Relaxing LMI Domination Matricially

J. William Helton, Igor Klep and Scott A. McCullough

***Abstract*—Given linear matrix inequalities (LMIs) L_1 and L_2 in the same number of variables it is natural to ask:**

(Q_1) does one dominate the other, that is, does $L_1(X) \succeq 0$ imply $L_2(X) \succeq 0$?

(Q_2) are they mutually dominant, that is, do they have the same solution set?

Such problems can be NP-hard. We describe a natural relaxation of an LMI, based on substituting matrices for the variables x_j. With this relaxation, the domination questions (Q_1) and (Q_2) have elegant answers, indeed reduce to semidefinite programs (SDP) which we show how to construct. For our "matrix variable" relaxation a positive answer to (Q_1) is equivalent to the existence of matrices V_j such that

$$L_2(x) = V_1^* L_1(x) V_1 + \cdots + V_\mu^* L_1(x) V_\mu. \tag{A_1}$$

As for (Q_2) we show that L_1 and L_2 are mutually dominant if and only if, up to certain redundancies described in the paper, L_1 and L_2 are unitarily equivalent.

An observation at the core of the paper is that the relaxed LMI domination problem is equivalent to a classical problem. Namely, the problem of determining if a linear map τ from a subspace of matrices to a matrix algebra is "completely positive".

I. Introduction

Semidefinite Programming, SDP, is one of the main techniques in studying linear control systems, cf. [16]. It is even used with nonlinear systems to find Lyapunov functions via sums of squares and in Linear Parameter Varying situations [17]. Less directly, SDP impacts systems and control through its importance in combinatorial optimization and application to statistical problems; [11] contains an excellent introduction. SDP is based on LMIs and this article surveys LMI results based on "matrix variable" relaxations.

For symmetric matrices $A_0, A_1, \ldots, A_g \in \mathbb{SR}^{d\times d}$, the expression

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j \in \mathbb{SR}^{d\times d}\langle x\rangle \tag{I.1}$$

in variables $x = (x_1, \ldots, x_g)$ is a **linear pencil**. If $A_0 = I$, then L is **monic**. If $A_0 = 0$, then L is a **truly linear pencil**. Call d the size of the pencil. An LMI is an inequality of the form $L(X) \succeq 0$ for $X \in \mathbb{R}^g$.

First observe that given such an LMI or equivalently a linear pencil L, it is mathematically natural to substitute symmetric matrices X_j for the variables. Formally, given $X = \mathrm{col}(X_1, \ldots, X_g) \in (\mathbb{SR}^{n\times n})^g$, the **evaluation** $L(X)$ is defined as

$$L(X) = A_0 \otimes I_n + \sum A_j \otimes X_j \in \mathbb{SR}^{dn\times dn}. \tag{I.2}$$

The tensor product in this expression is the usual (Kronecker) tensor product of matrices. For example, if $A = \begin{bmatrix} a_{ij}\end{bmatrix}_{i,j=1}^d$ and $X \in \mathbb{R}^{n\times n}$ then

$$A \otimes X = \begin{bmatrix} a_{11}X & \cdots & a_{1d}X \\ \vdots & \ddots & \vdots \\ a_{d1}X & \cdots & a_{dd}X \end{bmatrix} \in \mathbb{R}^{dn\times dn}.$$

Thus for each dimension n we have the set

$$\mathcal{D}_L(n) = \{X \in (\mathbb{SR}^{n\times n})^g \mid L(X) \succeq 0\}$$

of all matricial solutions X to an LMI. While one's main concern might be $n = 1$, using $n = 2$ or $n = 3$, etc gives a natural relaxation. Indeed the supreme relaxation of the solution set to an LMI, we call the **matricial linear matrix inequality (LMI) set** and it is

$$\mathcal{D}_L := \bigcup_{n\in\mathbb{N}} \mathcal{D}_L(n). \tag{I.3}$$

The set $\mathcal{D}_L(1) \subseteq \mathbb{R}^g$ is the feasibility set of the semidefinite program $L(X) \succeq 0$ and is called a **spectrahedron** by algebraic geometers.

