C}$ be set of all monomials in \mathfrak{C} which are not the leading monomial of an element in $I^{(i)}$.
- (4) For the polynomial $\theta^{(i)}$,

$$\theta^{(i)} = L \left(\sum_{m \in \mathfrak{C}} \alpha_m m \right),$$

where the α_m are ν -dimensional column vector variables, use the \mathfrak{C} -basis to solve for the space of α such that $\theta^{(i)}$ is in the left module generated by $I^{(i)}$. Using this solution, let $J^{(i)}$ be a basis for the following space:

$$\{\theta \in \mathbb{R}^{\nu \times 1} T^{(i)} \mid L\theta \in I^{(i)}\}.$$

Let $K^{(i)} \subseteq \mathbb{R}^{\nu \times 1} T^{(i)}$ be a maximal set of linearly independent polynomials not in $J^{(i)}$.

- (5) Let $\tau^{(i)} = (\tau_j^{(i)})_{1 \leq j \leq \pi^{(i)}}$ be a vector whose entries are the elements of $T^{(i)}$, and let $\kappa^{(i)} = (\kappa_j^{(i)})_{1 \leq j \leq \rho^{(i)}}$ be a vector whose entries are the elements of $K^{(i)}$. Define $L\kappa^{(i)} = (L\kappa_j^{(i)})_{1 \leq j \leq \rho}$.
- (6) Let $I^{(i)} = (\{\iota_j^{(i)}\}_{j=1}^{\mu^{(i)}}, \{\vartheta_j^{(i)}\}_{j=1}^{\sigma^{(i)}})$. For $1 \leq j \leq \mu^{(i)}$, let $s_j^{(i)} \in \mathfrak{C}$ be defined as

$$s_j^{(i)} := \sum_{c \in \mathfrak{C}} \gamma_{c,j}^{(i)} c,$$

where the $\gamma_{c,j}^{(i)}$ are real-valued variables. For $1 \leq j \leq \sigma^{(i)}$, let $\alpha_j^{(i)} \in \mathfrak{C}$ be

$$\alpha_j^{(i)} = \sum_{k \in \Gamma(\mathfrak{C})} \xi_{k,j}^{(i)} e_k \otimes 1,$$

where the $\xi_{k,j}^{(i)}$ are real-valued variables. Consider the problem of finding $A^{(i)}, B^{(i)} \succeq 0$ such that

$$(7.8) \quad (\tau^{(i)})^* A^{(i)} (\tau^{(i)}) + (\kappa^{(i)})^* B^{(i)} (L\kappa^{(i)}) = \sum_{j=1}^{\mu^{(i)}} ([s_j^{(i)}]^* \iota_j^{(i)} + [\iota_j^{(i)}]^* [s_j^{(i)}]) \\ = \sum_{j=1}^{\sigma^{(i)}} ([\alpha_j^{(i)}]^* \vartheta_j^{(i)} + [\vartheta_j^{(i)}]^* \alpha_j^{(i)})$$

for some values of $\gamma_{c,j}^{(i)}, \xi_{k,j}^{(i)} \in \mathbb{R}$. This can be solved by an algorithm similar to the SOS algorithm in [Nel, §9.1]; see §7.5.1 below.

(7) If (7.8) has a nonzero solution, then let $\iota_j^{(i)}$ and $\zeta_{(j,k)}^{(i)}$ be defined by

$$(7.9) \quad \begin{pmatrix} \iota_1^{(i)} \\ \vdots \\ \iota_{\sigma^{(i)}}^{(i)} \end{pmatrix} = \sqrt{A^{(i)}} \tau^{(i)} \\ \begin{pmatrix} \zeta_{(1,1)}^{(i)} & \zeta_{(1,2)}^{(i)} & \cdots & \zeta_{(1,\rho^{(i)})}^{(i)} \\ \zeta_{(2,1)}^{(i)} & \zeta_{(2,2)}^{(i)} & \cdots & \zeta_{(2,\rho^{(i)})}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{(\rho^{(i),1})}^{(i)} & \zeta_{(\rho^{(i),2})}^{(i)} & \cdots & \zeta_{(\rho^{(i),\rho^{(i)})}^{(i)}} \end{pmatrix} = \sqrt{B^{(i)}} L\kappa^{(i)}.$$

Let $I^{(i+1)}$ be the \mathfrak{C} -basis generated by the set

$$I^{(i)} \cup \{\iota_j\}_j^{\sigma^{(i)}} \cup \{\zeta_{(j,k)}^{(i)}\}_{j,k=1}^{\rho^{(i)}}.$$

Let $T^{(i+1)}$ be the space spanned by all monomials in $T^{(i)}$ which are not the leading monomial of an element of $I^{(i+1)}$. Set $i := i + 1$ and go to (6).

(8) If (7.8) has no nonzero solution, then stop and output $I^{(i)}$.

7.5.1. Modified SOS Algorithm. We now explain in detail how to solve the problem given in Step (6) of the above algorithm.

(a) Let $\mathcal{Z}^{(i)}$ be the space

$$\mathcal{Z}^{(i)} = \{(Z_\tau, Z_\kappa) \in \mathbb{S}^{\pi^{(i)}} \times \mathbb{S}^{\rho^{(i)}} \mid (\tau^{(i)})^* Z_\tau (\tau^{(i)}) + (\kappa^{(i)})^* Z_\kappa (\kappa^{(i)}) = 0\},$$

and let $(Z_{i,1,\tau}, Z_{i,1,\kappa}), \dots, (Z_{i,n^{(i)},\tau}, Z_{i,n^{(i)},\kappa})$ be a basis for $\mathcal{Z}^{(i)}$.

(b) Express the right hand side of (7.8) as

$$\begin{aligned} \sum_{c \in \mathfrak{C}} \sum_{j=1}^{\mu^{(i)}} \gamma_{c,j}^{(i)} ([\tau^{(i)}]^* F_{c,i,j,\tau} [\tau^{(i)}] + [\kappa^{(i)}]^* F_{c,i,j,\kappa} [\kappa^{(i)}]) \\ + \sum_{k \in \Gamma(\mathfrak{C})} \sum_{j=1}^{\sigma^{(i)}} \xi_{k,j}^{(i)} ([\tau^{(i)}]^* H_{k,i,j,\tau} [\tau^{(i)}] + [\kappa^{(i)}]^* H_{k,i,j,\kappa} [\kappa^{(i)}]), \end{aligned}$$

for some symmetric matrices $F_{c,i,j,\tau}$, $F_{c,i,j,\kappa}$, $H_{k,i,j,\tau}$, $H_{k,i,j,\kappa}$.

