

NONCOMMUTATIVE BALL MAPS

J. WILLIAM HELTON¹, IGOR KLEP², SCOTT MCCULLOUGH³, AND NICK SLINGLEND

ABSTRACT. In this paper, we analyze problems involving matrix variables for which we use a noncommutative algebra setting. To be more specific, we use a class of functions (called NC analytic functions) defined by power series in noncommuting variables and evaluate these functions on sets of matrices of all dimensions; we call such situations dimension-free. These types of functions have recently been used in the study of dimension-free linear system engineering problems [HMPV], [OHMP].

In this paper we characterize NC analytic maps that send dimension-free matrix balls to dimension-free matrix balls and carry the boundary to the boundary; such maps we call “NC ball maps”. We find that up to normalization, an NC ball map is the direct sum of the identity map with an NC analytic map of the ball into the ball. That is, “NC ball maps” are very simple, in contrast to the classical result of D’Angelo on such analytic maps in \mathbb{C} . Another mathematically natural class of maps carries a variant of the noncommutative distinguished boundary to the boundary, but on these our results are limited.

We shall be interested in several types of noncommutative balls, conventional ones, but also balls defined by constraints called Linear Matrix Inequalities (LMI). What we do here is a small piece of the bigger puzzle of understanding how LMIs behave with respect to noncommutative change of variables.

1. INTRODUCTION

In the introduction we will state some of our main results. For this we need to start with the definitions of NC polynomials (§1.1) and NC analytic maps (§1.2). We then proceed to define NC ball maps in §1.3, where we explain what it means for an NC ball map to map ball to ball with boundary to boundary. After that we can and do state our main results classifying NC ball maps in §1.3 and §1.4. Finally, the introduction concludes by considering two types of generalizations,

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the first being to balls defined by LMIs, the second being to NC analytic maps carrying special sets on the boundary of a ball to the boundary of a ball.

Before continuing with the more detailed introduction, we pause to offer some perspective and mention related significant contributions. Ball maps form a distinguished subset of the space of non-commutative analytic functions on a non-commutative domain and we direct the interested reader to the elegant general theory of non-commutative analytic functions developed in the articles [K-V, K-VV1, K-VV2, Vo1, Vo2, MS1, MS2] for a more complete account than found here and to the work of Popescu on free analytic functions. Some of these references are [Po2, Po3, Po4, Po5, Po6]. Non-commutative rational, Schur class and analytic functions are also intimately related to systems theory. A small sample of the references include [BGM, BBF, K-V]. The non-commutative balls that we consider are modeled on $g' \times g$ matrices, and in the special case that $g = 1$, they correspond to those studied by Popescu for operator, not just matrix, coefficients. Precomposition by an automorphism of the domain preserves ball maps and such automorphisms are studied at various levels of generality in [Po7, MS2]. Linear ball maps are an important special case, and these are identified with completely isometric mappings from one operator space into another. The books [Pa, Pi, BM] provide comprehensive introductions to the theory of operator systems, spaces, and algebras, and the papers [BH1] and [BH2] treat very generally complete isometries into a C-star algebra.

1.1. Words and NC polynomials. Let $g', g \in \mathbb{N}$. We write $\langle x \rangle$ for the monoid freely generated by x , i.e., $\langle x \rangle$ consists of **words** in the $g'g$ letters $x_{11}, \dots, x_{1g}, x_{21}, \dots, x_{g'g}$ (including the empty word \emptyset which plays the role of the identity 1). Let $\mathbb{C}\langle x \rangle$ denote the associative \mathbb{C} -algebra freely generated by x , i.e., the elements of $\mathbb{C}\langle x \rangle$ are polynomials in the noncommuting variables x with coefficients in \mathbb{C} . Its elements are called **NC polynomials**. An element of the form aw where $0 \neq a \in \mathbb{C}$ and $w \in \langle x \rangle$ is called a **monomial** and a its **coefficient**. Hence words are monomials whose coefficient is 1. Let $x^* = (x_{11}^*, \dots, x_{g'g}^*)$ denote another set of $g'g$ symbols. We shall also consider the free algebra $\mathbb{C}\langle x, x^* \rangle$ that comes equipped with the natural **involution** $x_{ij} \mapsto x_{ij}^*$. For example,

$$(1 + i x_{11}^2 x_{23}^* x_{34}^*)^* = 1 - i x_{34} x_{23} (x_{11}^*)^2.$$

(Here i denotes the imaginary unit $\sqrt{-1}$.)

1.1.1. NC matrix polynomials. A **matrix valued NC polynomial** is an NC polynomial with matrix coefficients. We shall use the phrase *scalar* NC polynomial if we want to emphasize the

absence of matrix constructions. Often when the context makes the usage clear we drop adjectives such as scalar, 1×1 , matrix polynomial, matrix of polynomials and the like.

1.1.2. *Polynomial evaluations.* If p is an NC polynomial in x and $X \in (\mathbb{C}^{n \times n})^{g' \times g}$, the evaluation $p(X)$ is defined by simply replacing x_{ij} by X_{ij} . For example, if $p(x) = Ax_{11}x_{21}$, where

$$A = \begin{bmatrix} -4 & 3 & 2 \\ 2 & -1 & 0 \end{bmatrix},$$

then

$$p\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = A \otimes \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 4 & 0 & -3 & 0 & -2 \\ -4 & 0 & 3 & 0 & 2 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, if $p(x) = A$ and $X \in (\mathbb{C}^{n \times n})^{g' \times g}$, then $p(X) = A \otimes I_n$.

The tensor product in the expressions above is the usual (Kronecker) tensor product of matrices. Thus we have reserved the tensor product notation for the tensor product of matrices and have eschewed the strong temptation of using $A \otimes x_{kl}$ in place of Ax_{kl} when x_{kl} is one of the noncommuting variables.

1.2. **Definition of NC analytic functions.** An elegant theory of noncommutative analytic functions is developed in the articles [K-V, K-VV1, K-VV2] and [Vo1, Vo2]; see also [Po2]. What we need in this article are specializations of definitions of these papers. In this section we summarize the definitions and properties needed in the sequel.

For $d', d \in \mathbb{N}$ define

$$(1.1) \quad \mathcal{B}_{d' \times d} := \bigcup_{n=1}^{\infty} \left\{ X \in (\mathbb{C}^{n \times n})^{d' \times d} \mid I_{dn} - X^*X \succeq 0 \right\},$$

$$(1.2) \quad \text{int } \mathcal{B}_{d' \times d} := \bigcup_{n=1}^{\infty} \left\{ X \in (\mathbb{C}^{n \times n})^{d' \times d} \mid I_{dn} - X^*X \succ 0 \right\},$$

$$(1.3) \quad \partial \mathcal{B}_{d' \times d} := \bigcup_{n=1}^{\infty} \left\{ X \in (\mathbb{C}^{n \times n})^{d' \times d} \mid \|X\| = 1 \right\},$$

$$(1.4) \quad \mathcal{M}_{d' \times d} := \bigcup_{n=1}^{\infty} (\mathbb{C}^{n \times n})^{d' \times d}.$$

We shall occasionally use the notation

$$\mathcal{B}_{d' \times d}(N) = \{X = [X_{j,\ell}]_{j,\ell=1}^{d',d} \mid X_{j,\ell} \in \mathbb{C}^{N \times N}, \|X\| \leq 1\},$$

$$\mathcal{M}_{d' \times d}(N) = \{X = [X_{j,\ell}]_{j,\ell=1}^{d',d} \mid X_{j,\ell} \in \mathbb{C}^{N \times N}\}.$$

Given $g', g \in \mathbb{N}$, the **noncommutative (NC) ε -neighborhood** of 0 in $\mathbb{C}^{g' \times g}$ is the (disjoint) union $\bigcup_{N \in \mathbb{N}} \{X \in \mathcal{M}_{g' \times g}(N) \mid \|X\| < \varepsilon\}$. An **open NC domain** \mathcal{D} containing 0 (in its interior) is a union $\bigcup_N \mathcal{D}_N$ of open sets $\mathcal{D}_N \subseteq \mathcal{M}_{g' \times g}(N)$ which is closed with respect to direct sums and such that there is an $\varepsilon > 0$ such that \mathcal{D} contains the NC ε -neighborhood of 0.

A $d' \times d$ **NC analytic function** f on an open NC domain \mathcal{D} containing 0 as follows:

- (1) f has an **NC power series**, for which there exists an NC $\varepsilon > 0$ neighborhood of 0 on which it is convergent. That is,

$$(1.5) \quad f = \sum_{w \in \langle x \rangle} a_w w$$

for $a_w \in \mathbb{C}^{d' \times d}$ and for every $N \in \mathbb{N}$ and every $g' \times g$ -tuple of square matrices $X \in \mathcal{B}_{g' \times g}$ with $\|X\| < \varepsilon$ the series

$$(1.6) \quad f(X) = \sum_{w \in \langle x \rangle} a_w \otimes w(X)$$

converges. We interpret convergence for a given X as conditional of the series

$$\sum_{\alpha=0}^{\infty} \sum_{|w|=\alpha} a_w \otimes w(X).$$

Thus the *order* of summation is over the homogeneous parts of the power series expansion. Thus with $f^{(\alpha)}$ equal to the α **homogeneous part** in the NC power series expansion of f , the series converges for a given X provided

$$(1.7) \quad \sum_{\alpha=0}^{\infty} f^{(\alpha)}(X)$$

converges. Since both a_w and $w(X)$ are matrices, the particular norm topology chosen has no influence on convergence. The radius ε of this ball of convergence (or sometimes, by abuse of notation, the ball itself) will be called the **series radius**.

- (2) If $a : \mathcal{W} \rightarrow \mathcal{D}$ is a matrix valued function analytic on a domain \mathcal{W} in \mathbb{C}^N , the composition $f \circ a$ is a matrix valued analytic function on \mathcal{W} and continuous to $\partial\mathcal{W}$.

Remark 1.1. Popescu [Po2] has a notion of free analytic function in g' variables (that is, $g = 1$) based upon power series expansions like that in (1.7). His definition allows for operator coefficients, but on the other hand requires convergence of the NC power series on all of $\text{int } \mathcal{B}_{g'}$ (the NC 1-neighborhood of 0 in $\mathbb{C}^{g'}$). It turns out that for *bounded* NC analytic functions with *matrix coefficients* the two notions are the same, see Lemma 6.1.

Here we have avoided extending the theory of free analytic functions to $\mathcal{B}_{g' \times g}$, looking forward to working on more general domains in the future.

1.2.1. Properties of NC analytic functions.

Proposition 1.2. *Let \mathcal{D} be an NC domain containing 0.*

- (i) *The sum of two $d' \times d$ NC analytic functions on \mathcal{D} is a $d' \times d$ NC analytic function on \mathcal{D} .*
- (ii) *The product of two $d' \times d$ NC analytic functions on \mathcal{D} is a $d' \times d$ NC analytic function on \mathcal{D} .*
- (iii) *The composition of two NC analytic functions is an NC analytic function. More precisely, if $f : \mathcal{D} \rightarrow \mathcal{D}'$ is a $d'_1 \times d_1$ NC analytic function, where \mathcal{D}' is an NC domain with $0 \in \mathcal{D}'$, and h is a $d'_2 \times d_2$ NC analytic function on \mathcal{D}' , then $h \circ f$ is a $d'_1 d'_2 \times d_1 d_2$ NC analytic function on \mathcal{D} .*

Proof. Properties (i) and (ii) are standard and we only consider (iii). The fact that $h \circ f$ admits an NC power series as in (1.5) was observed e.g. in [K-VV2, Vo1]. The composition property (2) of §1.2 is easily checked. ■

More is said about properties of NC analytic functions in §6.

1.3. NC ball maps f and their classification when $f(0) = 0$. A function

$$f : \text{int } \mathcal{B}_{g' \times g} \rightarrow \mathcal{M}_{d \times d'}$$

which is NC analytic will often be called an **NC analytic function on the ball** $\mathcal{B}_{g' \times g}$ and denoted $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{M}_{d \times d'}$. An NC analytic function $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ mapping the boundary to the boundary is called an **NC ball map**. The notion of f mapping boundary to boundary is a bit complicated (because of convergence issues) so requires explanation. For a given $X \in \mathcal{B}_{g' \times g}(N)$, define the function $f_X : \mathbb{D} \rightarrow \mathcal{M}_{d \times d'}(N)$ by $z \mapsto f(zX)$. (Here \mathbb{D} denotes the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ in the complex plane.) If

$$\lim_{r \nearrow 1} f_X(re^{it})$$

exists, denote that limit by $f(e^{it}X)$. The function f **maps the boundary to the boundary** if whenever $\|X\| = 1$ and $f(e^{it}X)$ exists, then

$$\|f(e^{it}X)\| = 1.$$

Since f is bounded, Fatou's Theorem implies that for each $X \in \mathcal{B}_{g' \times g}$ the limit $f_X(e^{it}) = f(e^{it}X)$ exists for almost every t . If f is an NC ball map, $X \in \partial\mathcal{B}_{g' \times g}$ and $f(X)$ is defined, then a (nonzero) vector γ such that $\|f(X)\gamma\| = \|\gamma\|$ is called a **binding vector** and this property **binding**.

Our main result on NC ball maps which map 0 to 0 is:

Theorem 1.3. *Let $h : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ be an NC ball map with $h(0) = 0$. Then there exist unitaries $U : \mathbb{C}^d \rightarrow \mathbb{C}^d$ and $V : \mathbb{C}^{d'} \rightarrow \mathbb{C}^{d'}$ such that*

$$(1.8) \quad h(x) = V \begin{bmatrix} x & 0 \\ 0 & \tilde{h}(x) \end{bmatrix} U^*,$$

where $\tilde{h} : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{(d'-g') \times (d-g)}$ is an NC analytic contraction-valued map with $\tilde{h}(0) = 0$.

Conversely, every NC analytic h satisfying (1.8) for unitaries U, V and an NC analytic contraction-valued map \tilde{h} fixing the origin, is an NC ball map $\mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ sending 0 to 0.