LMI domination problem: Given monic pencils L_1, L_2,
is $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$?
The relaxed problem asks: is $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$?

A very similar problem is the **LMI mutual domination problem**: is $\mathcal{D}_{L_1}(1) = \mathcal{D}_{L_2}(1)$? Also of interest is its relaxation: is $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$?

There are various problems which are not LMIs but have relaxations which are LMIs. Indeed, they have a chain of LMI relaxations that give finer and finer approximations (cf. Lasserre [8]). Whether this chain of relaxations becomes stationary is equivalent to an LMI mutual domination problem. See also [14].

A very special case of LMI domination (for $n = 1$) is the **matrix cube problem** of Ben-Tal and Nemirovski [10], [2], where $\mathcal{D}_{L_1}(1)$ is a cube. Its most fundamental application is to the Lyapunov stability analysis for uncertain dynamical systems. Also, maximizing a positive definite quadratic form over the unit cube can be formulated as a matrix cube problem. This implies the LMI domination for $n = 1$ is numerically NP-hard.

While the LMI domination problem algebraically has no clean answer, as we shall see the relaxed problem behaves

J. William Helton: Department of Mathematics, University of California San Diego, La Jolla, CA 92093-0112, USA. helton@math.ucsd.edu

Igor Klep: Univerza v Ljubljani, Fakulteta za matematiko in fiziko, Ljubljana, Slovenia. igor.klep@fmf.uni-lj.si

Scott A. McCullough: Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, USA. sam@math.ufl.edu

completely "perfectly". How closely the relaxed is to the unrelaxed has not been seriously explored.

This note treats relaxation of LMI domination and LMI mutual domination, giving precise algebraic characterizations §II and numerical algorithms for computation §IV. In [6] we prove that $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ is equivalent to the feasibility of a certain semidefinite program which we construct explicitly in §IV-A. We also have an algorithm (§IV-B) to determine if $\mathcal{D}_L$ is bounded and, if so, its "radius." Our algorithm thus yields an upper bound of the radius of $\mathcal{D}_L(1)$. Finally, given a matricial LMI set $\mathcal{D}_L$, §IV-C gives an algorithm to compute the linear pencil $\tilde{L} \in \mathbb{SR}^{d\times d}\langle x\rangle$ with smallest possible d satisfying $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$.

A by-product of this investigation is a very clean characterization of polynomials which are positive on a spectrahedron - a Putinar type Positivstellensatz §V.

Example I.1. Let $L_1(x_1, x_2) =$

$$I + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix}$$

which is in $\mathbb{SR}^{3\times 3}\langle x\rangle$ and let $L_2(x_1, x_2) =$

$$I + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{bmatrix}$$

which is in $\mathbb{SR}^{2\times 2}\langle x\rangle$. Then

$$\begin{aligned} \mathcal{D}_{L_1} &= \{(X_1, X_2) \mid 1 - X_1^2 - X_2^2 \succeq 0\}, \\ \mathcal{D}_{L_1}(1) &= \{(X_1, X_2) \in \mathbb{R}^2 \mid X_1^2 + X_2^2 \le 1\}, \\ \mathcal{D}_{L_2}(1) &= \{(X_1, X_2) \in \mathbb{R}^2 \mid X_1^2 + X_2^2 \le 1\}. \end{aligned}$$

Thus $\mathcal{D}_{L_1}(1) = \mathcal{D}_{L_2}(1)$. On one hand,

$$\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{bmatrix} \right) \in \mathcal{D}_{L_1} \setminus \mathcal{D}_{L_2},$$

so $L_1(X_1, X_2) \succeq 0$ does not imply $L_2(X_1, X_2) \succeq 0$. On the other hand, $L_2(X_1, X_2) \succeq 0$ does imply $L_1(X_1, X_2) \succeq 0$. We shall explain this later below, see Ex. III.3. □

The relaxed LMI domination problem is equivalent to the classical problem of determining if a linear map τ from a subspace of matrices to a matrix algebra is "completely positive", see §III. Complete positivity is one of the main techniques of modern operator theory and the theory of operator algebras. On one hand it provides tools for studying LMIs and on the other hand, since completely positive maps are not so far from representations and generally are much more tractable than their merely positive counterparts, the theory of completely positive maps provides perspective on the difficulties in solving LMI domination problems.