(c) If the linear pencil

$$(7.10) \quad \begin{aligned} L_i(\alpha^{(i)}, \gamma^{(i)}, \kappa^{(i)}) &= \sum_{j=1}^{n^{(i)}} \alpha^{(i)} (Z_{i,j,\tau} \oplus Z_{i,j,\kappa}) + \sum_{c,j} \gamma_{c,j} (F_{c,i,j,\tau} \oplus F_{c,i,j,\kappa}) \\ &+ \sum_{k,j} \xi_{k,j} (H_{k,i,j,\tau} \oplus H_{k,i,j,\kappa}) \end{aligned}$$

contains any 0 on its diagonal, set all entries in the row and column corresponding to the 0 diagonal entry to be 0. Use the resulting linear equations to reduce the number of variables. Repeat this step until there are no diagonal entries equal to 0.

(d) If we eventually get $L_i = 0$, stop and output that there is no nonzero solution.

(e) Solve the linear matrix inequality

$$L_i(\alpha^{(i)}, \gamma^{(i)}, \kappa^{(i)}) \succeq 0$$

to see if there is a nonzero solution $(\alpha^{(i)}, \gamma^{(i)}, \kappa^{(i)})$.

(f) If there is not, stop and output that there is no nonzero solution.

(g) Otherwise, output the obtained solution.

7.5.2. Properties of the L -Real Radical Algorithm for ${}^{(L,\mathfrak{C})}\sqrt{I}$.

Proposition 7.11. *Let $L \in \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle_{\sigma}$ be a symmetric polynomial, let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a finite right chip space, and let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be a left module. The L -Real Radical algorithm for ${}^{(L,\mathfrak{C})}\sqrt{I}$ in §7.5 has the following properties.*

- (1) *The algorithm terminates in a finite number of steps.*
- (2) *If I is generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$, then the algorithm involves only computations on polynomials in $\mathbb{R} \langle x, x^* \rangle_{\sigma} \mathfrak{C}$.*
- (3) *The algorithm outputs a \mathfrak{C} -basis for ${}^{(L,\mathfrak{C})}\sqrt{I}$.*

Proof. Given an index i , assume inductively that $I \subseteq \mathbb{R} \langle x, x^* \rangle I^{(i)} \subseteq {}^{(L,\mathfrak{C})}\sqrt{I}$, that $I^{(i)}$ is a \mathfrak{C} -basis, and that

$$(7.11) \quad \mathfrak{C} = (\text{span } I^{(i)} \cap \mathfrak{C}) \oplus \text{span } T^{(i)}.$$

For $i = 0$, this is true by construction since every monomial in \mathfrak{C} is either the leading monomial of an element in $I^{(0)}$ —in which case it is the leading monomial of an element of $I^{(0)} \cap \mathfrak{C}$ —or it is in $T^{(0)}$. We compute $J^{(i)}$ and $K^{(i)}$ so that, by Lemma 3.4, the left module generated by $I^{(i)}$ is (L, \mathfrak{C}) -real if and only if whenever

$$(7.12) \quad \sum_r^{\text{finite}} \tau_r^* \tau_r + \sum_j^{\text{finite}} \kappa_j^* L \kappa_j \in \mathbb{R}^{\ell \times 1} \langle x, x^* \rangle I^{(i)} + (I^{(i)})^* \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$$

for $\tau_r \in \text{span } T^{(i)}$ and $\kappa_j \in \text{span } K^{(i)}$, then each $\tau_r = 0$ and each $\kappa_j = 0$.

If (7.8) has a nonzero solution, then we see that

$$\sum_{j=1}^{\tau^{(i)}} (\iota_j^{(i)})^* \iota_j^{(i)} + \sum_{r=1}^k \begin{pmatrix} \zeta_{(1,r)}^{(i)} \\ \vdots \\ \zeta_{(\rho^{(i)},r)}^{(i)} \end{pmatrix}^* L \begin{pmatrix} \zeta_{(1,r)}^{(i)} \\ \vdots \\ \zeta_{(\rho^{(i)},r)}^{(i)} \end{pmatrix}$$

is in the space $\mathbb{R}^{\ell \times 1} \langle x, x^* \rangle I^{(i)} + (I^{(i)})^* \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, which implies that the $\iota_j^{(i)}, \zeta_{(j,k)}^{(i)} \in {}^{(L, \mathfrak{C})} \sqrt{I}$.

If (7.8) has no nonzero solution, then it follows that (7.12) holds if and only if each $\tau_r, \kappa_j = 0$. Therefore the left module generated by $I^{(i)}$ is (L, \mathfrak{C}) -real, and since $I \subseteq \mathbb{R} \langle x, x^* \rangle I^{(i)} \subseteq {}^{(L, \mathfrak{C})} \sqrt{I}$, we have that $I^{(i)}$ is a \mathfrak{C} -basis for ${}^{(L, \mathfrak{C})} \sqrt{I}$.

At each iteration of the algorithm, we either stop or we add at least one new polynomial in $\mathbb{R} \langle x, x^* \rangle_\sigma \mathfrak{C}$ to $I^{(i)}$. Therefore the algorithm takes at most $\dim(\mathbb{R} \langle x, x^* \rangle_\sigma \mathfrak{C})$ iterations. Also, if I is generated by polynomials in $\mathbb{R} \langle x, x^* \rangle_\sigma \mathfrak{C}$, then the generating set of each $I^{(i)}$ is made up of polynomials in $\mathbb{R} \langle x, x^* \rangle_\sigma \mathfrak{C}$. \blacksquare

7.6. Verifying if a Polynomial is Positive on a Spectrahedron. Given a left module $I \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$, a finite right chip space $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ and a linear pencil $L \subseteq \mathbb{R}^{\nu \times \nu} \langle x, x^* \rangle$, we now show how to algorithmically verify whether a symmetric polynomial $p \in \mathfrak{C}^* \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$ is of the form (1.5). In particular, by Theorem 1.9, this tells us whether p is positive where L is positive and each $\iota \in I$ vanishes.