The proof of the theorem is completed in §4. The general result is built off the linear version of the theorem, Theorem 3.3 which appears as Corollary 3.4 in [BH1]. See also [BH2]. As an illustration of Theorem 1.3 we describe a special case. For convenience, we adopt the notation $\mathcal{B}_{g'}$ for $\mathcal{B}_{g' \times 1}$.

Corollary 1.4. *If $h : \mathcal{B}_{g'} \rightarrow \mathcal{B}_{d'}$ is an NC ball map with $h(0) = 0$, then h is linear and there is a unique isometry $M \in \mathbb{C}^{d' \times g'}$ such that $h = Mx$. In particular, if $d' < g'$ then no such NC ball maps exist.*

Proof. When h maps $\mathcal{B}_{g'}$ to $\mathcal{B}_{d'}$ then the $\tilde{h}(x)$ column is gone. Moreover,

$$M = V^* \begin{bmatrix} I \\ 0 \end{bmatrix}$$

is an isometry. ■

1.4. NC Ball maps f when $f(0)$ is not necessarily 0. In the previous section we treated NC ball maps f with $f(0) = 0$, an assumption we drop in this section. The strategy is to compose f with a bianalytic automorphism of an NC ball to reduce the problem to the $f(0) = 0$ setting. §1.4.1 contains information on bianalytic mappings on a NC ball, while the main results appear in §1.4.2.

1.4.1. *Linear fractional transformations.* For a given $d' \times d$ scalar matrix v with $\|v\| < 1$, define $\mathcal{F}_v : \mathcal{B}_{d' \times d} \rightarrow \mathcal{B}_{d' \times d}$ by

$$(1.9) \quad \mathcal{F}_v(u) := v - (I_{d'} - vv^*)^{1/2}u(I_d - v^*u)^{-1}(I_d - v^*v)^{1/2}.$$

Of course it must be shown that \mathcal{F}_v actually takes values in $\mathcal{B}_{d' \times d}$. This is done in Lemma 1.6 below.

Linear fractional transformations such as \mathcal{F}_v are common in circuit and system theory, since they are associated with energy conserving pieces of a circuit (cf. [Wo]).

Lemma 1.5. *Suppose \mathcal{D} is an open NC domain containing 0. If $u : \mathcal{D} \rightarrow \mathcal{B}_{d' \times d}$ is NC analytic, then $\mathcal{F}_v(u(x))$ is an NC analytic function (in x) on \mathcal{D} .*

Proof. See §5. ■

Notice that if $d = d' = 1$, then v is a scalar and u is a scalar NC analytic function, hence

$$\mathcal{F}_v(u) = (v - u)(1 - u\bar{v})^{-1} = (1 - u\bar{v})^{-1}(v - u).$$

Now fix $v \in \mathbb{D}$ and consider the map $\mathbb{D} \rightarrow \mathbb{C}$, $u \mapsto \mathcal{F}_v(u)$. This map is a linear fractional map that maps the unit disc to the unit disc, maps the unit circle to the unit circle, and maps v to 0.

The geometric interpretation of the map in NC variables in (1.9) is similar. Suppose we fix $N \in \mathbb{N}$ and $V \in \mathcal{B}_{d' \times d}(N)$ with $\|V\| < 1$ and consider the map

$$(1.10) \quad U \mapsto \mathcal{F}_V(U).$$

The first part of Lemma 1.6 tells us that the map defined in (1.10) maps the unit ball of $d' \times d$ -tuples of $N \times N$ matrices to the unit ball of $d' \times d$ -tuples of $N \times N$ matrices carrying the boundary to the boundary. The third part of Lemma 1.6 tells us that $\mathcal{F}_V(V) = 0$; that is, the map given in (1.10) takes V to 0.

Lemma 1.6. *Suppose that $N \in \mathbb{N}$ and $V \in \mathcal{B}_{d' \times d}(N)$ with $\|V\| < 1$.*

- (1) $U \mapsto \mathcal{F}_V(U)$ maps the $\mathcal{B}_{d' \times d}(N)$ into itself with boundary to the boundary.
- (2) If $U \in \mathcal{B}_{d' \times d}(N)$, then $\mathcal{F}_V(\mathcal{F}_V(U)) = U$.
- (3) $\mathcal{F}_V(V) = 0$ and $\mathcal{F}_V(0) = V$.

Proof. See §5. ■

1.4.2. *Classification of NC Ball maps.* General NC ball maps - those where $f(0)$ is not necessarily 0 - are described using the linear fractional transformation \mathcal{F} .

Theorem 1.7. *Let $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ be an NC ball map with $f(0) \notin \partial \mathcal{B}_{d' \times d}$. Then*

$$(1.11) \quad f(x) = \mathcal{F}_{f(0)}(\varphi(x)),$$

where

$$(1.12) \quad \varphi(x) = \mathcal{F}_{f(0)}^{-1}(f(x)) = V \begin{bmatrix} x & 0 \\ 0 & \tilde{\varphi}(x) \end{bmatrix} U^*$$

for some unitaries $U : \mathbb{C}^d \rightarrow \mathbb{C}^d$ and $V : \mathbb{C}^{d'} \rightarrow \mathbb{C}^{d'}$ and an NC analytic contraction-valued map $\tilde{\varphi}$ with $\|\tilde{\varphi}(0)\| < 1$.

Conversely, every NC analytic f satisfying (1.11) and (1.12) for unitaries U, V and $\tilde{\varphi}$ as above, is an NC ball map $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ with $f(0) \notin \partial \mathcal{B}_{d' \times d}$.

Proof. Define $\varphi(x) := \mathcal{F}_{f(0)}^{-1}(f(x))$. Then $\varphi(0) = 0$. By Lemma 1.5, $\varphi(x)$ is an NC analytic map. Hence it is an NC ball map sending 0 to 0 and is thus classified by Theorem 1.3. Moreover, the equation (1.11) is implied by Lemma 1.6.(2). The converse easily follows from Lemmas 1.5 and 1.6. ■

The results of §1.3 and §1.4 are treated in Part I of this paper.

1.5. More generality. In this subsection we extend the main results presented so far in two directions. The first concerns LMIs. Our interest will be in properties of the set of all solutions to a given LMI. In §1.5.2 we will define what we mean by an LMI, then show that the set of solutions to a “monic” LMI equals a general type of matrix ball we call a pencil ball. Ultimately we would like to study maps from pencil balls to pencil balls and this paper is a beginning which handles the special case where the domain pencil ball is the ordinary NC ball $\mathcal{B}_{g' \times g}$ (see Corollary 1.10). Eventually we hope to understand which NC analytic change of variables takes one LMI to another. Work is in progress on such problems.

In the next generalization we do not have applications in mind, but do something that is mathematically natural. A basic notion in several complex variables is the Shilov or distinguished boundary. A natural problem is to classify NC analytic functions mapping the ball to the ball and carrying the distinguished boundary to the boundary. Classification of linear maps of this type proves to be an interesting challenge tackled in §7 and §9. For NC analytic maps we introduce the semi-distinguished boundary (a set larger than the distinguished boundary) and study the NC

analytic functions mapping the semi-distinguished boundary to the boundary. All of this we only do for balls of vectors, rather than balls of matrices, that is for $g = 1$.

1.5.1. *Linear pencils.* Let

$$(1.13) \quad L(x) := A_{11}x_{11} + \cdots + A_{g'g}x_{g'g}$$

denote an **NC analytic truly linear pencil** in x . If the matrices A_{ij} that are used to define it are in $\mathbb{C}^{d' \times d}$, then $L(x)$ is called a $d' \times d$ linear pencil. As an example, for $g' = 2$ and $g = 1$,

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

the linear pencil is

$$L(x) = \begin{bmatrix} x_{11} & 2x_{11} + x_{21} \\ 3x_{11} - x_{21} & 4x_{11} \end{bmatrix}.$$

1.5.2. *Linear matrix inequalities and (pencil) balls.* Let \tilde{L} be a $d \times d$ monic symmetric linear pencil. The positivity domain of \tilde{L} is defined to be

$$\mathcal{D}_{\tilde{L}} := \{X \in \mathcal{M}_{g' \times g} \mid \tilde{L}(X) \succeq 0\}.$$

In other words, it is the set of all solutions to the LMI $\tilde{L}(X) \succeq 0$. We wish to analyze this solution set and we can use results on balls which we have already obtained. Now we describe $\mathcal{D}_{\tilde{L}}$ as a type of ball. To do this write \tilde{L} as $\tilde{L} = I + L + L^*$ where L is a $d \times d$ NC analytic truly linear pencil, then to $L(x)$ we associate the **(pencil) ball**

$$(1.14) \quad \mathcal{B}_L := \bigcup_{n=1}^{\infty} \{X \in \mathcal{M}_{g' \times g}(n) \mid I_{dn} - L(X)^*L(X) \succeq 0\} = \bigcup_{n=1}^{\infty} \{X \in \mathcal{M}_{g' \times g}(n) \mid \|L(X)\| \leq 1\}.$$

Observe that $\mathcal{B}_{g' \times g} = \mathcal{B}_L$ for

$$L(x) = \sum_{i,j} E_{ij}x_{ij}$$

with E_{ij} being the elementary $g' \times g$ matrix with 1 located at position (i, j) .

Lemma 1.8. For $X \in \mathcal{M}_{g' \times g}$,

$$(1.15) \quad \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \in \mathcal{D}_{\tilde{L}} \quad \text{iff} \quad X \in \mathcal{B}_L.$$

Furthermore,

$$(1.16) \quad \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \in \partial \mathcal{D}_{\tilde{L}} \quad \text{iff} \quad X \in \partial \mathcal{B}_L.$$

Proof. By definition,

$$\tilde{L} \left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} I & L(X) \\ L(X)^* & I \end{bmatrix}.$$

■

1.5.3. *Pencil ball maps.* Now we turn to $\mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_L$ maps. As a generalization of NC ball map, given a linear pencil L , an NC analytic mapping $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_L$ will be called a **pencil ball map** provided $\|L(f(X))\| = 1$, whenever $\|X\| = 1$ and $f(X)$ is defined. Lemma 1.8 tells us that understanding pencil ball maps is equivalent to understanding maps on the sets of solutions to certain types of LMIs.

Theorem 1.9. *Let L be a $d' \times d$ NC analytic truly linear pencil and $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_L$ a pencil ball map with $f(0) = 0$. Write $h := L \circ f$. Then there exist unitaries $U : \mathbb{C}^d \rightarrow \mathbb{C}^d$ and $V : \mathbb{C}^{d'} \rightarrow \mathbb{C}^{d'}$ such that*

$$(1.17) \quad h(x) = V \begin{bmatrix} x & 0 \\ 0 & \tilde{h}(x) \end{bmatrix} U^*,$$

where \tilde{h} is an NC analytic contraction-valued map.

Proof. Follows easily by applying Theorem 1.3 to h . ■

Corollary 1.10. *Let L be a $d' \times d$ NC analytic truly linear pencil and $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_L$ a pencil ball map with $\|L \circ f(0)\| < 1$. Then*

$$(1.18) \quad L \circ f(x) = \mathcal{F}_{L \circ f(0)}(\varphi(x)),$$

where $\varphi(x) = \mathcal{F}_{L \circ f(0)}(L \circ f(x))$ is an NC ball map $\mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ taking 0 to 0 and is therefore completely described by Theorem 1.3.

Proof. Apply Theorem 1.7 to $L \circ f(x)$. ■

1.5.4. *Semi-distinguished pencil ball maps.* Many of our proofs with little extra effort work for a class of functions more general than pencil ball maps. These involve the notion of distinguished boundary which we now define.

The Shilov boundary or **distinguished boundary** of $\mathcal{B}_{g' \times g}(N)$ is the smallest closed subset Δ of $\mathcal{B}_{g' \times g}(N)$ with the following property: For $f : \mathcal{B}_{g' \times g}(N) \rightarrow \mathbb{C}^K$ analytic and continuous to the boundary $\partial \mathcal{B}_{g' \times g}(N)$, for any $X \in \mathcal{B}_{g' \times g}(N)$ we have

$$(1.19) \quad \|f(X)\| \leq \max_{U \in \Delta} \|f(U)\|.$$

In other words, the maximum of f over $\mathcal{B}_{g' \times g}(N)$ occurs in the distinguished boundary. We refer the reader to [Kr, p. 145] or [He, Ch. 4] for more details.

It is a Theorem [Ai, p. 77] that the distinguished boundary of $\mathcal{B}_g(N)$ is

$$\{X \in \mathcal{B}_g(N) \mid X^*X = I\}.$$

Accordingly, we let $\partial_{\text{dist}}\mathcal{B}_g$ denote the disjoint union of these distinguished boundaries and call this the distinguished boundary of \mathcal{B}_g . A further discussion of distinguished boundaries for $\mathcal{B}_{g' \times g}$ is in §6.3.

An NC analytic function $f : \mathcal{B}_g \rightarrow \mathcal{B}_L$ satisfying $f(0) \notin \partial\mathcal{B}_L$ and

$$(1.20) \quad f(\partial_{\text{dist}}\mathcal{B}_g) \subseteq \partial\mathcal{B}_L$$

is called a **distinguished pencil ball map**. Here, (1.20) means that for every isometry X for which $\lim_{\delta \nearrow 1} f(\delta X)$ exists, this limit lies in $\partial\mathcal{B}_L$.

A natural open question is: classify distinguished pencil ball maps. Our proof of Theorem 8.1 does something like this but a little weaker. A key distinction between the semi-distinguished maps and the case treated earlier in Theorems 1.3 and 1.9 occurs with *linear* distinguished ball maps. These we find much harder to classify than linear NC ball maps, which we leave as an interesting open question.