We shall focus on statement of results, algorithms and examples. For proofs and details see [6] on which this note is based.

II. Algebraic characterization of domination

Here we state our main theorems giving precise characterizations of LMI domination.

We call $\mathcal{D}_L$ **bounded** if there is an $N \in \mathbb{N}$ with $\|X\| \le N$ for all $X \in \mathcal{D}_L$. It turns out (see [6, Proposition 2.4]) that $\mathcal{D}_L$ is bounded if and only if $\mathcal{D}_L(1)$ is bounded.

Theorem II.1 (Linear Positivstellensatz). *Let L_j, $j = 1, 2$, be monic $d_j \times d_j$ linear pencils and assume $\mathcal{D}_{L_1}$ is bounded. Then $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ if and only if there is a $\mu \in \mathbb{N}$ and an isometry $V \in \mathbb{R}^{\mu d_1 \times d_2}$ such that*

$$L_2(x) = V^*\big(I_\mu \otimes L_1(x)\big)V = \sum_{j=1}^{\mu} V_j^* L_1(x) V_j. \qquad \text{(II.1)}$$

This is an algebraic statement in that we are thinking of the x_j in the pencil as noncommuting variables.

Now we turn to mutual domination. Suppose $L \in \mathbb{SR}^{d\times d}\langle x\rangle$,

$$L = I + \sum_{j=1}^{g} A_j x_j$$

is a monic linear pencil. A subspace $\mathcal{H} \subseteq \mathbb{R}^d$ is *reducing for* L if $\mathcal{H}$ reduces each A_j; i.e., if $A_j\mathcal{H} \subseteq \mathcal{H}$. Since each A_j is symmetric, it also follows that $A_j\mathcal{H}^\perp \subseteq \mathcal{H}^\perp$. Hence, with respect to the decomposition $\mathbb{R}^d = \mathcal{H} \oplus \mathcal{H}^\perp$, L can be written as the direct sum,

$$L = \tilde{L} \oplus \tilde{L}^\perp = \begin{bmatrix} \tilde{L} & 0 \\ 0 & \tilde{L}^\perp \end{bmatrix}, \quad \text{where} \quad \tilde{L} = I + \sum_{j=1}^{g} \widetilde{A}_j x_j,$$

and $\widetilde{A}_j$ is the restriction of A_j to $\mathcal{H}$. (The pencil $\tilde{L}^\perp$ is defined similarly.) If $\mathcal{H}$ has dimension ℓ, then by identifying $\mathcal{H}$ with $\mathbb{R}^\ell$, the pencil $\tilde{L}$ is a monic linear pencil of size ℓ. We say that $\tilde{L}$ is a *subpencil* of L. If moreover, $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$, then $\tilde{L}$ is a *defining subpencil* and if no proper subpencil of $\tilde{L}$ is defining subpencil for $\mathcal{D}_L$, then $\tilde{L}$ is a *minimal defining (sub)pencil*.

Theorem II.2 (Linear Gleichstellensatz). *Let $L_j \in \mathbb{SR}^{d\times d}\langle x\rangle$, $j = 1, 2$, be monic linear pencils with $\mathcal{D}_{L_1}$ bounded. Then $\mathcal{D}_{L_1} = \mathcal{D}_{L_2}$ if and only if minimal defining pencils $\tilde{L}_1$ and $\tilde{L}_2$ for $\mathcal{D}_{L_1}$ and $\mathcal{D}_{L_2}$ respectively, are unitarily equivalent. That is, there is a unitary matrix U such that*

$$\tilde{L}_2(x) = U^* \tilde{L}_1(x) U. \qquad \text{(II.2)}$$

III. Complete positivity by example

This section describes the map between domination and complete positivity and concludes with an example.

