Abstracts

The Procesi-Schacher conjecture and Hilbert's 17th problem for algebras with involution

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(joint work with Igor Klep)

Hilbert's 17th problem ("Is every nonnegative real polynomial a sum of squares of rational functions?") was solved in the affirmative by Artin in 1927. Starting with Helton's seminal paper [1], in which he proved that every positive semidefinite real or complex noncommutative polynomial is a sum of hermitian squares of *polynomials*, variants of Hilbert's 17th problem in a *noncommutative* setting have become a topic of current interest with wide-ranging applications. These results are often functional analytic in flavour.

A general framework to deal with this type of problem in an algebraic setting is the theory of central simple algebras with an involution. Let F be a formally real field with space of orderings X_F. Procesi and Schacher [3] consider central simple algebras A, equipped with an involution σ, which is *positive* with respect to an ordering $P \in X_F$ in the sense that its involution trace form $T_\sigma(x) := \operatorname{Trd}(\sigma(x)x)$, $\forall x \in A$ is positive semidefinite with respect to P. An ordering $P \in X_F$ for which σ is positive is called a *σ-ordering*. Let

$$X_F^\sigma := \{P \in X_F \mid P \text{ is a } \sigma\text{-ordering}\}.$$

An element $a \in \operatorname{Sym}(A, \sigma)$ is *σ-positive for $P \in X_F^\sigma$* if the scaled involution trace form $\operatorname{Trd}(\sigma(x)ax)$ is positive semidefinite with respect to $P \in X_F^\sigma$ and *totally σ-positive* if it is σ-positive for all $P \in X_F^\sigma$. It should be noted that Weil made a very detailed study of positive involutions [4]. For $x \in A$, $\sigma(x)x$ is called a *hermitian square.*

Procesi and Schacher prove the following result, which can be considered as a noncommutative analogue of Artin's solution to Hilbert's 17th problem:

Theorem 1. [3, Theorem 5.4] *Let A be a central simple F-algebra with involution σ. Let $T_\sigma \simeq \langle \alpha_1, \ldots, \alpha_m\rangle$ (with $\alpha_1, \ldots, \alpha_m \in F$) and let $a \in \operatorname{Sym}(A, \sigma)$. The following statements are equivalent:*

(i) *a is totally σ-positive;*
(ii) *there exist $x_{i,\varepsilon} \in A$ with*

$$a = \sum_{\varepsilon \in \{0,1\}^m} \alpha^\varepsilon \sum_i \sigma(x_{i,\varepsilon}) x_{i,\varepsilon}.$$

(Here α^ε denotes $\alpha_1^{\varepsilon_1} \cdots \alpha_m^{\varepsilon_m}$.)

In the case $\deg A = 2$ (i.e., quaternion algebras with arbitrary involution), Procesi and Schacher show that the weights α_j are superfluous. They conjecture that this is also the case for $\deg A > 2$:

Conjecture. [3, p. 404] *In a central simple algebra A with involution σ, every totally σ-positive element is a sum of hermitian squares.*

We show that the Procesi-Schacher conjecture is false in general. An elementary counterexample can already be obtained in degree 3:

Theorem 2. [2, Theorem 3.2] *Let F_0 be a formally real field and let $F = F_0(X,Y)$. Let $A = M_3(F)$ and $\sigma = \mathrm{ad}_q$, where $q = \langle X, Y, XY \rangle$. The ($\sigma$-symmetric) element XY is totally σ-positive, but is not a sum of hermitian squares in (A, σ).*

For symplectic involutions, we show that the conjecture is true for split, but false for non-split central simple algebras. The conjecture is also false for non-split central simple algebras with unitary involution.

For more details we refer to [2], where we also apply the results of Procesi and Schacher to study non-dimensionfree positivity of noncommutative polynomials. Our *Positivstellensatz* [2, Theorem 5.4] roughly says that a noncommutative polynomial all of whose evaluations in $n \times n$ matrices (for *fixed* n) are positive semidefinite, is a sum of hermitian squares with denominators and weights.

References

[1] J.W. Helton, *"Positive" noncommutative polynomials are sums of squares*, Ann. of Math. (2) **156** (2002), no. 2, 675–694.

[2] I. Klep, T. Unger, *The Procesi-Schacher conjecture and Hilbert's 17th problem for algebras with involution*, preprint (2008), `arXiv:0810.5254v1 [math.RA]`.

[3] C. Procesi, M. Schacher, *A non-commutative real Nullstellensatz and Hilbert's 17th problem*, Ann. of Math. (2) **104** (1976), no. 3, 395–406.

[4] A. Weil, *Algebras with involutions and the classical groups*, J. Indian Math. Soc. (N.S.) **24** (1961), 589–623.