7.6.1. Algorithm.

- (1) Compute a \mathfrak{C} -basis for ${}^{(L, \mathfrak{C})} \sqrt{I}$. Let $\tilde{\iota}$ be a vector whose entries are all polynomials in the \mathfrak{C} -basis.
- (2) Let c be a vector whose entries are all monomials in \mathfrak{C} .
- (3) Given a \mathfrak{C} -order, let τ be a vector whose entries are all monomials in \mathfrak{C} which are not divisible on the right by the leading monomial of an element of the \mathfrak{C} -basis for ${}^{(L, \mathfrak{C})} \sqrt{I}$.
- (4) Let $\tilde{\kappa}$ be a vector whose entries are all polynomials of the form $L e_i^* \tau_j$ for some entry τ_j of τ .

(5) Consider the equation

$$(7.13) \quad p = \tau^* A \tau + \tilde{\kappa}^* B \tilde{\kappa} + \tilde{\iota}^* C c + \tilde{\iota}^* C^* c$$

where A, B, C are unknowns. This equation amounts to a series of linear equations in the entries of A, B, C .

(6) The polynomial p is of the form (1.5) if and only if the following linear matrix inequality is feasible:

$$A, B \succeq 0 \quad \text{such that} \quad (7.13) \text{ holds}$$

8. COMPLETELY POSITIVE MAPS IN THE ABSENCE OF INVERTIBLE POSITIVE ELEMENTS

The theory we have developed can be used to strengthen the theory of complete positivity (CP). The theorem at the core of the subject represents a CP map τ between unital subspaces of matrix algebras as $\tau(A) = V^* \phi(A) V$, where ϕ is an isometric isomorphism. This can be thought of as an algebraic certificate for CP, and it is gotten by combining the Arveson extension theorem with the Stinespring representation theorem. In this section we give algebraic certificates for CP maps between nonunital subspaces of matrix algebras.

8.1. Completely Positive Maps. A subspace $\mathfrak{A} \subseteq \mathbb{R}^{\nu \times \nu}$ closed under the transpose will be called a **(nonunital) operator system**. We write $\mathbb{S}(\mathfrak{A})$ for the set of all symmetric elements $A = A^* \in \mathfrak{A}$, and $\mathbb{K}(\mathfrak{A}) = \{A \in \mathfrak{A} \mid A^* = -A\}$ denotes the skew-symmetric elements of \mathfrak{A} . Furthermore,

$$\mathfrak{A}_{\succeq 0} = \{A \in \mathbb{S}(\mathfrak{A}) \mid A \succeq 0\}.$$

Lemma 8.1. *We have*

$$\mathfrak{A} = \mathbb{S}(\mathfrak{A}) \oplus \mathbb{K}(\mathfrak{A}).$$

Proof. Observe that

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2}. \quad \blacksquare$$

Let $\mathfrak{B} \subseteq \mathbb{R}^{\ell \times \ell}$ be another operator system. A linear $*$ -map $\tau : \mathfrak{A} \rightarrow \mathfrak{B}$ is called **completely positive (CP)** if it is positive (i.e., $0 \preceq A \in \mathfrak{A}$ implies $\tau(A) \succeq 0$) and all its **ampliations** $\tau \otimes \text{Id}_k : M_k(\mathfrak{A}) \rightarrow M_k(\mathfrak{B})$, $k \in \mathbb{N}$, are positive.

Lemma 8.2. *Suppose $\mathfrak{A} \subseteq \mathbb{R}^{\nu \times \nu}$ and $\mathfrak{B} \subseteq \mathbb{R}^{\ell \times \ell}$ are operator systems, and let $\tau : \mathfrak{A} \rightarrow \mathfrak{B}$ be a linear $*$ -map. Suppose $\mathfrak{A}_{\succeq 0} = \{0\}$. Then:*

- (1) $M_k(\mathfrak{A})_{\succeq 0} = \{0\}$ for all $k \in \mathbb{N}$;
- (2) τ is CP.

Proof. If $0 \neq A \in M_k(\mathfrak{A})_{\geq 0}$, then at least one of the diagonal $\nu \times \nu$ blocks A_{jj} of A is positive semidefinite and nonzero, violating $\mathfrak{A}_{\geq 0} = \{0\}$. Item (2) now follows easily. ■

Remark 8.3. For this reason we may restrict our attention to the case of operator systems \mathfrak{A} with nonzero $\mathfrak{A}_{\geq 0}$. We point out that detecting whether $\mathfrak{A}_{\geq 0} = \{0\}$ is easily done using the machinery developed above. Indeed, choose a basis $\{A_1, \dots, A_s\}$ of $\mathbb{S}(\mathfrak{A})$, a basis $\{A_{s+1}, \dots, A_g\}$ of $\mathbb{K}(\mathfrak{A})$, and form the linear pencil

$$L(x) = A_1(x_1 + x_1^*) + \dots + A_s(x_s + x_s^*) + A_{s+1}(x_{s+1} - x_{s+1}^*) + \dots + A_g(x_g - x_g^*).$$

Consider the expanded pencil

$$(8.1) \quad \begin{aligned} \hat{L}(x) &= L(x) \oplus \left(\text{Tr}(L(x)) - 1 \right) \\ &= L(x) \oplus \left(\text{Tr}(A_1)(x_1 + x_1^*) + \dots + \text{Tr}(A_g)(x_s + x_s^*) - 1 \right). \end{aligned}$$

Note that $\mathfrak{A}_{\geq 0} = \{0\}$ iff $L(\mathcal{D}_L) = \{0\}$ iff $\mathcal{D}_{\hat{L}} = \emptyset$ iff $\sqrt[\hat{L}]{\{0\}} = \mathbb{R}\langle x, x^* \rangle$ (by Proposition 6.2), and the last condition is easily detected by the algorithms presented in §7.

Example 8.4. In general not every CP map $\tau : \mathfrak{A} \rightarrow \mathfrak{B}$ extends to a CP map $\hat{\tau} : \mathbb{R}^{\nu \times \nu} \rightarrow \mathbb{R}^{\ell \times \ell}$.