Given L_1 and L_2 monic linear pencils

$$L_j(x) = I + \sum_{\ell=1}^{g} A_{j,\ell} x_\ell \in \mathbb{SR}^{d_j\times d_j}\langle x\rangle, \quad j = 1, 2, \qquad \text{(III.1)}$$

we introduce subspaces to be used in our considerations:

$$\begin{aligned} \mathcal{S}_j &= \operatorname{span}\{I, A_{j,\ell} \mid \ell = 1, \dots, g\} \\ &= \operatorname{span}\{L_j(X) \mid X \in \mathbb{R}^g\} \subseteq \mathbb{SR}^{d_j\times d_j}. \end{aligned} \qquad \text{(III.2)}$$

The key tool in studying LMI domination is the mapping τ we now define.

Definition III.1. Let L_1, L_2 be monic linear pencils as in (III.1). If $\{I, A_{1,\ell} \mid \ell = 1, \dots, g\}$ is linearly independent (e.g. $\mathcal{D}_{L_1}$ is bounded), we define the linear map

$$\tau : \mathcal{S}_1 \to \mathcal{S}_2, \quad A_{1,\ell} \mapsto A_{2,\ell}, \quad I \mapsto I. \tag{III.3}$$

It turns out that the inclusion $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ is equivalent to the complete positivity of τ, a notion we now introduce. Let $\mathcal{S}_j \subseteq \mathbb{R}^{d_j \times d_j}$ be linear subspaces of matrices containing the identity matrix and invariant under the transpose, and $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ a unital (i.e., $\phi(I) = I$) linear $*$-map. For $n \in \mathbb{N}$, ϕ induces the map

$$\phi_n = I_n \otimes \phi : \mathbb{R}^{n\times n} \otimes \mathcal{S}_1 = \mathcal{S}_1^{n\times n} \to \mathcal{S}_2^{n\times n},$$
$$M \otimes A \mapsto M \otimes \phi(A),$$

called an **ampliation** of ϕ. Equivalently,

$$\phi_n\left(\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn}\end{bmatrix}\right) = \begin{bmatrix} \phi(T_{11}) & \cdots & \phi(T_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(T_{n1}) & \cdots & \phi(T_{nn})\end{bmatrix}$$

for $T_{ij} \in \mathcal{S}_1$. We say that ϕ is k**-positive** if ϕ_k is a positive map. If ϕ is k-positive for every $k \in \mathbb{N}$, then ϕ is **completely positive**.

Theorem III.2. *Let L_1, L_2 be linear pencils as in* (III.1) *with L_1 of size $d_1 \times d_1$. Assume the matricial LMI set $\mathcal{D}_{L_1}$ is bounded. Let $\tau : \mathcal{S}_1 \to \mathcal{S}_2$ be the unital linear map $A_{1,\ell} \mapsto A_{2,\ell}$ as in* (III.3).
(1) *τ is n-positive if and only if $\mathcal{D}_{L_1}(n) \subseteq \mathcal{D}_{L_2}(n)$;*
(2) *τ is completely positive if and only if $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$;*
(3) *$\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$ if and only if $\mathcal{D}_{L_1}(d_1) \subseteq \mathcal{D}_{L_2}(d_1)$.*

The proof is in [6].

Example III.3 (Example I.1 revisited)**.** The unital linear map $\tau : \mathcal{S}_2 \to \mathcal{S}_1$ in our example is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Consider the extension of τ to a unital linear $*$-map $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}^{3\times 3}$, defined by

$$E_{11} \mapsto \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{12} \mapsto \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

$$E_{21} \mapsto \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \; E_{22} \mapsto \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Here E_{ij} are the 2×2 matrix units.) Now we show the map ψ is completely positive. To do this, we use its Choi matrix defined as

$$C = \begin{bmatrix} \psi(E_{11}) & \psi(E_{12}) \\ \psi(E_{21}) & \psi(E_{22}) \end{bmatrix}. \tag{III.4}$$