Definition 1.11. The **semi-distinguished boundary** of $\mathcal{B}_{g'}$ is defined to be

$$\partial_{\text{dist}}^{1/2}\mathcal{B}_{g'} := \bigcup_{n=1}^{\infty} \left\{ X \in \mathcal{B}_{g'}(n) \mid X^*X \text{ is a projection of dimension } \geq \frac{1}{2}n \right\}.$$

An NC analytic function $f : \mathcal{B}_{g'} \rightarrow \mathcal{B}_L$ satisfying $f(0) \notin \partial\mathcal{B}_L$ and

$$(1.21) \quad f\left(\partial_{\text{dist}}^{1/2}\mathcal{B}_{g'}\right) \subseteq \partial\mathcal{B}_L$$

is called a **semi-distinguished pencil ball map**. Here, (1.21) means that for every $X \in \partial_{\text{dist}}^{1/2}\mathcal{B}_{g'}$ for which $\lim_{\delta \nearrow 1} f(\delta X)$ exists, this limit lies in $\partial\mathcal{B}_L$.

The study of semi-distinguished pencil ball maps is the subject of Part II of this article. For semi-distinguished pencil ball maps we get a weak version of the pencil ball map classification Theorem 1.9 – see Theorem 8.1.

Part I. Binding

2. MODELS FOR NC CONTRACTIONS

Let S denote the (g' -tuple of) shift(s) on **noncommutative Fock space** $\mathcal{F}_{g'}$. The Hilbert space $\mathcal{F}_{g'}$ is the Hilbert space with orthonormal basis consisting of words $\langle x \rangle$ in g' NC variables $x = (x_1, \dots, x_{g'})$. Then $S_j w = x_j w$ for a word $w \in \langle x \rangle$ and S_j extends by linearity and continuity to $\mathcal{F}_{g'}$. The key properties we need about S are:

$$(2.1) \quad \begin{aligned} S_j^* S_\ell &= \delta_j^\ell I \quad \text{for } j, \ell = 1, \dots, g' \\ I - \sum_{j=1}^{g'} S_j S_j^* &= P_0, \end{aligned}$$

where P_0 is the (rank one) projection onto the span of the empty word.

A **column contraction** is a g' -tuple of square matrices (operators),

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_{g'} \end{bmatrix}$$

such that $I - X^* X = I - \sum X_j^* X_j \succeq 0$. If X acts on finite dimensional space, then X is a column contraction if and only if $X \in \mathcal{B}_{g'}$. In general, X is a column contraction if and only if X^* is a row contraction. Row contractions (and so column contractions too) are well studied – e.g. by Popescu and also Arveson. A **strict column contraction** is a column contraction X for which there is an $\varepsilon > 0$ such that $I - \sum X_j^* X_j \succeq \varepsilon$. If X is acting on a finite dimensional space, this last condition is equivalent to $I - \sum X_j^* X_j \succ 0$, i.e., $X \in \text{int } \mathcal{B}_{g'}$. Column contractions are modeled by S^* , which is the content of Lemma 2.1 below and a major motivation for these definitions. We do not use this property of the S_j until proving Theorem 6.2.

Lemma 2.1 ([Fr],[Po1]). *If X is a strict column contraction acting on a Hilbert space \mathcal{H} , then there is a Hilbert space \mathcal{K} and an isometry $V : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{F}_{g'}$ such that $VX = (I \otimes S^*)V$; i.e., for each j , $VX_j = (I \otimes S_j^*)V$ and in particular, for each word $w \in \langle x \rangle$, $Vw(X) = (I \otimes w(S^*))V$. Here I is the identity on \mathcal{K} . Further, if $X \in \mathcal{B}_{g'}(N)$ (so is a tuple of matrices), then the dimension of \mathcal{K} can be assumed to be at most N .*

A natural generalization of the g' -tuple of shifts on Fock space to the $\mathcal{M}_{g' \times g}$ and its (sequence of) ball(s) is

$$\mathbb{X} = \left[S_j^* \otimes S_\ell \right]_{j,\ell=1}^{g',g}$$

(A word of *caution*: we have abused notation by using S_j to denote shifts on both $\mathcal{F}_{g'}$ and \mathcal{F}_g .) The operator \mathbb{X} should be compared to the *reconstruction operator* in [Po3].

Though we do not know if \mathbb{X} serves as a universal model for $\mathcal{B}_{g' \times g}$ in the same way that S does for $\mathcal{B}_{g'}$, it does serve as a type of boundary for NC analytic functions. The statement of the results requires approximating \mathbb{X} by matrices. The operator (not matrix) \mathbb{X} acts upon $\mathcal{F}_{g' \times g}$ – the Hilbert space with orthonormal basis consisting of words in $g'g$ NC variables $x = (x_{j,\ell})_{j,\ell=1}^{g',g}$. Given a natural number n , let $\mathcal{F}_g(n)$ denote the span of words of length at most n in \mathcal{F}_g , and set $\mathcal{F}_{g' \times g}(n) = \mathcal{F}_{g'}(n) \otimes \mathcal{F}_g(n)$. Let \mathbb{X}_n denote the compression of \mathbb{X} to the (semi-invariant finite dimensional) subspace $\mathcal{F}_{g' \times g}(n)$.

Lemma 2.2. *Let P_n denote the projection onto the complement of the span of \emptyset in $\mathcal{F}_{g'}(n)$ (and also in $\mathcal{F}_g(n)$) and let Q_n denote the projection onto the complement of the span of $\{w \mid w \text{ is a word of length } n\}$ in $\mathcal{F}_{g'}(n)$ (and also in $\mathcal{F}_g(n)$). Then:*

$$(2.2) \quad \begin{aligned} \mathbb{X}_n^* \mathbb{X}_n &= I_g \otimes P_n \otimes Q_n, \\ \mathbb{X}_n \mathbb{X}_n^* &= I_{g'} \otimes Q_n \otimes P_n. \end{aligned}$$

Remark 2.3. In view of the definition of $\mathcal{B}_{g' \times g}$, it is natural to think of an NC analytic function h on $\mathcal{B}_{g' \times g}$ as a function of the $g'g$ variables $x_{j,\ell}$, $1 \leq j \leq g'$ and $1 \leq \ell \leq g$. In turn, a monomial m in $(x_{j,\ell})$ can be viewed as a *homogeneous* monomial $u \otimes v$, where u and v are monomials of the same length (same as the length of m) and u and v monomials in NC variables y_j ($1 \leq j \leq g'$) and z_ℓ ($1 \leq \ell \leq g$) respectively. In this way,

$$h = \sum_{\alpha} \sum_{|u|=|v|=\alpha} a_{u \otimes v} u \otimes v = \sum_{\alpha} h^{(\alpha)}.$$

For instance, the monomial $x_{23}x_{41}$ is identified with $y_2y_4 \otimes z_3z_1$.

We want to evaluate NC analytic functions $\mathcal{B}_{g' \times g} \rightarrow \mathcal{M}_{d' \times d}$ on \mathbb{X}_n , which is a norm one matrix thereby causing power series convergence difficulties. However, evaluating NC analytic functions on *nilpotent* tuples $X \in \mathcal{B}_{g' \times g}$ behaves especially well. Here a tuple X is called **nilpotent** of order β if $w(X) = 0$ for every word w of length $\geq \beta$.

Lemma 2.4. *If $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{M}_{d \times d'}$ is NC analytic and $X \in \mathcal{B}_{g' \times g}$ is nilpotent of order β , then $f(X)$ is defined and moreover,*

$$f(X) = \sum_{\alpha \leq \beta} f^{(\alpha)}(X).$$

In particular, if f is an NC ball map, $f(0) = 0$, and $Y \in \partial \mathcal{B}_{g' \times g}$ is nilpotent of order two, then

$$f(Y) = f^{(1)}(Y).$$

Proof. Let $X \in \mathcal{B}_{g' \times g}$ be given and let r denote the series radius for f . For $z \in \mathbb{D}$ with $|z| < r$ the power series expansion for $f(zX)$ converges. The nilpotent hypothesis gives,

$$f(zX) = \sum_{\alpha \leq \beta} f^{(\alpha)}(X)z^\alpha.$$

Since $f(zX)$ is analytic for $|z| < 1$ and is equal to the polynomial on the right hand side above for $|z| < r$, equality holds for all z .

If $Y \in \partial\mathcal{B}_{g' \times g}$ and Y is nilpotent of order two, the argument above shows,

$$f(zY) = \sum_{\alpha \leq 1} f^{(\alpha)}(Y)z^\alpha.$$

Moreover, the assumption $f(0) = 0$ implies $f^{(0)} = 0$. Choosing $z = 1$ gives $f(Y) = f^{(1)}(Y)$. ■

Lemma 2.5.

(a) *Suppose p is an NC polynomial of degree N with $\mathbb{C}^{d' \times d}$ coefficients in $g'g$ variables and $p(0) = 0$.*

(1) *If*

$$0 \preceq I - \mathbb{X}_n^* \mathbb{X}_n - p(\mathbb{X}_n)^* p(\mathbb{X}_n)$$

for each $n \leq N$, then $p = 0$.

(2) *If*

$$0 \preceq I - \mathbb{X}_n \mathbb{X}_n^* - p(\mathbb{X}_n) p(\mathbb{X}_n)^*$$

for each $n \leq N$, then $p = 0$.

(b) *Suppose $h : \mathcal{B}_{g' \times g} \rightarrow \mathcal{M}_{d' \times d}$ is NC analytic. If $h(\mathbb{X}_n) = 0$ for each n , then $h = 0$.*

Proof. (a) Write

$$p = \sum_{\alpha=0}^m p^{(\alpha)}$$

as in Remark 2.3. In particular,

$$p^{(\alpha)} = \sum_{|u|=|v|=\alpha} a_{u \otimes v} u \otimes v,$$

and $a_{u \otimes v} \in \mathbb{C}^{d' \times d}$.

By hypothesis $a_\emptyset = 0$, so that $p^{(0)} = 0$. Now suppose $p_k = 0$ for $k < n$. Let w be a word of length n and $\gamma \in \mathbb{C}^g$ be given. From Lemma 2.2, we have

$$0 = (I - \mathbb{X}_n^* \mathbb{X}_n) \gamma \otimes w \otimes \emptyset.$$

Hence,

$$0 = p(\mathbb{X}_n)\gamma \otimes w \otimes \emptyset = p^{(n)}(\mathbb{X}_n)\gamma \otimes w \otimes \emptyset = \sum_{|v|=n} p_{w \otimes v} \gamma \otimes \emptyset \otimes v.$$

Thus $p_{w \otimes v} = 0$ and it follows that $p^{(n)} = 0$.

(b) This proof is similar. Here is a brief outline. First note that $0 = h(0)$. Let r denote the series radius for h . Fix N . For $|z| < r$ and for any $n \leq N$, by Lemma 2.4 we have (since \mathbb{X}_n is nilpotent of order $n \leq N$)

$$h(\mathbb{X}_n) = \sum_{\alpha=1}^N h^{(\alpha)}(\mathbb{X}_n).$$

If we now let p denote the polynomial $\sum_{\alpha=1}^N h^{(\alpha)}$ of degree N , it follows from (a) that $p = 0$. Since this is true for all N , we see $h = 0$. ■

3. NC ISOMETRIES

This section has two parts. The first shows that the linear part of an NC ball map is an NC ball map, i.e., it is what is commonly known as a complete isometry. The second subsection classifies these linear NC ball maps. Recall that an NC analytic function $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ is an NC ball map provided it is NC analytic and contraction-valued in the interior of $\mathcal{B}_{g' \times g}$ and for $X \in \mathcal{B}_{g' \times g}(N)$ with $\|X\| = 1$, $\|f(e^{it}X)\| = 1$ for almost every $t \in \mathbb{R}$.

3.1. Pencil ball maps have isometric derivatives. A linear mapping $\psi : \mathbb{C}^{g' \times g} \rightarrow \mathbb{C}^{d' \times d}$ is completely determined by its action on the matrix units $E_{j,\ell} \in \mathbb{C}^{g' \times g}$ with a 1 in the (j, ℓ) position and 0 elsewhere. The mapping ψ then naturally extends to a mapping, still denoted ψ , on $\mathbb{C}^{n \times n} \otimes \mathbb{C}^{g' \times g}$ by the formula

$$(3.1) \quad \psi \left(\left[X_{j,\ell} \right]_{j,\ell} \right) = \sum X_{j,\ell} \otimes \psi(E_{j,\ell}) \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{d' \times d}.$$

For notational simplicity, the formula above is written $\psi(X)$. The mapping ψ is **completely isometric** if $\|\psi(X)\| = \|X\|$ for each $X \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{g' \times g}$ and each n , and is **completely contractive** if $\|\psi(X)\| \leq \|X\|$ for all X .

Proposition 3.1. *Suppose $f : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ is an NC analytic map with $f(0) = 0$. If f is an NC ball map, then $f^{(1)}$, the linear part of f , is a complete isometry.*

Proof. We start by observing that, in view of Lemma 2.4,

$$(3.2) \quad f^{(1)} \left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) = f \left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right)$$

for every $X \in \mathcal{B}_{g' \times g}$.

If f is an NC ball map, then for $X \in \partial \mathcal{B}_{g' \times g}$

$$(3.3) \quad 1 = \|X\| = \left\| \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right\| = \left\| f \left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) \right\|$$

by the binding property. Now by (3.2),

$$(3.4) \quad \left\| f \left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) \right\| = \left\| f^{(1)} \left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} 0 & f^{(1)}(X) \\ 0 & 0 \end{bmatrix} \right\| = \|f^{(1)}(X)\|.$$

From (3.3) and (3.4) we obtain $\|f^{(1)}(X)\| = 1$ for all X with $\|X\| = 1$. ■

Remark 3.2. This remark does not contribute to the proofs, rather it is for the sake of reconciling the definitions of complete isometries and contractions given here with what is typically found in the literature (cf. [Pa]).