(1) Suppose $\mathfrak{A}_{\geq 0} = \{0\}$, and let $\tau : \mathfrak{A} \rightarrow \mathbb{R}^{\ell \times \ell}$ be any map *not* of the Arveson-Stinespring form (i.e., of the form $X \mapsto \sum_j V_j^* X V_j$). Then τ is CP by Lemma 8.2, but cannot be extended to the full matrix algebra.

(2) For a slightly less trivial example, let $\mathfrak{A} = \begin{pmatrix} 0 & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix} \subseteq \mathbb{R}^{2 \times 2}$, and consider $\tau : \mathfrak{A} \rightarrow \mathbb{R}$

given by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{12} + a_{21}$. Then $\tau(\mathfrak{A}_{\geq 0}) = \{0\}$, so τ is positive. Since it maps into \mathbb{R} , an easy exercise (or see [Pau02]) now shows τ is CP. But τ does not admit an extension to a positive map on $\mathbb{R}^{2 \times 2}$.

8.2. Pencils Associated with Operator Systems. Retain the notation of Lemma 8.2. We assume $\mathfrak{A}_{\geq 0} \neq \{0\}$. As in Remark 8.3 we can select a basis $A_0, \dots, A_{s-1}, A_s, \dots, A_g$ of \mathfrak{A} consisting solely of symmetric and skew symmetric elements. Here, $A_0, \dots, A_{s-1} \in \mathbb{S}(\mathfrak{A})$ and $A_s, \dots, A_g \in \mathbb{K}(\mathfrak{A})$ for $s \in \mathbb{N}$. Let us call this a **symmetric basis** for \mathfrak{A} . To such a basis we associate the linear pencil

$$(8.2) \quad \begin{aligned} L_{\mathfrak{A}}(x) &= A_0 + A_1(x_1 + x_1^*) + \dots + A_{s-1}(x_{s-1} + x_{s-1}^*) \\ &\quad + A_s(x_s - x_s^*) + \dots + A_g(x_g - x_g^*). \end{aligned}$$

Proposition 8.5. *Assume that $A_0 \neq 0$ is a maximum rank positive semidefinite matrix of \mathfrak{A} , and that A_0, \dots, A_{s-1} are pairwise orthogonal, i.e., $\text{Tr}(A_i^* A_j) = 0$. Then $\mathcal{D}_{L_{\mathfrak{A}}}(1)$ is bounded.*

Proof. Without loss of generality we may assume

$$A_0 = \begin{bmatrix} \text{Id}_r & 0 \\ 0 & 0_{\nu-r} \end{bmatrix}$$

for some $1 \leq r \leq \nu$.

Claim. If for some $A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \in \mathfrak{A}$ with $A_{11} \in \mathbb{S}^r$ we have $\langle A_0, A_1 \rangle = 0$ and

$$(8.3) \quad A_0 + \lambda A_1 = \begin{bmatrix} \text{Id}_r + \lambda A_{11} & \lambda A_{12} \\ \lambda A_{12}^* & \lambda A_{22} \end{bmatrix} \succeq 0 \quad \text{for all } \lambda \in \mathbb{R}_{\geq 0},$$

then $A_1 = 0$.

Explanation. Since $\langle A_0, A_1 \rangle = 0$, $\text{Tr}(A_{11}) = 0$. This means that either $A_{11} = 0$ or A_{11} has both positive and negative eigenvalues. In the latter case, fix an eigenvalue $\mu < 0$ of A_{11} . Then for every $\lambda \in \mathbb{R}$ with $\lambda > -\mu^{-1} > 0$, we have that $\text{Id}_r + \lambda A_{11} \not\succeq 0$, contradicting (8.3). So $A_{11} = 0$. If $r = \nu$ we are done. Hence assume $r < \nu$.

Now

$$(8.4) \quad A_0 + \lambda A_1 = \begin{bmatrix} \text{Id}_r & \lambda A_{12} \\ \lambda A_{12}^* & \lambda A_{22} \end{bmatrix} \succeq 0$$

for all $\lambda \in \mathbb{R}_{\geq 0}$. Using Schur complements, (8.4) is equivalent to

$$\lambda A_{22} - \lambda^2 A_{12}^* A_{12} \succeq 0.$$

Hence $A_{22} - \lambda A_{12}^* A_{12} \succeq 0$ for all $\lambda \in \mathbb{R}_{\geq 0}$. Equivalently, $A_{12} = 0$ and $A_{22} \succeq 0$. If $A_{22} \neq 0$, then $0 \preceq A_0 + A_1 \in \mathfrak{A}$, and

$$r = \text{rank}(A_0) < \text{rank}(A_0 + A_1),$$

contradicting the maximality of the rank of A_0 . \square

We now show that $\mathcal{D}_{L_{\mathfrak{A}}}(1)$ is bounded. Assume otherwise. Then there exists a sequence $(x^{(k)})_k$ in \mathbb{R}^{s-1} such that $\|x^{(k)}\| = 1$ for all k , and an increasing sequence $t_k \in \mathbb{R}_{>0}$ tending to ∞ such that $L_{\mathfrak{A}}(t_k x^{(k)}) \succeq 0$. By convexity this implies $t_k x^{(j)} \in \mathcal{D}_{L_{\mathfrak{A}}}$ for all $j \geq k$. Without loss of generality we assume the sequence $(x^{(k)})_k$ converges to a vector $x = (x_1, \dots, x_{s-1}) \in \mathbb{R}^{s-1}$. Clearly, $\|x\| = 1$. For any $t \in \mathbb{R}_{\geq 0}$, $t x^{(k)} \rightarrow t x$, and for k big enough, $t x^{(k)} \in \mathcal{D}_{L_{\mathfrak{A}}}$ by convexity. So x satisfies $L_{\mathfrak{A}}(t x) \succeq 0$ for all $t \in \mathbb{R}_{\geq 0}$. In other words,

$$A_0 + 2t(A_1 x_1 + \dots + A_{s-1} x_{s-1}) \succeq 0$$

for all $t \in \mathbb{R}_{\geq 0}$. But now the claim implies $A_1 x_1 + \cdots + A_{s-1} x_{s-1} = 0$, contradicting the linear independence of the A_j s. \blacksquare

Lemma 8.6. *Let $L(x)$ be as in (8.2), and assume the A_j satisfy the assumptions of Proposition 8.5. Then:*

(1) *if $\Lambda \in \mathbb{R}^{d \times d}$ and $Z \in (\mathbb{R}^{d \times d})^g$, and if*

$$(8.5) \quad S := \Lambda \otimes A_0 + (Z_1 + Z_1^*) \otimes A_1 + \cdots + (Z_{s-1} + Z_{s-1}^*) \otimes A_{s-1} \\ + (Z_s - Z_s^*) \otimes A_s + \cdots + (Z_g - Z_g^*) \otimes A_g$$

is symmetric, then $\Lambda = \Lambda^$;*

(2) *if $S \succeq 0$, then $\Lambda \succeq 0$.*

Proof. To prove item (1), suppose S is symmetric. Then $0 = S - S^* = (\Lambda - \Lambda^*) \otimes A_0$. (Here we have used that A_0, \dots, A_{s-1} are symmetric and A_s, \dots, A_g are skew-symmetric.) Since $A_0 \neq 0$, we get $\Lambda = \Lambda^*$.