[13, Theorem 3.14] says ψ is completely positive if and only if $C \succeq 0$. We will use the Choi matrix again in §IV for computational algorithms. To see that C is positive semidefinite, note

$$C = \frac{1}{2}W^*W \quad \text{for} \quad W = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

Now ψ has a very nice representation:

$$\psi(S) = \frac{1}{2}V_1^* S V_1 + \frac{1}{2}V_2^* S V_2 = \frac{1}{2}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^* \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{III.5}$$

for all $S \in \mathbb{R}^{2\times 2}$. Here

$$V_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad V_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

thus $W = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$. In particular,

$$2L_1(x,y) = V_1^* L_2(x,y) V_1 + V_2^* L_2(x,y) V_2. \tag{III.6}$$

Hence $L_2(X_1, X_2) \succeq 0$ implies $L_1(X_1, X_2) \succeq 0$, i.e., $\mathcal{D}_{L_2} \subseteq \mathcal{D}_{L_1}$.

The formula (III.6) illustrates our linear Positivstellensatz (Theorem II.1). The construction of the formula in this example is a concrete implementation of the theory leading up to the general result, see [6]. $\square$

The proof of the Gleichstellensatz (Theorem II.2) is more involved and uses Arveson's theory of the noncommutative Choquet boundary from operator algebras [1].

IV. Computational algorithms

In this section we present numerical algorithms using semidefinite programming (SDP) [18] based on the theory explained in the preceding section. Given L_1 and L_2 monic linear pencils

$$L_j(x) = I + \sum_{\ell=1}^{g} A_{j,\ell} x_\ell \in \mathbb{SR}^{d_j \times d_j}\langle x \rangle, \quad j = 1, 2, \tag{IV.1}$$

with bounded matricial LMI set $\mathcal{D}_{L_1}$, we present an algorithm to test whether $\mathcal{D}_{L_1} \subseteq \mathcal{D}_{L_2}$; see §IV-A. Of course this numerical test yields a sufficient condition for containment of the spectrahedra $\mathcal{D}_{L_1}(1) \subseteq \mathcal{D}_{L_2}(1)$. We refer the reader to §IV-B for a test of boundedness of LMI sets, which works both for commutative LMIs and matricial LMIs, and computes the radius of a matricial LMI set. We then in §IV-C discuss give a (generically successful) algorithm for computation of a minimal representing pencil. In [6, §4.3] we also give a matricial version of the classical matrix cube problem of Ben-Tal and Nemirovski [2].

A. Checking inclusion of matricial LMI sets

By Theorem III.2, L_2 dominates L_1 if and only if there is a completely positive unital map

$$\tau : \mathbb{R}^{d_1 \times d_1} \to \mathbb{R}^{d_2 \times d_2} \tag{IV.2}$$

satisfying

$$\tau(A_{1,\ell}) = A_{2,\ell} \quad \text{for} \quad \ell = 1, \dots, g. \tag{IV.3}$$

To determine the existence of such a map, consider the Choi matrix $C = \big(\tau(E_{ij})\big)_{i,j=1}^{d_1} \in (\mathbb{R}^{d_2\times d_2})^{d_1\times d_1}$ of τ. (Here, E_{ij} are the $d_1\times d_1$ matrix units.) For convenience of notation we consider C to be a $d_1\times d_1$ matrix with $d_2\times d_2$ entries c_{ij}. It is well-known that τ is completely positive if and only if C is positive semidefinite [13, Theorem 3.14].