Often a completely contractive (resp. isometric) mapping $\psi : \mathbb{C}^{g' \times g} \rightarrow \mathbb{C}^{d' \times d}$ is defined as follows. Given n , let $(\mathbb{C}^{g' \times g})^{n \times n}$ denote the $n \times n$ matrices with entries from $\mathbb{C}^{g' \times g}$ and define $1_n \otimes \psi : (\mathbb{C}^{g' \times g})^{n \times n} \rightarrow (\mathbb{C}^{d' \times d})^{n \times n}$ by

$$1_n \otimes \psi \left(\left[Y_{\alpha, \beta} \right]_{\alpha, \beta=1}^n \right) = \left[\psi(Y_{\alpha, \beta}) \right]_{\alpha, \beta=1}^n.$$

In this definition, the block matrix $Y = \left[Y_{\alpha, \beta} \right]_{\alpha, \beta=1}^n$ is written as

$$Y = \sum E_{\alpha, \beta} \otimes Y_{\alpha, \beta},$$

where $E_{\alpha, \beta} \in \mathbb{C}^{n \times n}$ are the $n \times n$ matrix units. Evaluating ψ on Y becomes,

$$1_n \otimes \psi(Y) = \sum E_{\alpha, \beta} \otimes \psi(Y_{\alpha, \beta}),$$

By using the matrix units basis $E_{j, \ell}$ of $\mathbb{C}^{g' \times g}$, Y can be rewritten as

$$Y = \sum X_{j, \ell} \otimes E_{j, \ell},$$

for some $X_{j, \ell}$. Evaluating $1_n \otimes \psi$ on Y expressed as above gives equation (3.1). Passing between these two expressions for Y is known as the **canonical shuffle** in [Pa].

Letting $A_{j, \ell} = \psi(E_{j, \ell})$, equation (3.1) becomes,

$$\psi(X) = \sum X_{j, \ell} \otimes A_{j, \ell}.$$

3.2. Completely isometric maps on $\mathbb{C}^{g' \times g}$. The following theorem which classifies completely isometric maps on $\mathbb{C}^{g' \times g}$ is the main result of this section. It appears as Corollary 3.4 in [BH1] (see also [BH2]). For the readers convenience, we provide an elementary self-contained proof.

Theorem 3.3. *A linear mapping $\psi : \mathbb{C}^{g' \times g} \rightarrow \mathbb{C}^{d' \times d}$ is completely isometric if and only if there exist unitaries $U : \mathbb{C}^d \rightarrow \mathbb{C}^d$, $V : \mathbb{C}^{d'} \rightarrow \mathbb{C}^{d'}$ and a completely contractive (linear) mapping $\varphi : \mathbb{C}^{g' \times g} \rightarrow \mathbb{C}^{(d'-g') \times (d-g)}$ such that*

$$\psi(Y) = V \begin{bmatrix} Y & 0 \\ 0 & \varphi(Y) \end{bmatrix} U^*.$$

Throughout this subsection let $\psi : \mathbb{C}^{g' \times g} \rightarrow \mathbb{C}^{d' \times d}$ denote a completely isometric mapping. It is convenient to make use of the matrix units in $\mathbb{C}^{g' \times g}$. Let $\{e'_j\}$ and $\{e_j\}$ denote the standard basis for $\mathbb{C}^{g'}$ and \mathbb{C}^g respectively. Let $A_{j,\ell} = \psi(e'_j e_\ell^*)$ for $1 \leq j \leq g'$ and $1 \leq \ell \leq g$ be as in Remark 3.2. We have represented ψ in terms of the matrix

$$A = [A_{j,\ell}]_{j,\ell=1}^{g',g} \in (\mathbb{C}^{d' \times d})^{g' \times g}$$

This matrix has the formal block transpose given by

$$A^* = [A_{\ell,j}]_{j,\ell}.$$

Lemma 3.4. *If ψ is completely contractive, then A^* is a contraction.*

Proof. Choose $X = \sum_{j,\ell=1}^{g',g} e'_j e_\ell^* \otimes e_\ell (e'_j)^*$. Direct computation reveals that $X^*X = I_{g'g}$ and thus the block matrix X is a contraction. Hence

$$\psi(X) = A^*$$

is also a contraction. ■

Remark 3.5.

(1) That the converse of Lemma 3.4 is not true in general can be seen by considering the mapping

$\psi : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}$ defined by $\psi(e'_j e_\ell^*) = \delta_j^\ell$. In this case,

$$A^* = I_2,$$

but $\psi(E_{11} + E_{22}) = 2$, so that ψ is not even contraction-valued.

(2) For $g = 1$ the converse does hold. We leave this as an exercise for the interested reader.

Proposition 3.6. *A completely contractive mapping $\psi : \mathbb{C}^{g' \times g} \rightarrow \mathbb{C}^{d' \times d}$ is a complete isometry if and only if there exist there is a set $\{f_1, \dots, f_g\} \subseteq \mathbb{C}^d$ of unit vectors satisfying*

$$(3.5) \quad \langle A_{\alpha,s}f_u, A_{\beta,t}f_v \rangle = \begin{cases} 1 & \text{if } (\alpha, s, t) = (\beta, u, v) \\ 0 & \text{otherwise.} \end{cases}$$

Here $1 \leq u, v \leq g$, $1 \leq s, t \leq g$, and $1 \leq \alpha, \beta \leq g'$.

The following Lemma is an important ingredient in the proof.

Lemma 3.7. *Under the hypotheses of Proposition 3.6, the set $\{f_1, \dots, f_g\}$ is orthonormal. Moreover,*

$$(3.6) \quad h_\alpha = A_{\alpha,j}f_j \in \mathbb{C}^d \quad (1 \leq \alpha \leq g')$$

is independent of j .

Proof. Let f_j be a set of unit vectors satisfying equation (3.5). Notice first that, for fixed j , the set $\{A_{\alpha,j}f_j \mid 1 \leq \alpha \leq g'\}$ is an orthonormal set. Let \mathcal{S}_j denote the span of this set. Given j, ℓ and α ,

$$A_{\alpha,j}f_j = \sum c_\beta A_{\beta,\ell}f_\ell + \zeta$$

for some ζ orthogonal to \mathcal{S}_ℓ (and where the dependence of the coefficients c_β on α, j, ℓ has been suppressed). Taking the inner product with $A_{\gamma,\ell}f_\ell$ it follows that $c_\beta = 1$ if $\beta = \alpha$ and $c_\beta = 0$ otherwise; i.e.,

$$A_{\alpha,j}f_j = A_{\alpha,\ell}f_\ell + \zeta.$$

On the other hand both $A_{\alpha,j}f_j$ and $A_{\alpha,\ell}f_\ell$ are unit vectors and thus $\zeta = 0$. Hence, $A_{\alpha,j}f_j$ is independent of j and

$$h_\alpha = A_{\alpha,j}f_j$$

is unambiguously defined.

Since $A_{\alpha,j}$ is a contraction (as it is, by definition, $\varphi(E_{\alpha,j})$) and since $\|f_j\| = 1$ and

$$\|A_{\alpha,j}f_j\| = 1,$$

it follows that

$$f_j = A_{j,\alpha}^* h_\alpha,$$

and is thus independent of α .

Using this last claim, consider, for $j \neq \ell$,

$$2 \geq \|(A_{j,\alpha}^* + e^{it} A_{\ell,\alpha}^*) h_\alpha\|^2 = 2 + 2 \operatorname{Re} e^{it} \langle f_j, f_\ell \rangle.$$

It follows that $\langle f_j, f_\ell \rangle = 0$. Here we have used

$$\varphi((e_j e_\alpha^* \otimes e_\alpha e_j^* + e^{-it} e_\ell e_\alpha^* \otimes e_\alpha e_\ell^*)) = A_{\alpha,j} + e^{-it} A_{\alpha,\ell}$$

is a contraction, $\left\| \begin{bmatrix} 1 & e^{it} \end{bmatrix} \right\|^2 = 2$, and $\|h_\alpha\| = 1$. ■

Proof of Proposition 3.6. Suppose such f 's exist. Let $X \in \mathbb{C}^{g' \times g} \otimes \mathbb{C}^{n \times n}$ with $\|X\| = 1$ be given. Thus X is a contraction and there is a unit vector $x = \sum x_j \otimes e_j$ such that $\|Xx\| = 1$. In particular,

$$\sum x_t^* X_{\alpha,t}^* X_{\alpha,s} x_s = 1.$$

Thus,

$$\begin{aligned} \langle \psi(X)^* \psi(X) \sum_u x_u \otimes f_u, \sum_v x_v \otimes f_v \rangle &= \sum (f_v^* A_{\beta,t}^* A_{\alpha,s} f_u) (x_v^* X_{\beta,t}^* X_{\alpha,s} x_u) \\ &= \sum x_t^* X_{\alpha,t}^* X_{\alpha,s} x_s = 1. \end{aligned}$$

Of course we also must be careful to check, in view of the orthonormality of $\{f_1, \dots, f_g\}$ of Lemma 3.7,

$$\langle \sum_u x_u \otimes f_u, \sum_v x_v \otimes f_v \rangle = \sum_u x_u^* x_u = 1.$$

Thus, if $\|X\| = 1$, then $\|\psi(X)\| \geq 1$. Since ψ assumed to be a contraction, the proof that ψ is completely isometric follows.

Let us now turn to the converse. Suppose ψ is completely isometric. Fix α and choose $X = \sum_\ell e_\alpha (e'_\ell)^* \otimes e_\alpha \otimes (e'_\ell)^*$. Then,

$$XX^* = g' e_\alpha e_\alpha^* \otimes e_\alpha e_\alpha^*.$$

Thus, $\varphi(X) = \sum A_{1,\ell} \otimes e_1 (e'_\ell)^*$ has norm at most $\sqrt{g'}$. Equivalently,

$$\Delta_\alpha = \begin{bmatrix} A_{\alpha,1} & \dots & A_{\alpha,g'} \end{bmatrix}$$

has norm at most $\sqrt{g'}$. Suppose now that

$$h = \begin{bmatrix} h_1 \\ \vdots \\ h_{g'} \end{bmatrix}$$

and $\|\Delta_\alpha h\|^2 = g'$. Then, using the fact that each $A_{\alpha,s}$ is a contraction,

$$\begin{aligned} g' &= \left\| \sum A_{\alpha,s} h_s \right\|^2 = \left| \sum_{s,t} \langle A_{\alpha,s} h_s, A_{\alpha,t} h_t \rangle \right| \leq \sum_{s,t} |\langle A_{\alpha,s} h_s, A_{\alpha,t} h_t \rangle| \\ &\leq \sum_{s,t} \|A_{\alpha,s} h_s\| \|A_{\alpha,t} h_t\| \leq \sum_{s,t} \|h_s\| \|h_t\| = \left(\sum_{s,t} \|h_s\| \right)^2 \leq g' \|h\|^2 = g'. \end{aligned}$$

The Cauchy-Schwartz inequality was used in two of the inequalities. Because equality prevails in the end, we must have equality in the inequalities. Therefore, $\|h_s\|^2 = \frac{1}{g'}$ for each s and moreover,

$$\langle A_{\alpha,s}h_s, A_{\alpha,t}h_t \rangle = \frac{1}{g'}$$

for each s .

Choose $X = \sum_{j,\ell} e_j(e'_\ell)^* \otimes e_j(e'_\ell)^*$ and note $\|X\|^2 = gg'$. Then

$$\varphi(X) = \sum A_{j,\ell} e_j(e'_\ell)^*$$

has norm squared exactly gg' . In particular, there is a unit vector

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_{g'} \end{bmatrix}$$

such that $\|\varphi(X)f\|^2 = gg'$. Hence

$$gg' = \sum_{\alpha} \|\Delta_{\alpha}f\|^2.$$

From the paragraph above $\|\Delta_{\alpha}\|^2 \leq g'$ and thus for each α we must have $\|\Delta_{\alpha}f\|^2 = g'$. Again in view of the preceding paragraph, it follows that $\|f_s\|^2 = \frac{1}{g'}$ for each s and moreover

$$(3.7) \quad \langle A_{\alpha,s}f_s, A_{\alpha,t}f_t \rangle = \frac{1}{g'}$$

for every α, s, t .

Fix α . Applying the matrix A^* of Lemma 3.4 to the vector $f_1 \otimes e_{\alpha}$ produces the vector

$$\begin{bmatrix} A_{\alpha,1}f_1 \\ A_{\alpha,2}f_1 \\ \vdots \\ A_{\alpha,g'}f_1 \end{bmatrix}.$$

Since the first entry has norm $\sqrt{\frac{1}{g'}}$ and the whole vector itself has norm at most $\sqrt{\frac{1}{g'}}$, it follows that $A_{\alpha,s}f_1 = 0$ whenever $s \neq 1$. Applying the same argument to the other indices u shows

$$(3.8) \quad A_{\alpha,s}f_u = 0 \quad \text{for } s \neq u.$$

For the final ingredient, fix $\alpha \neq \beta$ and let

$$Y = e_{\alpha}(e'_1)^* \otimes e_1^* + e_{\beta}(e'_2)^* \otimes e_2^*.$$

Since

$$Y^*Y = e_1e_1^* \otimes e_1e_1^* + e_2e_2^* \otimes e_2e_2^*,$$

Y is a contraction. Therefore,

$$\varphi(Y) = A_{\alpha,1} \otimes e_1^* + A_{\beta,2} \otimes e_2^*$$

is also a contraction. Let

$$F(t) = f_1 \otimes e_1 + e^{it} f_2 \otimes e_2.$$

With these notations,

$$\varphi(Y)F(t) = A_{\alpha,1}f_1 + A_{\beta,2}f_2,$$

which gives the second equality in

$$\begin{aligned} 2 &= \frac{1}{2\pi} \int (2 + e^{-it} \langle A_{\alpha,1}f_1, A_{\beta,2}f_2 \rangle + e^{it} \langle A_{\beta,2}f_2, A_{\alpha,1}f_1 \rangle) dt \\ &= \frac{1}{2\pi} \int \|\varphi(Y)F(t)\|^2 dt \leq 2. \end{aligned}$$

The inequality is a consequence of the hypothesis $\|\varphi(Y)\| \leq 1$ and $\|F(t)\|^2 = 2$. It follows that $\|\varphi(Y)F(t)\| = 1$ for every t and thus $\langle A_{\alpha,1}f_1, A_{\beta,2}f_2 \rangle = 0$ whenever $\alpha \neq \beta$.