For (2), if $\Lambda \not\succeq 0$, then there is a unit vector v such that $v^* \Lambda v < 0$. Consider the orthogonal projection $P \in S\mathbb{R}^{d\nu \times d\nu}$ from $\mathbb{R}^d \otimes \mathbb{R}^\nu$ onto $\mathbb{R}v \otimes \mathbb{R}^\nu$, and let $Y = ((v^* Z_i v) P_v)_{i=1}^g \in (\mathbb{S}^d)^g$. Here $P_v \in \mathbb{S}^d$ is the orthogonal projection from \mathbb{R}^d onto $\mathbb{R}v$. Note that $P = P_v \otimes \text{Id}_\nu$. Then the compression

$$PSP = P \left(\Lambda \otimes A_0 + (Z_1 + Z_1^*) \otimes A_1 + \cdots + (Z_{s-1} + Z_{s-1}^*) \otimes A_{s-1} \right. \\ \left. + (Z_s - Z_s^*) \otimes A_s + \cdots + (Z_g - Z_g^*) \otimes A_g \right) P \\ = (v^* \Lambda v) P_v \otimes A_0 + \sum_{i=1}^{s-1} 2Y_i \otimes A_i \succeq 0,$$

which implies that $0 \neq \sum_{i=1}^{s-1} Y_i \otimes A_i \succeq 0$ since $0 \neq A_0 \succeq 0$ and $v^* \Lambda v < 0$. This implies $0 \neq tY \in \mathcal{D}_L$ for all $t > 0$. In particular, the spectrahedron $\mathcal{D}_{\hat{L}}$ of the commutative collapse \hat{L} of L is unbounded. Hence $\mathcal{D}_L(1)$ is unbounded (cf. [KS11, §4.1]) contradicting Proposition 8.5. \blacksquare

8.3. Characterizing Completely Positive Maps. Suppose $\mathfrak{A} \subseteq \mathbb{R}^{\nu \times \nu}$ and $\mathfrak{B} \subseteq \mathbb{R}^{\ell \times \ell}$ are operator systems, and $\tau : \mathfrak{A} \rightarrow \mathfrak{B}$ is a linear $*$ -map. Assume $\mathfrak{A}_{\succeq 0} \neq \{0\}$ and select a basis $A_0, \dots, A_{s-1}, A_s, \dots, A_g$ of \mathfrak{A} consisting of symmetric and skew-symmetric elements and satisfying the assumptions of Proposition 8.5. Consider the linear pencil $L_{\mathfrak{A}}(x)$ given by (8.2), and let

$$L_{\mathfrak{B}}(x) = \tau(A_0) + \tau(A_1)(x_1 + x_1^*) + \cdots + \tau(A_{s-1})(x_{s-1} + x_{s-1}^*) \\ + \tau(A_s)(x_s - x_s^*) + \cdots + \tau(A_g)(x_g - x_g^*).$$

Theorem 8.7. *The following are equivalent:*

- (i) τ is CP;
- (ii) $\mathcal{D}_{L_{\mathfrak{A}}} \subseteq \mathcal{D}_{L_{\mathfrak{B}}}$;
- (iii) $\mathcal{D}_{L_{\mathfrak{A}}}(\ell) \subseteq \mathcal{D}_{L_{\mathfrak{B}}}(\ell)$.

[HKM13, Theorem 3.5] obtained this result for $A_0 \succ 0$, and [KS11, §4] considered (complete) positivity of linear functionals $\tau : \mathfrak{A} \rightarrow \mathbb{R}$.

Proof. The implication (i) \Rightarrow (ii) is obvious. We next prove its converse.

Fix $d \in \mathbb{N}$. Suppose $S \in M_d(\mathfrak{A})$ is positive semidefinite. Then it is of the form (8.5) for some $\Lambda \in \mathbb{R}^{d \times d}$ and $Z \in (\mathbb{R}^{d \times d})^g$. By Lemma 8.6, $\Lambda \succeq 0$. If we replace Λ by $\Lambda + \epsilon I$ for some $\epsilon > 0$, the resulting $S = S_\epsilon$ is still in $M_d(\mathfrak{A})$, so without loss of generality we may assume $\Lambda \succ 0$. Hence,

$$(\Lambda^{-\frac{1}{2}} \otimes I)S(\Lambda^{-\frac{1}{2}} \otimes I) = I \otimes A_0 + \sum_{i=1}^{s-1} \Lambda^{-\frac{1}{2}}(Z_i + Z_i^*)\Lambda^{-\frac{1}{2}} \otimes A_i + \sum_{i=s}^g \Lambda^{-\frac{1}{2}}(Z_i - Z_i^*)\Lambda^{-\frac{1}{2}} \otimes A_i \succeq 0.$$

Since $\mathcal{D}_{L_{\mathfrak{A}}} \subseteq \mathcal{D}_{L_{\mathfrak{B}}}$, this implies

$$I \otimes \tau(A_0) + \sum_{i=1}^{s-1} \Lambda^{-\frac{1}{2}}(Z_i + Z_i^*)\Lambda^{-\frac{1}{2}} \otimes \tau(A_i) + \sum_{i=s}^g \Lambda^{-\frac{1}{2}}(Z_i - Z_i^*)\Lambda^{-\frac{1}{2}} \otimes \tau(A_i) \succeq 0.$$

Multiplying on the left and right by $\Lambda^{\frac{1}{2}} \otimes I$ shows

$$\begin{aligned} \tau(S_\epsilon) &= \Lambda \otimes \tau(A_0) + (Z_1 + Z_1^*) \otimes \tau(A_1) + \cdots + (Z_{s-1} + Z_{s-1}^*) \otimes \tau(A_{s-1}) \\ &\quad + (Z_s - Z_s^*) \otimes \tau(A_s) + \cdots + (Z_g - Z_g^*) \otimes \tau(A_g) \succeq 0. \end{aligned}$$

A straightforward approximation argument now implies that if $S \succeq 0$, then $\tau(S) \succeq 0$ and hence τ is CP. This proves (ii) \Rightarrow (i).