Let $\alpha^\ell_{p,q}$ denote the (p,q) entry of $A_{1,\ell}$, that is, $A_{1,\ell} = \sum_{p,q}\alpha^\ell_{pq}E_{pq}$. Note

$$\tau(A_{1,\ell}) = \sum_{p,q}\alpha^\ell_{pq}\tau(E_{pq}) = \sum_{p,q}\alpha^\ell_{pq}c_{pq}.$$

The inclusion algorithm

Solve the following (feasibility) SDP:

$$(c_{pq})_{p,q=1}^{d_1} := C \succeq 0, \quad \sum_p c_{pp} = I_{d_2}, \qquad \text{(IV.4)}$$

$$\forall \ell = 1,\dots,g: \ \sum_{p,q}\alpha^\ell_{pq}c_{pq} = A_{2,\ell},$$

for the unknown C. This can be, in practice, done numerically with standard SDP solvers. The matrix C of unknown variables is of size $d_1d_2\times d_1d_2$ and there are $(1+g)d_2^2$ (scalar) linear constraints.

Clearly, $\mathcal{D}_{L_1}\subseteq\mathcal{D}_{L_2}$ if and only if the SDP (IV.4) is feasible, i.e., has a solution, since if a solution C has been obtained, then a Positivstellensatz-type certificate for the inclusion of the matricial LMI sets $\mathcal{D}_{L_1}\subseteq\mathcal{D}_{L_2}$ can be obtained; cf. Example III.3.

B. Computing the radius of matricial LMI sets

Let L be a monic linear pencil,

$$L(x) = I + \sum_{\ell=1}^{g} A_\ell x_\ell \in \mathbb{SR}^{d\times d}\langle x\rangle. \qquad \text{(IV.5)}$$

We present an algorithm based on semidefinite programming to compute the radius of a matricial LMI set $\mathcal{D}_L$ (and at the same time check whether it is bounded). The idea is simply to use the test in §IV-A to check if $\mathcal{D}_L$ is contained in the ball of radius N. The smallest such N will be the matricial radius, and also an upper bound on the radius of the spectrahedron $\mathcal{D}_L(1)$.

Consider the monic linear pencil

$$\begin{aligned}\mathcal{J}_N(x) &= \frac{1}{N}\begin{bmatrix} N & x^* \\ x & NI_g\end{bmatrix} \\ &= I + \frac{1}{N}\sum (E'_{1,j+1}+E'_{j+1,1})x_j, \\ \mathcal{J}_N(x) &\in \mathbb{SR}^{(g+1)\times(g+1)}\langle x\rangle.\end{aligned}$$

Then $\mathcal{D}_L$ is bounded if and only if there is an N such that $\mathcal{D}_{J_N}\supseteq\mathcal{D}_L$ (and in this case its radius is at most N). As in the previous subsection, we need to determine whether there is a completely positive unital map $\tau:\mathbb{R}^{d\times d}\to\mathbb{R}^{(g+1)\times(g+1)}$ satisfying

$$\tau(A_j) = \frac{1}{N}(E'_{1,j+1}+E'_{j+1,1})$$

for some N.

The Choi matrix here is

$$C = (\tau(E_{ij}))_{i,j} \in (\mathbb{R}^{(g+1)\times(g+1)})^{d\times d}.$$

Let $A_\ell = \sum_{r,s}\alpha^\ell_{rs}E_{rs}$. Then the linear constraints we need to consider say that

$$\tau(A_\ell) = \sum_{r,s}\alpha^\ell_{rs}c_{rs}$$

has all entries 0 except for the $(1,\ell+1)$ and $(\ell+1,1)$ entries which are the same; indeed they are all equal to $\frac{1}{N}$. Thus we arrive at a feasibility SDP.

The matricial radius algorithm

Let $\alpha^\ell_{r,s}$ denote the (r,s) entry of A_ℓ, that is, $A_\ell = \sum_{r,s}\alpha^\ell_{rs}E_{rs}$. Solve the SDP:

(RM$_1$) $(c_{rs})_{r,s=1}^d := C \succeq 0$,

(RM$_2$) $\sum_{r=1}^{d} c_{rr} = I_{g+1}$,

(RM$_3$) $\forall\ell=1,\dots,g,\ \forall p,q=1,\dots,g+1:$

$$\sum_{r,s}\alpha^\ell_{rs}(c_{rs})_{p,q} = 0 \quad \text{for}$$

$$(p,q)\notin\{(1,\ell+1),\,(\ell+1,1)\},$$

(RM$_4$) $\sum_{r,s}\alpha^1_{rs}(c_{rs})_{1,2} = \sum_{r,s}\alpha^1_{rs}(c_{rs})_{2,1}$
$= \sum_{r,s}\alpha^2_{rs}(c_{rs})_{1,3} = \sum_{r,s}\alpha^2_{rs}(c_{rs})_{3,1} = \cdots$
$= \sum_{r,s}\alpha^g_{rs}(c_{rs})_{1,g+1} = \sum_{r,s}\alpha^g_{rs}(c_{rs})_{g+1,1}$

for the unknown C.

This SDP is always feasible (for $b := \sum_{r,s}\alpha^1_{rs}(c_{rs})_{1,2} = 0$). Clearly, $\mathcal{D}_L$ is bounded if and only if this SDP has a positive solution. In fact, any value of $b>0$ obtained gives an upper bound of $\frac{1}{b}$ for the norm of an element in $\mathcal{D}_L$. The size of the matrix of unknown variables is $d(g+1)\times d(g+1)$ and there are g^3+3g^2+3g (scalar) linear constraints. To reduce the number of unknowns, solve the linear system of g^3+2g^2-g equations given in (RM$_3$).

If one is interested in the biggest norm of an $X\in\mathcal{D}_L$, then one maximizes the linear objective function $\sum_{r,s}\alpha^1_{rs}(c_{rs})_{1,2}$ over the spectrahedron given above. If its optimal value is $b\in\mathbb{R}_{>0}$, then $\|X\|\le\frac{1}{b}$ for all $X\in\mathcal{D}_L$, and this bound is sharp.

Checking boundedness of $\mathcal{D}_L(1)$ is a classical, fairly basic semidefinite programming problem. Indeed, given a monic linear pencil L, $\mathcal{D}_L(1)$ is bounded (equivalently, $\mathcal{D}_L$ is bounded) if and only if the following SDP is feasible:

$$L^{(1)}(X)\succeq 0, \quad \operatorname{tr}\left(L^{(1)}(X)\right) = 1.$$

(Here, $L^{(1)}$ denotes the truly linear part $\sum_{j=1}^g A_jx_j$ of L.)

However, computing the radius of $\mathcal{D}_L(1)$ is harder. Thus our algorithm, yielding a convenient upper bound on the radius, might be of broad interest, motivating us to spend more time describing its implementation. The algorithm can be written entirely in a matricial form which is both elegant and easy to code in Matlab or Mathematica. The matricial component of the algorithm is as follows. Let $\mathbf{e}_n$ denote the vector of length n with all ones, let $\mathbf{E}_n = \mathbf{e}_n \otimes \mathbf{e}_n^t$ be the $n \times n$ matrix of all ones. Then (RM$_2$) is (using $\bullet_{\rm H}$ for the Hadamard product)

$$\big(\mathbf{e}_{g+1} \otimes I_d\big)^t \big((I_d \otimes \mathbf{E}_{g+1}) \bullet_{\rm H} C\big)\big(\mathbf{e}_{g+1} \otimes I_d\big) = I_{g+1},$$

while the left hand side of (RM$_3$) can be presented as the (p,q) entry of

$$\big(\mathbf{e}_{g+1} \otimes I_d\big)^t \big((A_\ell \otimes \mathbf{E}_{g+1}) \bullet_{\rm H} C\big)\big(\mathbf{e}_{g+1} \otimes I_d\big).$$

Equations (RM$_3$) and (RM$_4$) give constraints on these matrices.

As an example we computed the matricial radius of an ellipse. The corresponding Mathematica notebook is available at `http://srag.fmf.uni-lj.si`.

C. Minimal pencils

This section describes an algorithm aimed at constructing from a given pencil L a pencil $\tilde{L}$ of minimal size with $\mathcal{D}_L = \mathcal{D}_{\tilde{L}}$.