Repeating the argument with other indices shows,

$$(3.9) \quad \langle A_{\alpha,s}f_s, A_{\beta,t}f_t \rangle = 0 \text{ if } \alpha \neq \beta.$$

(Here $s = t$ is ok so long as $\alpha \neq \beta$.)

Combining equations (3.7), (3.8), and (3.9) gives the desired (3.5). ■

3.2.1. Characterization of complete isometries. In this subsection, Theorem 3.3 is deduced from Proposition 3.6. We begin with a lemma which follows readily from Lemma 2.5.

Lemma 3.8. *Suppose the linear map $\Sigma : \mathbb{C}^{g' \times g} \rightarrow \mathbb{C}^{d' \times d}$ has the form*

$$\Sigma(x) = \begin{bmatrix} x & \sigma_1(x) \\ \sigma_2(x) & \sigma_3(x) \end{bmatrix}.$$

If Σ is a completely contractive, then $\sigma_1 = 0$ and $\sigma_2 = 0$.

Proof. For a given n we have

$$\begin{aligned} 0 &\preceq I - \Sigma(\mathbb{X}_n)^* \Sigma(\mathbb{X}_n) \\ &= \begin{bmatrix} I - \mathbb{X}_n^* \mathbb{X}_n - \sigma_2(\mathbb{X}_n)^* \sigma_2(\mathbb{X}_n) & * \\ * & * \end{bmatrix} \end{aligned}$$

Thus the upper left hand corner in the block matrix above is positive semidefinite and Lemma 2.5 implies $\sigma_2 = 0$. Reversing the order of the products shows $\sigma_1 = 0$. ■

Proof of Theorem 3.3. If ψ has the given form, then ψ is evidently completely isometric.

Conversely, suppose ψ is completely isometric. Let f_j be a set of unit vectors satisfying equation (3.5). By Lemma 3.7, the set $\{f_1, \dots, f_g\}$ is orthonormal and moreover, $h_\alpha = A_{\alpha,j}f_j$ is independent of j and $\{h_1, \dots, h_{g'}\}$ is also an orthonormal set.

Let

$$F = \begin{bmatrix} f_1 & \dots & f_g \end{bmatrix}, \quad H = \begin{bmatrix} h_1 & \dots & h_{g'} \end{bmatrix}.$$

The mappings F, H are isometries $\mathbb{C}^g \rightarrow \mathbb{C}^d$ and $\mathbb{C}^{g'} \rightarrow \mathbb{C}^d$ respectively. Further, for given β, u ,

$$h_\beta^* \sum_{\alpha,s} x_{\alpha,s} A_{\alpha,s} f u = \sum_{\alpha,s} x_{\alpha,s} h_\beta^* A_{\alpha,s} f u = x_{\alpha,s} h_\alpha^* A_{\alpha,s} f u = x_{\alpha,s}.$$

It follows that,

$$H^* \varphi(x) F = x.$$

This proves the first part of this direction of the theorem.

The isometries H and F extend to unitaries V and U respectively which produces the representation

$$\varphi(x) = V \begin{bmatrix} x & \sigma_1 \\ \sigma_2 & \sigma_3 \end{bmatrix} U^*,$$

where the block matrix $\Sigma = \begin{bmatrix} x & \sigma_1 \\ \sigma_2 & \sigma_3 \end{bmatrix}$ is completely contractive since the same is true of φ . Now Lemma 3.8 completes the proof. ■

4. PROOF OF THEOREM 1.3

In this section we prove Theorem 1.3. Accordingly, suppose $h : \mathcal{B}_{g' \times g} \rightarrow \mathcal{B}_{d' \times d}$ is an NC ball map and $h(0) = 0$. From Lemma 3.1, $h^{(1)}$, the linear part of h , is a complete isometry. By Theorem 3.3, there exists unitaries U and V and a completely contractive mapping $\tilde{h}^{(1)}$ such that

$$(4.1) \quad h^{(1)}(x) = V \begin{bmatrix} x & 0 \\ 0 & \tilde{h}^{(1)}(x) \end{bmatrix} U^*.$$

We claim that $Vh(x)U^*$ is of the desired form (1.8).

For the sake of convenience we replace $h(x)$ by $V^*h(x)U$. For $X \in \mathcal{B}_{g' \times g}(N)$ consider $\mathbb{D} \rightarrow \mathcal{M}_{d' \times d}(N)$, $z \mapsto h(zX)$, which is analytic (in z). This is a function of one complex variable, so the classical Schwarz lemma applies. Hence for all $0 \leq \delta < 1$ and $0 \leq \theta \leq 2\pi$ we have

$$(4.2) \quad 0 \preceq \delta^2 I - h(\delta e^{i\theta} X)^* h(\delta e^{i\theta} X).$$

If δ is in the series radius, we may write

$$h(\delta e^{i\theta} X) = h^{(1)}(\delta e^{i\theta} X) + h^{(\infty)}(\delta e^{i\theta} X) = \sum_{\alpha=1}^{\infty} h^{(\alpha)}(\delta e^{i\theta} X).$$

We integrate (4.2) for such δ to obtain

$$\begin{aligned} 0 &\preceq \frac{1}{2\pi} \int_0^{2\pi} (\delta^2 I - h(\delta e^{i\theta} X)^* h(\delta e^{i\theta} X)) d\theta \\ (4.3) \quad &= \delta^2 I - \delta^2 h^{(1)}(X)^* h^{(1)}(X) - \frac{1}{2\pi} \int_0^{2\pi} h^{(\infty)}(\delta e^{i\theta} X)^* h^{(\infty)}(\delta e^{i\theta} X) d\theta \\ &= \delta^2 I - \delta^2 h^{(1)}(X)^* h^{(1)}(X) - \sum_{\alpha=2}^{\infty} \delta^{2\alpha} h^{(\alpha)}(X)^* h^{(\alpha)}(X), \end{aligned}$$

where the last equality uses the homogeneity (of order α) of $h^{(\alpha)}$.

Fix an $\alpha \geq 2$ and write $\delta^{\alpha-1} h^{(\alpha)} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ for NC analytic polynomials b_j . Then by equations (4.1) and (4.3) and because the b_j are polynomials,

$$\begin{aligned} 0 &\preceq \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} X^* X & 0 \\ 0 & \tilde{h}^{(1)}(X)^* \tilde{h}^{(1)}(X) \end{bmatrix} - \begin{bmatrix} b_1(X)^* & b_3(X)^* \\ b_2(X)^* & b_4(X)^* \end{bmatrix} \begin{bmatrix} b_1(X) & b_2(X) \\ b_3(X) & b_4(X) \end{bmatrix} \\ (4.4) \quad &= \begin{bmatrix} I - X^* X & 0 \\ 0 & I - \tilde{h}^{(1)}(X)^* \tilde{h}^{(1)}(X) \end{bmatrix} \\ &\quad - \begin{bmatrix} b_1(X)^* b_1(X) + b_3(X)^* b_3(X) & b_1(X)^* b_2(X) + b_3(X)^* b_4(X) \\ b_2(X)^* b_1(X) + b_4(X)^* b_3(X) & b_2(X)^* b_2(X) + b_4(X)^* b_4(X) \end{bmatrix}. \end{aligned}$$

It follows that

$$I - \mathbb{X}_n^* \mathbb{X}_n - b_j(\mathbb{X}_n)^* b_j(\mathbb{X}_n) \succeq 0$$

for $j = 1, 3$ and all n . Lemma 2.5 thus implies $b_1 = 0$ and $b_3 = 0$.

We now multiply in the other order (consider say $\mathbb{X}_n \mathbb{X}_n^*$ instead of $\mathbb{X}_n^* \mathbb{X}_n$) to conclude that $b_2 = 0$ (also $b_1 = 0$, but that we already knew.) This shows h has the desired form and completes the proof.

5. LINEAR FRACTIONAL TRANSFORMATION OF A BALL

It is well known that the bianalytic maps on the unit disk \mathbb{D} are exactly the linear fractional maps. These act transitively on the unit disk. That is, if $w, z \in \mathbb{D}$, then there is a linear fractional map \mathcal{F} which maps w to z . It is standard in classical several complex variables that this generalizes

to special domains in \mathbb{C}^n [He]. In this subsection we give basic properties of linear fractional maps on $\mathcal{B}_{d' \times d}$.

Given a $d' \times d$ matrix v with $\|v\| < 1$, define $\mathcal{F}_v : \mathcal{B}_{d' \times d} \rightarrow \mathcal{B}_{d' \times d}$ by

$$(5.1) \quad \mathcal{F}_v(u) := v - (I_{d'} - vv^*)^{1/2} u (I_d - v^* u)^{-1} (I_d - v^* v)^{1/2}.$$

Lemma 5.1. *Suppose \mathcal{D} is an open NC domain containing 0. If $u : \mathcal{D} \rightarrow \mathcal{B}_{d' \times d}$ is NC analytic, then $\mathcal{F}_v(u(x))$ is an NC analytic function (in x) on \mathcal{D} .*

Proof. Since v is a matrix with $\|v\| < 1$, the expressions $(I_{d'} - vv^*)^{1/2}$ and $(I_d - v^* v)^{1/2}$ are constant NC analytic functions. As sums and products of NC analytic functions are NC analytic, it suffices to show that $\zeta := (I_d - v^* u)^{-1}$ is NC analytic. Note that $v^* u$ is NC analytic on \mathcal{D} . Thus ζ being the composition of the NC analytic function $(1 - z)^{-1}$ on \mathbb{D} and the NC analytic function $v^* u$ on \mathcal{D} is NC analytic as well. \blacksquare

For the convenience of the reader, we now give the basic, and known (see for instance [Po8]) properties of \mathcal{F} in a lemma generalizing Lemma 1.6. For this we define \mathcal{U}_k to be the set of all $U \in \mathcal{B}_{d' \times d}(N)$ which are isometric on a space of dimension at least Nk . For example, \mathcal{U}_d denotes the isometries in $\mathcal{B}_{d' \times d}(N)$.

Lemma 5.2. *Suppose that $N \in \mathbb{N}$ and $V \in \mathcal{B}_{d' \times d}(N)$ with $\|V\| < 1$.*

- (1) $U \mapsto \mathcal{F}_V(U)$ maps the unit ball $\mathcal{B}_{d' \times d}(N)$ into itself with boundary to the boundary. Furthermore, for each $k \leq d$, \mathcal{U}_k maps onto \mathcal{U}_k .
- (2) If $U \in \mathcal{B}_{d' \times d}(N)$, then $\mathcal{F}_V(\mathcal{F}_V(U)) = U$.
- (3) $\mathcal{F}_V(V) = 0$ and $\mathcal{F}_V(0) = V$.

Proof. The proof is motivated by linear system theory but an understanding of system theory is not needed to read the proof.

Let $y \in \mathbb{C}^{Nd}$ be given. Define

$$i = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} (I - V^*V)^{-\frac{1}{2}}(I - V^*U)y \\ -Uy \end{pmatrix} \in \mathbb{C}^{Nd} \oplus \mathbb{C}^{Nd'}.$$

Let M denote the matrix

$$(5.2) \quad M := \begin{bmatrix} (I - V^*V)^{1/2} & -V^* \\ V & (I - VV^*)^{1/2} \end{bmatrix}.$$

Straightforward computation shows M is unitary; i.e., $M^*M = I = MM^*$. Let

$$o = \begin{pmatrix} o_1 \\ o_2 \end{pmatrix} = Mi = \begin{pmatrix} y \\ (I - VV^*)^{-\frac{1}{2}}(V - U)y \end{pmatrix} \in \mathbb{C}^{Nd} \oplus \mathbb{C}^{Nd'}.$$

The relation $V(I - VV^*)^{-\frac{1}{2}} = (I - V^*V)^{-\frac{1}{2}}V$ was used in computing Mi .

Since M is unitary,

$$(5.3) \quad \|i_1\|^2 + \|i_2\|^2 = \|o_1\|^2 + \|o_2\|^2.$$

On the other hand, computations give

$$\mathcal{F}_V(U)i_1 = o_2.$$

Combining the last two equations gives

$$(5.4) \quad \|i_1\|^2 - \|\mathcal{F}_V(U)i_1\|^2 = \|o_1\|^2 - \|i_2\|^2 = \|y^2\|^2 - \|Uy\|^2 \geq 0.$$

Since the mapping $y \mapsto i_1 = (I - V^*V)^{-\frac{1}{2}}(I - V^*U)y$ is onto, the matrix $\mathcal{F}_V(U)$ is a contraction and the first part of item (1) of the lemma is proved.

To prove the second part of item (1), notice that from equation (5.4) and the fact that both $\mathcal{F}_V(U)$ and U are contractions, the dimension of the space on which $\mathcal{F}_V(U)$ is isometric is the same as the dimension of the space on which U is isometric.

We now turn to the proof of item (2). Define

$$F := \mathcal{F}_V(U) = V - (I - VV^*)^{1/2}U(I - V^*U)^{-1}(I - V^*V)^{1/2}.$$

First notice that

$$\begin{aligned} I - V^*F &= I - V^*V + V^*(I - VV^*)^{1/2}U(I - V^*U)^{-1}(I - V^*V)^{1/2} \\ &= (I - V^*V) + (I - V^*V)^{1/2}V^*U(I - V^*U)^{-1}(I - V^*V)^{1/2} \\ &= (I - V^*V)^{1/2}(I - V^*U)(I - V^*U)^{-1}(I - V^*V)^{1/2} + \\ &\quad + (I - V^*V)^{1/2}V^*U(I - V^*U)^{-1}(I - V^*V)^{1/2} \\ &= (I - V^*V)^{1/2}(I - V^*U)^{-1}(I - V^*V)^{1/2}. \end{aligned}$$

So

$$(I - V^*F)^{-1} = (I - V^*V)^{-1/2}(I - V^*U)(I - V^*V)^{-1/2}.$$

We use this and elementary calculations to obtain

$$\begin{aligned}
\mathcal{F}_V(F) &= V - (I - VV^*)^{1/2}F(I - V^*F)^{-1}(I - V^*V)^{1/2} \\
&= V - (I - VV^*)^{1/2}F(I - V^*V)^{-1/2}(I - V^*U) \\
&= V - (I - VV^*)^{1/2}V(I - V^*V)^{-1/2}(I - V^*U) + (I - VV^*)U \\
&= V - V(I - V^*U) + U - VV^*U = U.
\end{aligned}$$

For (3), compute

$$\begin{aligned}
\mathcal{F}_V(V) &= V - (I - VV^*)^{1/2}V(I - V^*V)^{-1}(I - V^*V)^{1/2} \\
&= V - (I - VV^*)^{1/2}V(I - V^*V)^{-1/2} \\
&= V - V(I - V^*V)^{1/2}(I - V^*V)^{-1/2} = 0.
\end{aligned}$$

■

Part II. Clinging

In this, and the sections to follow, we turn our attention to semi-distinguished ball maps introduced in §1.5.4. In particular, attention is restricted to the NC domains $\mathcal{B}_{g'}$.