The implication (ii) \Rightarrow (iii) is obvious, and (iii) \Rightarrow (ii) is given in Corollary 4.7. \blacksquare

8.4. Algorithm for Determining Complete Positivity. Given are operator systems $\mathfrak{A} \subseteq \mathbb{R}^{\nu \times \nu}$, $\mathfrak{B} \subseteq \mathbb{R}^{\ell \times \ell}$, and a linear $*$ -map $\tau : \mathfrak{A} \rightarrow \mathfrak{B}$.

- (1) If $\mathfrak{A}_{\succeq 0} = \{0\}$, then τ is CP. Stop.
- (2) Find the maximum rank positive semidefinite $A_0 \in \mathfrak{A}$.
- (3) Compute a basis $A_0, \dots, A_{s-1}, A_s, \dots, A_g$ of \mathfrak{A} consisting only of symmetric and skew-symmetric elements, satisfying the assumptions of Proposition 8.5.
- (4) Form $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$ as in §8.3.
- (5) Is $L_{\mathfrak{B}}|_{\mathcal{D}_{L_{\mathfrak{A}}}} \succeq 0$? If yes, then τ is CP. If not, τ is not CP.

The correctness of this algorithm follows from Theorem 8.7.

Remark 8.8.

- (1) How to implement (1) is explained in Remark 8.3.
- (2) To find a matrix A_0 with maximum possible rank one solves the strictly feasible SDP

$$\lambda^* = \min \{ \lambda : \hat{L}(x) + \lambda I \succeq 0, \lambda \geq 0, x \in \mathbb{R}^g \},$$

where \hat{L} is as in (8.1), using a path-following interior-point method. As shown in [dKTR00, Theorem 5.6.1] (see also [dKl02, §3]), the limit of the central path is maximally complementary, therefore when $\lambda^* = 0$, the solution of this problem will produce A_0 with maximal rank. Note that if $\lambda^* > 0$, then no feasible A_0 exists.

- (3) The algorithmic interpretation §7.6 of Theorem 1.9 enables us to compute a certificate for $L_{\mathfrak{B}}|_{\mathcal{D}_{L_{\mathfrak{A}}}} \succeq 0$, yielding at the same time a certificate for complete positivity of τ .

9. ADAPTING THE THEORY TO SYMMETRIC VARIABLES

In some contexts, it is desirable to work with NC polynomials in symmetric variables. Define $\langle x \rangle$ to be the monoid freely generated by x with identity the empty word, and let $\mathbb{R}\langle x \rangle$ denote the \mathbb{R} -algebra freely generated by $\langle x \rangle$. Define the involution $*$ on $\mathbb{R}\langle x \rangle$ to be linear such that $x_i^* = x_i$ and such that $(pq)^* = q^*p^*$ for each $p, q \in \mathbb{R}\langle x \rangle$. We say that elements of $\mathbb{R}\langle x \rangle$ are NC polynomials in **symmetric variables**. We henceforth refer to polynomials in $\mathbb{R}\langle x, x^* \rangle$ as polynomials in **non-symmetric variables**.

There are direct analogs of the results of this paper to the case of symmetric variables. It turns out that essentially the same proofs given throughout this paper work for symmetric variables. Alternately, some results for symmetric variables can be proved directly from our existing results on non-symmetric variables. In this section, we will prove the analog of Theorem 1.9 for symmetric variables.

Lemma 9.1. *Let $I \subseteq \mathbb{R}^{1 \times \ell}\langle x \rangle$ be a left module and let $L \in \mathbb{R}^{\nu \times \nu}\langle x \rangle$ be a linear pencil. Then*

$$J = \{ p \in \mathbb{R}^{1 \times \ell}\langle x \rangle \mid p(X)v = 0 \text{ whenever } (X, v) \in V(I) \text{ and } L(X) \succeq 0 \}$$

is an L -real left module containing I .

Proof. Suppose

$$\sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j \in \mathbb{R}^{\ell \times 1} J + J^* \mathbb{R}^{1 \times \ell}.$$

Now if $(X, v) \in V(I)$ is such that $L(X) \succeq 0$, then

$$v^* \left(\sum_i^{\text{finite}} p_i(X)^* p_i(X) + \sum_j^{\text{finite}} q_j(X)^* L(X) q_j(X) \right) v = \sum_i^{\text{finite}} \|p_i(X)v\|^2 + \sum_j^{\text{finite}} \|\sqrt{L}(X)q_j(X)\|^2 = 0$$

which implies that each $p_i \in J$ and each $Lq_j \in \mathbb{R}^{\nu \times 1} J$. \blacksquare

For $p \in \mathbb{R}^{a \times b} \langle x, x^* \rangle$ define $\text{Sym}(p) \in \mathbb{R}^{a \times b} \langle x \rangle$ to be the polynomial produced by setting each x_i^* equal to x_i . If $q \in \mathbb{R}^{a \times b} \langle x \rangle$, define $\text{Fr}(q) \in \mathbb{R}^{a \times b} \langle x, x^* \rangle$ to be

$$\text{Fr}(q) = q \left(\frac{1}{2}(x + x^*) \right).$$

Here is the symmetric analog of Theorem 1.9.