Let L be a monic linear pencil,

$$L(x) = I + \sum_{\ell=1}^{g} A_\ell x_\ell \in \mathbb{SR}^{d\times d}\langle x\rangle \tag{IV.6}$$

with bounded $\mathcal{D}_L$. We present a probabilistic algorithm based on semidefinite programming that computes a minimal pencil $\tilde{L}$ with the same matricial LMI set. Our algorithm depends on the decomposition of a semisimple algebra into a direct sum of simple algebras, a classical technique in computational algebra, cf. Friedl and Rońyal [4], Eberly and Giesbrecht [3], or Murota, Kanno, Kojima, and Kojima [9] for a recent treatment.

The two-step procedure goes as follows. In Step 1, we applied the probabilistic method of [9] to find a unitary matrix $U \in \mathbb{R}^{d\times d}$ that simultaneously transforms the A_ℓ into block diagonal form, that is,

$$U^* A_\ell U = \oplus_{j=1}^s B_\ell^j \quad \text{ for all } \quad \ell.$$

For each j, the set $\{I, B_1^j, \dots, B_g^j\}$ generates a simple real algebra. Define the monic linear pencils

$$L^j(x) = I + \sum_{\ell=1}^{g} B_\ell^j x_\ell, \quad L'(x) = U^* L(x) U = \oplus_{j=1}^s L^j(x).$$

Given ℓ, let $\tilde{L}_\ell = \oplus_{j\neq\ell} L^j$. If there is no ℓ such that

$$L^\ell|_{\mathcal{D}_{\tilde{L}_\ell}} \succeq 0,$$

(this can be tested using SDP as explained in §IV-A) then the pencil is minimal. If there is such an ℓ remove such the (one) corresponding block from L' to obtain a new pencil and repeat the process. Once we have no more redundant blocks in L', the obtained pencil $\tilde{L}$ is minimal, and satisfies $\mathcal{D}_{\tilde{L}} = \mathcal{D}_L$ by construction.

V. Positivstellensatz

Algebraic characterizations of polynomials p which are positive on $\mathcal{D}_L$ are called Positivstellensätze and are classical for polynomials on $\mathbb{R}^g$. This theory underlies the main approach currently used for global optimization of polynomials, cf. [7], [8], [12]. The generally noncommutative techniques of this paper and [6] lead to a cleaner and more powerful commutative Putinar-type Positivstellensatz [15] for p strictly positive on a bounded spectrahedron $\mathcal{D}_L(1)$. In the theorem which follows, $\mathbb{SR}^{d\times d}[y]$ is the set of symmetric $d\times d$ matrices with entries from $\mathbb{R}[y]$, the algebra of (commutative) polynomials with coefficients from $\mathbb{R}$. Note that an element of $\mathbb{SR}^{d\times d}[y]$ may be identified with a polynomial (in commuting variables) with coefficients from $\mathbb{SR}^{d\times d}$.

Theorem V.1. *Suppose $L \in \mathbb{SR}^{d\times d}[y]$ is a monic linear pencil and $\mathcal{D}_L(1)$ is bounded. Then for every symmetric matrix polynomial $p \in \mathbb{R}^{\ell\times\ell}[y]$ with $p|_{\mathcal{D}_L(1)} \succ 0$, there are $A_j \in \mathbb{R}^{\ell\times\ell}[y]$, and $B_k \in \mathbb{R}^{d\times\ell}[y]$ satisfying*

$$p = \sum_j A_j^* A_j + \sum_k B_k^* L B_k. \tag{V.1}$$

For perspective we mention that the proof of Theorem V.1 actually relies on the linear Positivstellensatz. For experts we point out that the key reason LMI sets behave better is that the quadratic module associated to a monic linear pencil L with bounded $\mathcal{D}_L$ is archimedean.

A general Positivstellensatz for matrix polynomials positive definite on compact semialgebraic sets has been given by Hol and Scherer [5]. The main difference with our Theorem V.1 is the archimedeanity which is an additional assumption in [5], but is a consequence of the boundedness of $\mathcal{D}_L(1)$ in our setting.

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