6. NC FUNCTIONS REVISITED

This section gives several basic facts about NC analytic functions on the ball, most of which are used in the remainder of the paper. We feel several of the main results here also are of interest in their own right. A few of the results are included purely for their own sake.

6.1. Series radius of convergence. This section shows that NC power series expansions of NC analytic functions on a ball have good convergence properties. As a consequence of this convergence, bounded NC analytic functions are free analytic in the sense of Popescu [Po2].

Lemma 6.1. *If $h : \mathcal{B}_g \rightarrow \mathcal{B}_{d' \times d}$ is an NC analytic function, with NC power series expansion*

$$h = \sum_w a_w w,$$

then

$$\sum_w \|a_w\|^2 \leq d.$$

Moreover, if Z is a strict column contraction acting on a separable Hilbert space or if $Z = I \otimes S^*$ where S is the shift of Fock space \mathcal{F}_g , and $z \in \mathbb{D}$, then

$$h(zZ) = \sum a_w \otimes (zZ)^w$$

converges absolutely, $h(zZ)$ is a contraction and $z \mapsto h(zZ)$ is an analytic function on \mathbb{D} .

Proof. Let S denote the shifts introduced in §2. Let $\mathcal{F}_g(n)$ denote the span of the words of length at most n in the NC Fock space \mathcal{F}_g . Let $W_n : \mathcal{F}_g(n) \rightarrow \mathcal{F}_g$ denote the inclusion. Thus, for any finite dimensional Hilbert space \mathcal{K} , $I \otimes S_j(n) = I \otimes W_n^*(I \otimes S_j)I \otimes W_n$ is the compression of $I \otimes S_j$ to the (semi-invariant finite dimensional) subspace $\mathcal{K} \otimes \mathcal{F}_g(n)$. Here I is the identity on \mathcal{K} .

In view of the hypotheses (and since the $S_j(n)$ are nilpotent of order n),

$$h(S(n)^*) = \sum_{|w| \leq n} a_w \otimes w(S(n))^*.$$

Thus, for any vector $\gamma \in \mathbb{C}^d$,

$$\|\gamma\|^2 \geq \|h(S(n)^*)^* \gamma \otimes \emptyset\|^2 = \left\| \sum_{|w| \leq n} a_w^* \gamma \otimes w \right\|^2 = \sum_{|w| \leq n} \|a_w^* \gamma\|^2.$$

It follows that,

$$d \geq \sum_w \sum_j \|a_w^* e_j\|^2,$$

where $\{e_1, \dots, e_d\}$ is an orthonormal basis for \mathbb{C}^d . (Note that the sums over j terms on the right hand side are the squares of the Hilbert-Schmidt norms of the a_w). Since $\|a_w\|^2 = \|a_w^*\|^2 \leq \sum_j \|a_w^* e_j\|^2$, it follows that

$$d \geq \sum \|a_w\|^2.$$

Consequently, if $|z| < 1$ and $Z = (Z_1, \dots, Z_g)$ is a g tuple of operators on Hilbert space (potentially infinite dimensional) satisfying $\sum Z_j^* Z_j \leq I$ and if $|z| < 1$, then

$$h(zZ) := \sum a_w \otimes (zZ)^w$$

converges (absolutely). A favorite choice is $Z = I \otimes S^*$.

For $|z| < 1$, we have $I \otimes W_n W_n^* h(zI \otimes S^*) I \otimes W_n W_n^*$ converges in the SOT to $h(zI \otimes S^*)$. On the other hand, $I \otimes W_n^* h(zI \otimes S^*) I \otimes W_n = h(zI \otimes S(n)^*)$ which is assumed to be a contraction. Thus, $h(zI \otimes S^*)$ is a contraction.

For a general strict column contraction X , represent X as $VX = (I \otimes S^*)V$ by Lemma 2.1. For $|z| < 1$, it follows that $h(I \otimes zS^*)V = Vh(zX)$ and hence $\|h(zX)\| \leq 1$. ■

6.2. The NC Schwarz lemma. The classical Schwarz lemma from complex variables states the following: if $f : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and $f(0) = 0$, then $\|f(z)\| \leq \|z\|$ for $z \in \mathbb{D}$. There are several ways to extend this to NC analytic functions, for example Popescu [Po2, Theorem 2.4] gives one and other results of this type can be found in [Po7]. In this subsection we give two extensions of our own.

Theorem 6.2. *Suppose $f : \mathcal{B}_{g'} \rightarrow \mathcal{B}_{d' \times d}$ is an NC analytic function on $\mathcal{B}_{g'}$. If $f(0) = 0$ and $\|f(X)\| \leq 1$ for each $X \in \text{int } \mathcal{B}_{g'}$, then*

$$(6.1) \quad X^*X - f(X)^*f(X) \succeq 0,$$

for $X \in \text{int } \mathcal{B}_{g'}$.

Proof. The proof relies on the model for column contractions and the convergence result for bounded NC analytic functions f of Lemma 6.1 which allows us to evaluate bounded NC analytic functions on operators, not just matrices.

Since f maps into $\mathcal{B}_{d' \times d}$, if $|z| < 1$, then

$$(6.2) \quad I - f(zS^*)^*f(zS^*) \succeq 0,$$

by Lemma 6.1. Thus,

$$(6.3) \quad \sum_j S_j S_j^* + P_0 - f(zS^*)^*f(zS^*) \succeq 0.$$

(Here P_0 is the projection onto the span of the empty word.) From $f(0) = 0$, we obtain $f(S^*)P_0 = 0$. Hence (6.3) transforms into

$$(6.4) \quad \begin{aligned} & (I - P_0) \left(\sum_j S_j S_j^* + P_0 - f(zS^*)^*f(zS^*) \right) (I - P_0) + P_0 \left(\sum_j S_j S_j^* + P_0 - f(zS^*)^*f(zS^*) \right) P_0 = \\ & (I - P_0) \left(\sum_j S_j S_j^* + P_0 - f(zS^*)^*f(zS^*) \right) (I - P_0) + P_0 \succeq 0. \end{aligned}$$

As

$$(I - P_0) \left(\sum_j S_j S_j^* + P_0 - f(zS^*)^*f(zS^*) \right) (I - P_0) = (I - P_0) \left(\sum_j S_j S_j^* - f(zS^*)^*f(zS^*) \right) (I - P_0)$$

and

$$P_0 \left(\sum_j S_j S_j^* - f(zS^*)^*f(zS^*) \right) = 0 = \left(\sum_j S_j S_j^* - f(zS^*)^*f(zS^*) \right) P_0,$$

(6.4) is equivalent to

$$(6.5) \quad \sum_j S_j S_j^* - f(zS^*)^* f(zS^*) \succeq 0,$$

for $|z| < 1$. Replacing S by $I \otimes S$ in the argument above yields,

$$(6.6) \quad I \otimes \sum_j S_j S_j^* - f(zI \otimes S^*)^* f(zI \otimes S^*) \succeq 0.$$

Given $X \in \mathcal{B}_{g'}$ with $\|X\| < 1$, we can write $VX = (I \otimes S^*)V$, where I is the identity on a finite dimensional Hilbert space, by Lemma 2.1. Moreover, by Lemma 6.1, for $|z| < 1$,

$$Vf(zX) = f(zI \otimes S^*)V.$$

Multiply (6.6) by V^* on the left and V on the right to obtain

$$V^* \left(\sum_j I \otimes S_j S_j^* - f(zI \otimes S^*)^* f(zI \otimes S^*) \right) V = X^* X - f(zX)^* f(zX) \succeq 0,$$

for $|z| < 1$. Since $\|X\| < 1$, letting $z \nearrow 1$ completes the proof. \blacksquare

Remark 6.3. Popescu [Po2, Theorem 2.4] formulates and proves a Schwarz lemma for free analytic functions, which in our context implies that if f is a contraction-valued NC analytic function with $f(0) = 0$, then $\|f(X)\| \leq \|X\|$ for $\|X\| < 1$ and further, $\sum_{|w|=\alpha} a_w a_w^* \leq I$ for all α . (This inequality remains true even with operator coefficients a_w .)

A classical complex variables statement equivalent to Schwarz's lemma is the following: if $f : \mathbb{D} \rightarrow \mathbb{D}$ is analytic and $f(0) = 0$, then $h(z) = \frac{f(z)}{z}$ is also analytic and $h : \mathbb{D} \rightarrow \mathbb{D}$. We give a noncommutative analog of this result, which does not appear to be an immediate consequence of Theorem 6.2.

Theorem 6.4. *Suppose that $H = \begin{bmatrix} H_1 & \dots & H_{g'} \end{bmatrix}$ is a row of $d' \times d$ NC analytic functions on $\mathcal{B}_{g'}$. If for each $X \in \text{int } \mathcal{B}_{g'}$,*

$$(6.7) \quad \|H(X)X\| = \left\| \sum_j H_j(X)X_j \right\| \leq 1,$$

i.e.,

$$(6.8) \quad I - H(X)X (H(X)X)^* \succeq 0,$$

then for each $X \in \text{int } \mathcal{B}_{g'}$

$$(6.9) \quad I - H(X)H(X)^* \succeq 0.$$

Equivalently, $\|H(X)\| \leq 1$.

Proof. This proof depends upon both Lemmas 2.1 and 6.1.

Let $G(x) = H(x)x$. The hypotheses imply $G : \mathcal{B}_g \rightarrow \mathcal{M}_{d \times d'}$ is contraction-valued. Hence Lemma 6.1 applies. Denote the power series expansions for H_j by

$$H_j = \sum_{\alpha} h_j^{(\alpha)}.$$

It follows that the power series expansion (by homogeneous terms) for G is then

$$G = \sum_{\alpha} \sum_j h_j^{(\alpha)} x_j.$$

Hence, also by Lemma 6.1, for each j the power series expansion for H_j converges for any strict column contraction Z (even for operators on an infinite dimensional Hilbert space) and for such Z ,

$$G(Z) = \sum_j H_j(Z)Z_j.$$

In particular, for $|z| < 1$ and $Z = zI \otimes S^*$, (where S is an in Lemma 2.1 and I is the identity on a finite dimensional Hilbert space),

$$G(zS^*) = \sum_j H_j(zI \otimes S^*)S_j^*.$$

Because $\|G(zI \otimes S^*)\| \leq 1$,

$$\begin{aligned} (6.10) \quad & 0 \preceq I - G(zI \otimes S^*)G(zI \otimes S)^* \\ & = I - \sum_j H_j(zI \otimes S^*)I \otimes S_j^* \sum_{\ell} I \otimes S_{\ell} H_{\ell}(zI \otimes S^*)^* \\ & = I - \sum_j H_j(zI \otimes S^*)H_j(zI \otimes S^*)^*. \end{aligned}$$

Let $X \in \text{int } \mathcal{B}_{g'}$ be a strict column contraction acting on a finite dimensional space. Express $X = V^*(I \otimes S^*)V$ according to Lemma 2.1, where I is the identity on a finite dimensional Hilbert space. For every NC analytic polynomial f and $|z| < 1$, $f(zX) = V^*f(zI \otimes S^*)V$. Hence the same holds true for NC analytic functions and in particular,

$$H_j(zI \otimes S^*)V = VH_j(zX).$$

Thus, applying V on the right and V^* on the left of equation (6.10) gives,

$$0 \preceq I - \sum_j H_j(zX)H_j(zX)^*.$$

Letting $z \nearrow 1$ concludes the proof. ■

6.3. The distinguished boundary for $\mathcal{B}_{g' \times g}$. Fix N . The distinguished (Shilov) boundary of the algebra $\mathcal{A}(\mathcal{B}_{g' \times g}(N))$, the functions which are analytic in $\text{int } \mathcal{B}_{g' \times g}(N)$ and continuous on $\mathcal{B}_{g' \times g}(N)$ is the smallest closed subset Δ of $\mathcal{B}_{g' \times g}(N)$ so that each element of $\mathcal{A}(\mathcal{B}_{g' \times g}(N))$ takes its maximum on Δ . That a smallest, as opposed simply minimal, such sets exists is a standard fact in complex analysis and the theory of uniform algebras; see [Kr, p. 145] or [He, Ch. 4] for more details.

While not needed in the sequel, the following known result explains the *distinguished* terminology in the definitions of distinguished isometry and semi-distinguished pencil ball map.

Proposition 6.5. *The distinguished boundary of $\mathcal{A}(\mathcal{B}_{g' \times g}(N))$ is $\{X \in \mathcal{B}_{g' \times g}(N) \mid X^*X = I\}$.*

That the distinguished boundary of $\mathcal{A}(\mathcal{B}_{g' \times g}(N))$ must be contained in $\{X \in \mathcal{B}_{g' \times g}(N) \mid X^*X = I\}$ follows readily from 5.2; see Proposition 6.6. For the fact that no smaller set can serve as a distinguished boundary, we refer to reader to [Ai, p. 77].