Theorem 9.2. *Suppose $L \in \mathbb{R}^{\nu \times \nu} \langle x \rangle$ is a linear pencil. Let $\mathfrak{C} \subseteq \mathbb{R}^{1 \times \ell} \langle x \rangle$ be a finite chip space, let $I \subseteq \mathbb{R}^{1 \times \ell} \langle x \rangle$ be a left $\mathbb{R} \langle x \rangle$ -module generated by polynomials in $\mathbb{R} \langle x \rangle_1 \mathfrak{C}$, and let $p \in \mathfrak{C}^* \mathbb{R} \langle x \rangle_1 \mathfrak{C}$ be a symmetric polynomial.*

(1) $v^* p(X)v \geq 0$ whenever $(X, v) \in V(I)$ and $L(X) \succeq 0$ if and only if p is of the form

$$(9.1) \quad p = \sum_i^{\text{finite}} p_i^* p_i + \sum_j^{\text{finite}} q_j^* L q_j + \sum_k^{\text{finite}} (r_k^* \iota_k + \iota_k^* r_k)$$

where each $p_i, r_k \in \mathfrak{C}$, each $q_j \in \mathbb{R}^{\ell \times 1} \mathfrak{C}$ and each $\iota_k \in {}^{(L, \mathfrak{C})} \sqrt{I} \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$.

(2) $v^* p(X)v \geq 0$ whenever $(X, v) \in V(I)$ and $L(X) \succ 0$, if and only if p is of the form (9.1) where each $p_i, r_k \in \mathfrak{C}$, each $q_j \in \mathbb{R}^{\nu \times 1} \mathfrak{C}$ and each $\iota_k \in {}^{(L, \mathfrak{C})+} \sqrt{I} \cap \mathbb{R} \langle x, x^* \rangle_1 \mathfrak{C}$

Proof. We will only prove (1). The proof of (2) is similar.

If p is of the form (9.1), then Lemma 9.1 implies that $v^* p(X)v \geq 0$ whenever $(X, v) \in I$ and $L(X) \succeq 0$.

Conversely, suppose $v^* p(X)v \geq 0$ whenever $(X, v) \in I$ and $L(X) \succeq 0$. Let $\tilde{\mathfrak{C}} \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be the right chip space spanned by all monomials in $\text{Sym}^{-1}(\mathfrak{C})$. That is, the monomials in $\tilde{\mathfrak{C}}$ have the property that when all of the $*$ are removed, one is left with a monomial in \mathfrak{C} . Therefore, it is easy to see that $\tilde{\mathfrak{C}}$ is finite.

Let $J \subseteq \mathbb{R}^{1 \times \ell} \langle x, x^* \rangle$ be the left module generated by $\text{Sym}^{-1}(I) \cap \mathbb{R} \langle x, x^* \rangle_1 \tilde{\mathfrak{C}}$. Note that $\text{Sym}(J) = I$. Further, if $(Y, v) \in V(J)$, and $X = \frac{1}{2}(Y + Y^*)$, then for each $\iota \in I \cap \mathbb{R} \langle x \rangle_1 \mathfrak{C}$ we see $\iota(X)v = \text{Fr}(\iota)(Y)v = 0$. Also, $\text{Fr}(L)(Y) = L(X)$ and $\text{Fr}(p)(Y) = p(X)$. Therefore $v^* \text{Fr}(p)(Y)v \geq 0$ whenever $(Y, v) \in V(J)$ and $\text{Fr}(L)(Y) \succeq 0$. Theorem 1.9 implies that

$$\text{Fr}(p) = \sum_i^{\text{finite}} a_i^* a_i + \sum_j^{\text{finite}} b_j^* \text{Fr}(L) b_j + \sum_k^{\text{finite}} (c_k^* \theta_k + \theta_k^* c_k)$$

where each $a_i, c_k \in \tilde{\mathfrak{C}}$, $b_i \in \mathbb{R}^{\nu \times 1} \tilde{\mathfrak{C}}$ and each $\theta_k \in {}^{(\text{Fr}(L), \tilde{\mathfrak{C}})}\sqrt{J}$. Therefore

$$p = \sum_i^{\text{finite}} \text{Sym}(a_i)^* \text{Sym}(a_i) + \sum_j^{\text{finite}} \text{Sym}(b_j)^* L \text{Sym}(b_j) \\ + \sum_k^{\text{finite}} (\text{Sym}(c_k)^* \text{Sym}(\theta_k) + \text{Sym}(\theta_k)^* \text{Sym}(c_k))$$

Hence it suffices to show that $\text{Sym}(\theta_k) \in {}^{(L, \mathfrak{C})}\sqrt{I}$.

Let $K \subseteq \mathbb{R}^{1 \times \ell} \langle x \rangle$ be the left module generated by $\text{Sym}^{-1}({}^{(L, \mathfrak{C})}\sqrt{I}) \cap \mathbb{R} \langle x \rangle_1 \tilde{\mathfrak{C}}$, i.e., the set of polynomials in $\mathbb{R} \langle x \rangle_1 \tilde{\mathfrak{C}}$ which map into ${}^{(L, \mathfrak{C})}\sqrt{I}$ under Sym . Suppose

$$\sum_i^{\text{finite}} f_i^* f_i + \sum_j^{\text{finite}} g_j^* \text{Fr}(L) g_j \in \mathbb{R}^{\ell \times 1} K + K^* \mathbb{R}^{1 \times \ell},$$

where each $f_i \in \tilde{\mathfrak{C}}$ and each $g_j \in \mathbb{R}^{\nu \times 1} \tilde{\mathfrak{C}}$. Then

$$\sum_i^{\text{finite}} \text{Sym}(f_i)^* \text{Sym}(f_i) + \sum_j^{\text{finite}} \text{Sym}(g_j)^* L \text{Sym}(g_j) \in \mathbb{R}^{\ell \times 1} {}^{(L, \mathfrak{C})}\sqrt{I} + {}^{(L, \mathfrak{C})}\sqrt{I}^* \mathbb{R}^{1 \times \ell},$$

each $\text{Sym}(f_i) \in \mathfrak{C}$, and each $\text{Sym}(g_j) \in \mathbb{R}^{\nu \times 1} \mathfrak{C}$. By definition, each $\text{Sym}(f_i) \in {}^{(L, \mathfrak{C})}\sqrt{I}$ and each $\text{Sym}(g_j) \in \mathbb{R}^{\nu \times 1} {}^{(L, \mathfrak{C})}\sqrt{I}$. This implies that each $f_i \in K$ and each $g_j \in \mathbb{R}^{\nu \times 1} K$. Further, it is clear by definition that $J \subseteq K$. Therefore

$$\text{Sym}({}^{(\text{Fr}(L), \tilde{\mathfrak{C}})}\sqrt{J}) \subseteq \text{Sym}(K) = {}^{(L, \mathfrak{C})}\sqrt{I}. \quad \blacksquare$$

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APPENDIX A.