Proposition 6.6. *Fix $N \in \mathbb{N}$. If $f : \mathcal{B}_{g' \times g}(N) \rightarrow \mathbb{C}^{d' \times d}$ is continuous and analytic in $\text{int } \mathcal{B}_{g' \times g}(N)$, then for any $X \in \mathcal{B}_{g' \times g}(N)$ we have*

$$(6.11) \quad \|f(X)\| \leq \max_{U \in \mathcal{U}_k} \|f(U)\|$$

for any $0 < k \leq \min\{g', g\}$. Thus if $f(X) = 0$ for all $X \in \mathcal{B}_{g' \times g}(N)$ such that $X^*X = I$ (if $g' \geq g$) or $XX^* = I$ (if $g' < g$), then $f = 0$. For example, if $g' \geq g$, then the set of isometries \mathcal{U}_g contains the distinguished boundary of $\mathcal{B}_{g' \times g}(N)$.

Proof. First suppose $f : \mathcal{B}_{g' \times g}(N) \rightarrow \mathbb{C}$ (so that $d = 1 = d'$). Pick any $U \in \mathcal{U}_k$. By the maximum principle, the function $h(z) = f(zU)$ takes its maximum value on $|z| = 1$.

Now we use linear fractional automorphisms of the ball to prove that such an inequality holds for any X in the interior of $\mathcal{B}_{g' \times g}(N)$. Select \mathcal{F} as in Lemma 5.2 which maps 0 to X . Then $h(Z) := f(\mathcal{F}(Z))$ is analytic and maps 0 to $f(X)$. The previous paragraph applies to give

$$\|f(X)\| = \|h(0)\| \leq \max_{|z|=1} \|h(zU)\| = \max_{|z|=1} \|f(\mathcal{F}(zU))\|.$$

By Lemma 5.2(1), $\mathcal{F}(zU) \in \mathcal{U}_k$ for $|z| = 1$; so we have proved that the maximum of f occurs on \mathcal{U}_k .

To prove the statement for matrix-valued f , simply note that given unit vectors $\gamma \in \mathbb{C}^d$ and $\eta \in \mathbb{C}^{d'}$, the function $F(X) = \eta^* f(X) \gamma$ takes its maximum on \mathcal{U}_k . It follows that

$$|F(X)| \leq \max_{U \in \mathcal{U}_k} \|f(U)\|.$$

Since γ and η are arbitrary, the result follows. ■

Remark 6.7. This proposition has more content for larger k and in particular $k = \min\{g, g'\}$ is optimal.

6.4. Matrix Linksnullstellensatz. For scalar NC analytic polynomials there is an elegant Linksnullstellensatz whose proof is due to Bergman, cf. [HM]. Now we generalize it to matrices with entries which are NC analytic polynomials.

Theorem 6.8. *Given a $m \times d$ matrix P over $\mathbb{C}\langle x \rangle$ and a $n \times d$ matrix Q over $\mathbb{C}\langle x \rangle$, suppose that $P(X)v = 0$ implies $Q(X)v = 0$ for every matrix g -tuple X and vector v . Then for some $G \in \mathbb{C}\langle x \rangle^{n \times m}$ we obtain $Q = GP$.*

Proof. The rows of a matrix A will be denoted by $A_j = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jd} \end{bmatrix}$. In particular, P_j is a $1 \times d$ matrix over $\mathbb{C}\langle x \rangle$.

Let $V_d = \mathbb{C}\langle x \rangle^{1 \times d}$ denote the left $\mathbb{C}\langle x \rangle$ -module of $1 \times d$ matrices of polynomials. Note $P_j \in V_d$. Let I_d be the $\mathbb{C}\langle x \rangle$ -submodule of V_d generated by the P_j , i.e.,

$$I_d = \left\{ \sum r_j P_j \mid r_j \in \mathbb{C}\langle x \rangle \right\}.$$

I_d is the smallest subspace of V_d containing the P_j and invariant with respect to M_j =left multiplication by x_j (for each j).

Note that M_j determines a well defined linear mapping Y_j on the quotient:

$$Y_j : V_d/I_d \rightarrow V_d/I_d.$$

Let W_k denote the image of polynomials of degree at most k in the quotient V_d/I_d . These spaces are finite dimensional and $W_{k-1} \subseteq W_k$. So W_{k-1} is complemented in W_k .

Choose $N > \max$ degree of all polynomials in P and Q . Define $X_j = Y_j : W_{N-1} \rightarrow W_N$ and extend X_j to a linear mapping $W_N \rightarrow W_N$ in any way (on a complementary subspace).

Let v_j denote the element of W_N determined by the row with the polynomial 1 in the j -entry and 0 elsewhere. Define $v = \oplus v_j \in W_N^d$.

For a polynomial q , $q(X)v_j = \begin{bmatrix} 0 & \cdots & 0 & q & 0 & \cdots & 0 \end{bmatrix}$ (j -th spot). Hence $Q_j(X)v = Q_j v_j$. A similar statement is true for P_j ; i.e., $P_j(X)v = P_j \in I_d$ and so $P_j(X)v = 0$. So $Q_j(X)v = Q_j v_j$ is 0 too which means $Q_j \in I_d$. Thus there exists G_{sj} such that

$$Q_j = \sum G_{js} P_s.$$

Hence $Q = GP$, as desired. ■

7. THE LINEAR PART OF SEMI-DISTINGUISHED BALL MAPS

Having established preliminary results, we turn our attention to semi-distinguished ball maps, introduced in §1.5.4. First we show that semi-distinguished ball maps have very distinctive linear parts. And then we set about to give properties of these linear maps.

A linear map $L : \mathbb{C}^g \rightarrow \mathbb{C}^{d' \times d}$ is a **distinguished isometry** if it maps the *distinguished* boundary of \mathcal{B}_g to the boundary of $\mathcal{B}_{d' \times d}$; i.e., if for each $X \in \mathcal{B}_g$ with $X^*X = I$ we have that $\|L(X)\| = 1$. In this case a (nonzero) vector γ such that $\|L(X)\gamma\| = \|\gamma\|$ is called a **clinging vector** and this property **clinging**.

Proposition 7.1. *If f is a semi-distinguished ball map, then $f^{(1)}$, the linear part of f , is a distinguished isometry.*

Proof. The proof is the same as that of Proposition 3.1. ■

7.1. Properties of distinguished isometries. The remainder of this section is devoted to giving properties of distinguished isometries.

Proposition 7.2. *Let $L : \mathbb{C}^g \rightarrow \mathbb{C}^{d' \times d}$ be a linear map.*

(1) *L is a distinguished isometry if and only if*

$$(7.1) \quad \Delta_L(X) := (X_1^*X_1 + \cdots + X_g^*X_g) \otimes I_d - L^*(X)L(X) \succeq 0$$

and clings (i.e., $\Delta_L(X)$ is always positive semidefinite and never positive definite).

(2) *If L is completely isometric, then it is a distinguished isometry. The converse is not true.*

Proof. For the implication (\Rightarrow) in (1), given any X_i , choose a W satisfying $W^*W = X_1^*X_1 + \cdots + X_g^*X_g$. Note that it suffices to show (7.1) on a dense subset of $\mathcal{B}_{g'}$. Thus we may assume that W is invertible. Then $(X_1W^{-1})^*X_1W^{-1} + \cdots + (X_gW^{-1})^*X_gW^{-1} = I$, so by assumption, $I - L^*(XW^{-1})L(XW^{-1}) \succeq 0$ and it binds. Since L is truly linear, we multiply this inequality with W^* on the left and with W on the right: $W^*W - L^*(X)L(X) \succeq 0$ and it binds. The converse (\Leftarrow) is obvious.

First part of (2) is trivial. To finish the proof it suffices to exhibit an example of a distinguished isometry which is not a complete isometry. Consider $L(x, y) = Ax + By$ with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

$$\text{For } X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\left\| \begin{bmatrix} X \\ Y \end{bmatrix} \right\| = \sqrt{2} > \sqrt{\frac{3}{2}} = \|L(X, Y)\|$$

This shows that L is not a complete isometry.

It remains to be seen that L satisfies (7.1). We compute

$$\Delta_L(x, y) = \begin{bmatrix} \frac{1}{2}y^*y & -\frac{1}{2}y^*x & 0 \\ -\frac{1}{2}x^*y & \frac{1}{2}x^*x & 0 \\ 0 & 0 & \frac{1}{2}(x-y)^*(x-y) \end{bmatrix}.$$

The top left 2×2 block of $\Delta_L(x, y)$ can be factored as

$$\begin{bmatrix} -y^*x^* & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}x^*x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -x^{-1}y & 1 \\ 1 & 0 \end{bmatrix}.$$

This immediately implies that for invertible X , $\Delta_L(X, Y)$ is always positive semidefinite and never positive definite. For noninvertible X the same holds true by a standard density argument. \blacksquare

Remark 7.3. By way of contrast, every contractive $L : \mathbb{C}^g \rightarrow \mathbb{C}^{d \times d}$ is completely contractive. For related results see §9.

7.1.1. *The Gram representation.* A powerful tool used is a matrix representation of a quadratic NC polynomial. A key property of this representation is that matrix positivity of the quadratic NC polynomial is equivalent to the positive semidefiniteness of the representing matrix. The following lemma is needed to establish this.

Lemma 7.4. *For large enough n the set*

$$(7.2) \quad \{Xw \mid X \in (\mathbb{C}^{n \times n})^g, w \in \mathbb{C}^n\}$$

is all \mathbb{C}^{ng} .

Proof. Given $w, x_1, \dots, x_g \in \mathbb{C}^n$ with $x_j \neq 0$, choose $X_j \in \mathbb{C}^{n \times n}$ such that $X_j w = x_j$. For instance $X_j = x_j \frac{w^*}{\|w\|^2}$ will do. ■

Note Lemma 7.4 is true even with a *fixed* $w \neq 0$ and parametrizing over all X .

Proposition 7.5. *Let*

$$p = \sum_{1 \leq i, j \leq g} x_i^* B_{ij} x_j$$

be a homogeneous quadratic NC polynomial with $B_{ij} \in \mathbb{C}^{d' \times d}$. Then there is a unique matrix $G \in (\mathbb{C}^{d' \times d})^{g \times g}$ with

$$(7.3) \quad p = x^* G x.$$

Moreover, $p(X) = \sum_{i, j} B_{ij} \otimes X_i^ X_j$ is positive semidefinite for all $N \in \mathbb{N}$ and all $X \in (\mathbb{C}^{N \times N})^g$ iff $G \succeq 0$.*

Proof. In $d' \times d$ block form, $G = [B_{ij}]_{i, j}$. If $G \geq 0$, then $G = H^* H$ for some matrix H . Hence $p = (Hx)^* (Hx)$ is a sum of hermitian squares, so $p(X) \geq 0$ for all $N \in \mathbb{N}$ and $X \in (\mathbb{C}^{N \times N})^g$. The converse follows from Lemma 7.4. ■

7.1.2. Orthotropicity. In this subsection we establish a basic property of distinguished isometries L (that is, of those L for which Δ_L is positive semidefinite and clinging), which we call orthotropicity.

A $d' \times d$ linear pencil $L = A_1 x_1 + \dots + A_g x_g : \mathbb{C}^g \rightarrow \mathbb{C}^{d' \times d}$ is called **orthotropic** if for every $X \in \mathbb{C}^g$ and $w \in \mathbb{C}^d$ satisfying $\|L(X)w\| = \|w\|$, the vector $L(X)w$ is orthogonal to the image of $L(X^\perp)$.

Proposition 7.6. *Every distinguished isometry is orthotropic.*

To continue our analysis of distinguished isometries we write L in a special form. We multiply L with a unitary V on the left and a unitary U^* on the right. Thus without loss of generality, A_1 is the block matrix

$$(7.4) \quad \begin{bmatrix} 1 & 0 \\ 0 & (A_1)_{22} \end{bmatrix}$$

and A_j for $j \geq 2$ equals

$$(7.5) \quad \begin{bmatrix} 0 & (A_j)_{12} \\ (A_j)_{21} & (A_j)_{22} \end{bmatrix}.$$

Proof of Proposition 7.6. Suppose $L = \sum_{i=1}^g A_i x_i$ is a distinguished isometry and without loss of generality write L in the special form described above. Clearly, orthotropicity of L is equivalent to $(A_j)_{12} = 0$ for $j \geq 2$. In order to prove this we set all variables except for X_1, X_j to 0. For convenience we use X, Y (resp. x, y) instead of X_1, X_j (resp. x_1, x_j) and A, B instead of A_1, A_j . Thus

$$L(x, y) = \begin{bmatrix} x & B_{12}(I_{d-1} \otimes y) \\ B_{21}y & A_{22}(I_{d-1} \otimes x) + B_{22}(I_{d-1} \otimes y) \end{bmatrix}.$$

A straightforward computation shows we can represent $\Delta_L(x, y) = x^*x + y^*y - L(x, y)^*L(x, y)$ as $\begin{bmatrix} x \\ y \end{bmatrix}^* G \begin{bmatrix} x \\ y \end{bmatrix}$ (cf. Proposition 7.5), where $\begin{bmatrix} x \\ y \end{bmatrix}$ stands for

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & I_{d-1} \otimes x \\ y & 0 \\ 0 & I_{d-1} \otimes y \end{bmatrix}$$

and

$$G = \begin{bmatrix} 0 & 0 & 0 & -B_{12} \\ 0 & I - A_{22}^*A_{22} & -A_{22}^*B_{21} & -A_{22}^*B_{22} \\ 0 & -B_{21}^*A_{22} & 1 - B_{21}^*B_{21} & -B_{21}^*B_{22} \\ -B_{12}^* & -B_{22}^*A_{22} & -B_{22}^*B_{21} & I - B_{12}^*B_{12} - B_{22}^*B_{22} \end{bmatrix}.$$

If $\Delta_L(X, Y)$ is positive semidefinite for all X, Y , then by Proposition 7.5, G is positive semidefinite. In particular, $B_{12} = 0$. (Note if $\Delta_L(X, Y)$ is only positive semidefinite for scalars X, Y , then B_{12} need not be 0.)