BACKGROUND FROM [HV07] ON ALGEBRAIC INTERIORS AND THEIR DEGREE

This lemma and its proof are copied directly from [HV07]. To mesh perfectly with this the current paper take $x^0 = 0$ and $\mathcal{C}_p(x^0) = \mathcal{C}_p(0) = \mathcal{C}$.

Lemma A.1 ([HV07, Lemma 2.1]). *A polynomial p of the lowest degree for which $\mathcal{C} = \mathcal{C}_p(x^0)$ is unique (up to a multiplication by a positive constant), and any other polynomial q such that $\mathcal{C} = \mathcal{C}_q(x^0)$ is given by $q = ph$ where h is an arbitrary polynomial which is strictly positive on a dense connected subset of \mathcal{C} .*

Proof. We shall be using some properties of algebraic and semi-algebraic sets in \mathbb{R}^m , so many readers may want to skip over it and go to our main results which are much more widely understandable; our reference is [BCR98]. We notice first that \mathcal{C} is a semi-algebraic set (since it is the closure of a connected component of a semi-algebraic set, see [BCR98, Proposition 2.2.2 and Theorem 2.4.5]). Therefore the interior $\text{int } \mathcal{C}$ of \mathcal{C} is also semi-algebraic, and so is the boundary $\partial \mathcal{C} = \mathcal{C} \setminus \text{int } \mathcal{C}$. Notice also that \mathcal{C} equals the closure of its interior.

We claim next that *for each $x \in \partial \mathcal{C}$, the local dimension $\dim \partial \mathcal{C}_x$ equals $m - 1$* . On the one hand, we have

$$\dim \partial \mathcal{C}_x \leq \dim \partial \mathcal{C} < \dim \text{int } \mathcal{C} = m;$$

here we have used [BCR98, Proposition 2.8.13 and Proposition 2.8.4], and the fact that $\partial \mathcal{C} = \text{clos int } \mathcal{C} \setminus \text{int } \mathcal{C}$, since \mathcal{C} equals the closure of its interior. On the other hand, let B be an open ball in \mathbb{R}^m around x ; then

$$B \cap \partial \mathcal{C} = B \setminus [(B \cap (\mathbb{R}^m \setminus \mathcal{C})) \cup (B \cap \text{int } \mathcal{C})].$$

Since \mathcal{C} equals the closure of its interior, every point of $\partial \mathcal{C}$ is an accumulation point of both $\mathbb{R}^m \setminus \mathcal{C}$ and $\text{int } \mathcal{C}$; therefore $B \cap (\mathbb{R}^m \setminus \mathcal{C})$ and $B \cap \text{int } \mathcal{C}$ are disjoint open nonempty semi-algebraic subsets of B . Using [BCR98, Lemma 4.5.2] we conclude that $\dim B \cap \partial \mathcal{C} \geq m - 1$, hence $\dim \partial \mathcal{C}_x \geq m - 1$.

Let now V be the Zariski closure of $\partial \mathcal{C}$, and let $V = V_1 \cup \dots \cup V_k$ be the decomposition of V into irreducible components. We claim that $\dim V_i = m - 1$ for each i . Assume by contradiction that V_1, \dots, V_l have dimension $m - 1$ while V_{l+1}, \dots, V_k have smaller dimension. Then there exists $x \in \partial \mathcal{C}$ such that $x \notin V_1, \dots, V_l$, and consequently there exists an open ball B in \mathbb{R}^m around x such that

$$B \cap \partial \mathcal{C} = (B \cap \partial \mathcal{C} \cap V_{l+1}) \cup \dots \cup (B \cap \partial \mathcal{C} \cap V_k).$$

By assumption each set in the union on the right hand side has dimension smaller than $m - 1$, hence it follows (by [BCR98, Proposition 2.8.5, I]) that $\dim B \cap \partial \mathcal{C} < m - 1$, a contradiction with $\dim \partial \mathcal{C}_x = m - 1$.

Suppose now that p is a polynomial of the lowest degree with $\mathcal{C} = \mathcal{C}_p(x^0)$. Lowest degree implies that p can have no multiple irreducible factors, i.e., $p = p_1 \cdots p_s$, where p_1, \dots, p_s are distinct irreducible polynomials; we may assume without loss of generality that every p_i is non-negative on \mathcal{C} . Since p vanishes on $\partial\mathcal{C}$ it also vanishes on $V = V_1 \cup \cdots \cup V_k$. We claim that for every V_i there exists a p_j so that p_j vanishes on V_i : otherwise $\mathcal{Z}(p_j) \cap V_i$ is a proper algebraic subset of V_i for every $j = 1, \dots, s$, therefore (since V_i is irreducible) $\dim \mathcal{Z}(p_j) \cap V_i < \dim V_i$ for every j and thus also $\dim \mathcal{Z}(p) \cap V_i < \dim V_i$, a contradiction since p vanishes on V_i . If p_j vanishes on V_i , it follows (since p_j is irreducible and $\dim V_i = m - 1$) that $\mathcal{Z}(p_j) = V_i$. The fact that p is a polynomial of the lowest degree with $\mathcal{C} = \mathcal{C}_p(x^0)$ implies now (after possibly renumbering the irreducible factors of p) that $p = p_1 \cdots p_k$ where $\mathcal{Z}(p_i) = V_i$ for every i . Since $\dim V_i = m - 1$ it follows from the real Nullstellensatz for principal ideals [BCR98, Theorem 4.5.1] that the irreducible polynomials p_i are uniquely determined (up to a multiplication by a positive constant), hence so is their product p . This proves the uniqueness of p .

The rest of the lemma now follows easily. If $\mathcal{C} = \mathcal{C}_q(x^0)$, then the polynomial q vanishes on $\partial\mathcal{C}$ hence also on $V = V_1 \cup \cdots \cup V_k$. Since q vanishes on $\mathcal{Z}(p_i) = V_i$ and $\dim V_i = m - 1$, the real Nullstellensatz for principal ideals implies that q is divisible by p_i ; this holds for every i hence q is divisible by $p = p_1 \cdots p_k$, i.e., $q = ph$. It is obvious that h must be strictly positive on a dense connected subset of \mathcal{C} . ■

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