Alternative proof of $B_{12} = 0$. By density, we may assume Y is invertible. $\Delta_L(X, Y)$ multiplied on the right by $\begin{bmatrix} Y^{-1} & 0 \\ 0 & I \end{bmatrix}$ and on the left by the transpose of the same matrix yields $M_2 := \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ for

$$\begin{aligned} m_{11} &= 1 - B_{21}^*B_{21} \\ m_{12} &= -B_{21}^*A_{22}(I_{d-1} \otimes X) - B_{21}^*B_{22}(I_{d-1} \otimes Y) + Y^{-*}X^*B_{12}(I_{d-1} \otimes Y) \\ (7.6) \quad m_{21} &= m_{12}^* = -(I_{d-1} \otimes X^*)A_{22}^*B_{21} - (I_{d-1} \otimes Y^*)B_{22}^*B_{21} + (I_{d-1} \otimes Y^*)B_{12}^*XY^{-1} \\ m_{22} &= I_{d-1} \otimes (X^*X + Y^*Y) - (I_{d-1} \otimes Y^*)B_{12}^*B_{12}(I_{d-1} \otimes Y) - \\ &\quad -((I_{d-1} \otimes X^*)A_{22}^* + (I_{d-1} \otimes Y^*)B_{22}^*)(A_{22}(I_{d-1} \otimes X) + B_{22}(I_{d-1} \otimes Y)). \end{aligned}$$

Consider m_{12} and note that

$$Y^{-*}X^*B_{12}(I_{d-1} \otimes Y) = Y^{-*}X^*YB_{12}.$$

Suppose $B_{12} \neq 0$. Then it is easy to construct $X = X(\varepsilon)$, $Y = Y(\varepsilon)$ sending this term to ∞ as $\varepsilon \rightarrow 0$ while keeping all the remaining terms bounded. This contradiction yields $B_{12} = 0$. \blacksquare

8. CHARACTERIZATION OF SEMI-DISTINGUISHED BALL MAPS

The following theorem summarizes what we know about semi-distinguished pencil ball maps. Both the hypotheses and the conclusions are weaker than those of Theorem 1.9. The relationship between both results is made precise by Corollary 8.2 of this section.

Theorem 8.1. *Let L be a $d' \times d$ NC analytic truly linear pencil and $f : \mathcal{B}_{g'} \rightarrow \mathcal{B}_L$ a semi-distinguished pencil ball map with $f(0) = 0$. Clearly, $h := L \circ f$ maps $\mathcal{B}_{g'} \rightarrow \mathcal{B}_{d' \times d}$. Write h as $h = h^{(1)} + h^{(\infty)}$, where $h^{(1)}$ is the linear homogeneous component in the NC power series expansion of h and $h^{(\infty)} = \sum_{\alpha=2}^{\infty} h^{(\alpha)}$. Then there is a unique contraction $M \in (\mathbb{C}^{d' \times d})^{g'}$ and a unique nontrivial subspace $\mathcal{S} \subseteq \mathbb{C}^{dg'}$ such that:*

- (1) $h^{(1)}(x) = Mx$, $M|_{\mathcal{S}}$ is an isometry and $M\Pi_{\mathcal{S}^\perp}$ is a strict contraction.
- (2) Each $h^{(\alpha)}(x)$ for $\alpha \geq 2$ is of the form $P_\alpha\Pi_{\mathcal{S}^\perp}x$ for a matrix P_α of NC polynomials.
- (3) For the formal NC power series $P^{(\infty)} := \sum_{\alpha=2}^{\infty} P_\alpha$, $P^{(\infty)}(X)v := \sum_{\alpha=2}^{\infty} P_\alpha(X)v$ converges for $v \in \mathcal{S}^\perp \otimes \mathbb{C}^N$ and $X \in (\mathbb{C}^{N \times N})^g$ in an NC ε -neighborhood of 0. Also, $h^{(\infty)}(x) = P^{(\infty)}(x)\Pi_{\mathcal{S}^\perp}x$.
- (4) $\|(M \otimes I_N + P^{(\infty)}(X))\Pi_{\mathcal{S}^\perp \otimes \mathbb{C}^N}\| \leq 1$ for $X \in (\mathbb{C}^{N \times N})^g$ with $\|X\| < 1$ and $(M \otimes I_N)(\mathcal{S} \otimes \mathbb{C}^N)$ is orthogonal to $(M \otimes I_N)(\mathcal{S}^\perp \otimes \mathbb{C}^N)$, to $P_\alpha(X)(\mathcal{S}^\perp)$ for all $\alpha \geq 2$ and to $P^{(\infty)}(X)(\mathcal{S}^\perp)$.

Proof. Let $\Delta_{h^{(1)}}(x) := x^*x - h^{(1)}(x)^*h^{(1)}(x) = x^*Gx$ be as in Proposition 7.5, where $G \geq 0$. Write $h^{(1)}(x) = Mx$ and note that $G = 1 - M^*M$.

Let $\mathcal{S} := \ker G = \ker(I - M^*M) = \text{range}(I - M^*M)^\perp$. By the clinging property, $\mathcal{S} \neq \{0\}$. By definition, $M|_{\mathcal{S}}$ is an isometry. Conversely, if v satisfies $\|Mv\| = \|v\|$, then $\langle Mv, Mv \rangle = \langle v, v \rangle$ and hence $\langle v, (I - M^*M)v \rangle = 0$. Since M is a contraction, $I - M^*M$ is positive semidefinite. Thus $(I - M^*M)v = 0$, that is, $v \in \mathcal{S}$. This proves (1) and also the uniqueness of M and \mathcal{S} .

For (2) fix $N \geq 1$ and let $X \in \mathcal{B}_{g'}(N)$ such that $\|X\| = 1$ be given. By equation (6.1) of Schwarz's lemma (Theorem 6.2) applied to $h(zX)$, $|z| < 1$ for all $0 \leq \delta < 1$ and $0 \leq \theta \leq 2\pi$ we have

$$(8.1) \quad 0 \leq \delta^2 X^*X - h(\delta e^{i\theta} X)^*h(\delta e^{i\theta} X).$$

If δ is in the series radius, we may write $h(\delta e^{i\theta} X) = h^{(1)}(\delta e^{i\theta} X) + h^{(\infty)}(\delta e^{i\theta} X) = \sum_{\alpha=1}^{\infty} h^{(\alpha)}(\delta e^{i\theta} X)$.

We integrate (8.1) to obtain

$$\begin{aligned}
(8.2) \quad 0 &\preceq \frac{1}{2\pi} \int_0^{2\pi} (\delta^2 X^* X - h(\delta e^{i\theta} X)^* h(\delta e^{i\theta} X)) d\theta \\
&= \delta^2 X^* X - \delta^2 h^{(1)}(X)^* h^{(1)}(X) - \frac{1}{2\pi} \int_0^{2\pi} h^{(\infty)}(\delta e^{i\theta} X)^* h^{(\infty)}(\delta e^{i\theta} X) d\theta \\
&= \delta^2 X^* X - \delta^2 h^{(1)}(X)^* h^{(1)}(X) - \sum_{\alpha=2}^{\infty} h^{(\alpha)}(\delta X)^* h^{(\alpha)}(\delta X).
\end{aligned}$$

By Proposition 7.2, $\Delta_{h^{(1)}}(X) \succeq 0$ with clinging. Thus by (8.2), for every w satisfying

$$(8.3) \quad (X^* X - h^{(1)}(X)^* h^{(1)}(X))w = 0$$

we have $h^{(\alpha)}(\delta X)w = 0$ for $\alpha \geq 2$ and δ in the series radius. In particular, by Proposition 7.5, (8.3) is equivalent to $\sqrt{G}xw = 0$ and this implies that $h^{(\alpha)}(X)w = 0$ for $\alpha \geq 2$ and every X in the series radius. By a scaling argument ($h^{(\alpha)}$ is homogeneous), the same holds true for every X and w . Hence the matrix NC Nullstellensatz Theorem 6.8 applies and implies that there is a matrix of NC polynomials \tilde{P}_α with $\tilde{P}_\alpha(x)\sqrt{G}x = h^{(\alpha)}(x)$. Since $\sqrt{G} = \sqrt{G}(\Pi_{\mathcal{S}} + \Pi_{\mathcal{S}^\perp}) = \sqrt{G}\Pi_{\mathcal{S}^\perp}$, we set $P_\alpha = \tilde{P}_\alpha\sqrt{G}$. Then $h^{(\alpha)}(x) = P_\alpha(x)\Pi_{\mathcal{S}^\perp}x$.

(3) The second part is clear and for the first statement we refer the reader to [K-VV2].

(4) Let $v \in \mathcal{S}$ and $w \in \mathcal{S}^\perp$. Then

$$(8.4) \quad \langle Mv, Mw \rangle = \langle M^*Mv, w \rangle = \langle v, w \rangle = 0$$

since $(1 - M^*M)v = 0$. This shows that $M(\mathcal{S}^\perp)$ is orthogonal to $M(\mathcal{S})$. For the strengthening with tensor products, let $s_i \in \mathcal{S}$, $t_j \in \mathcal{S}^\perp$, $v_i, u_j \in \mathbb{C}^N$. Then

$$\begin{aligned}
\langle (M \otimes I_N) \left(\sum_i s_i \otimes v_i \right), (M \otimes I_N) \left(\sum_j t_j \otimes u_j \right) \rangle &= \left\langle \sum_i (Ms_i \otimes v_i), \sum_j (Mt_j \otimes u_j) \right\rangle \\
&= \sum_{i,j} \langle Ms_i, Mt_j \rangle \langle v_i, u_j \rangle = 0.
\end{aligned}$$

Let $h(x) = \tilde{h}(x)x$, where

$$\tilde{h}(x) = M + \sum_{\alpha \geq 2} P_\alpha(x)\Pi_{\mathcal{S}^\perp}.$$

By Theorem 6.4 (applied with $H = \tilde{h}$), $\|\tilde{h}(X)\| \leq 1$ for all X with $\|X\| < 1$.

Rewrite $\tilde{h}(x)$ as

$$(8.5) \quad \tilde{h}(x) = M\Pi_{\mathcal{S}} + \left(M + \sum_{\alpha \geq 2} P_\alpha(x) \right) \Pi_{\mathcal{S}^\perp}.$$

Both summands have norm ≤ 1 for X with $\|X\| < 1$. In particular,

$$\|(M \otimes I_N + \sum_{\alpha \geq 2} P_\alpha(X))\Pi_{\mathcal{S}^\perp \otimes \mathbb{C}^N}\| \leq 1,$$

as desired.

Clearly, $\tilde{h}(x)|_{\mathcal{S}} = M|_{\mathcal{S}}$ is an isometry and thus $\tilde{h}(X)(\mathcal{S} \otimes \mathbb{C}^N)$ is orthogonal to $\tilde{h}(X)(\mathcal{S}^\perp \otimes \mathbb{C}^N) = (M \otimes I_N + P^{(\infty)}(X))(\mathcal{S}^\perp \otimes \mathbb{C}^N)$. Since $(M \otimes I_N)(\mathcal{S} \otimes \mathbb{C}^N)$ is orthogonal to $(M \otimes I_N)(\mathcal{S}^\perp \otimes \mathbb{C}^N)$, this implies $(M \otimes I_N)(\mathcal{S}^\perp \otimes \mathbb{C}^N) \perp P^{(\infty)}(X)(\mathcal{S}^\perp \otimes \mathbb{C}^N)$.

Suppose $w \in (M \otimes I_N)(\mathcal{S}^\perp \otimes \mathbb{C}^N)$. Then $w^*P^{(\infty)}(tX)(\mathcal{S}^\perp \otimes \mathbb{C}^N) = \{0\}$ for small enough t . Then

$$0 = w^*P^{(\infty)}(tX) = w^* \sum_{\alpha \geq 2} t^\alpha P_\alpha(X) = \sum_{\alpha \geq 2} t^\alpha (w^*P_\alpha(X))$$

implies $(M \otimes I_N)(\mathcal{S}^\perp \otimes \mathbb{C}^N) \perp P_\alpha(X)(\mathcal{S}^\perp \otimes \mathbb{C}^N)$. ■

Let us note in passing that under the conditions of the previous theorem, (8.3) implies $h^{(\infty)}(X)w = 0$ for all $X \in \mathcal{B}_g$. Indeed, let us consider the analytic function $z \mapsto h^{(\infty)}(zX)w$ on \mathbb{D} . Clearly, (8.3) holds for X replaced by δX due to homogeneity. If δ is in the series radius, then by the NC power series expansion and the lemma,

$$h^{(\infty)}(\delta X)w = \sum_{\alpha=2}^{\infty} h^{(\alpha)}(\delta X)w = 0.$$

Thus by analytic continuation, $h^{(\infty)}(X)w = 0$.

Next we give a corollary which makes the relationship between Theorem 8.1 and Theorem 1.9 clearer.

Corollary 8.2. *Keep the assumptions and notation from Theorem 8.1. If, in addition, f is a pencil ball map, then M is a complete isometry.*

Conversely, $h = L \circ f$ satisfying (1), (2), (3), (4) for a complete isometry M is an NC ball map $\mathcal{B}_{g'} \rightarrow \mathcal{B}_{d' \times d}$ sending 0 to 0.

Proof. Suppose f is a pencil ball map. Then $h^{(1)}$ is a (linear) NC ball map by Proposition 3.1. Hence $h^{(1)} = Mx$ with M a complete isometry (see Theorem 3.3).

For the converse, suppose h satisfies (1)–(4). By (1) and (3), $h(x) = \tilde{h}(x)x$, where $\tilde{h}(x)$ is given by:

$$\tilde{h}(x) = M\Pi_{\mathcal{S}} + (M + \sum_{\alpha \geq 2} P_\alpha(x))\Pi_{\mathcal{S}^\perp}.$$

(1) and (4) implies that $\tilde{h}(X)$ is for $X \in \mathcal{B}_{g'}$ an orthogonal sum of two contractions, thus $\tilde{h}(X)$ is a contraction for $X \in \mathcal{B}_{g'}$, i.e., $\|X\| \leq 1$. Hence $h(X) = \tilde{h}(X)X$ is a contraction.

For the binding property of h we use that M is a complete isometry. Let e denote the distinguished vector associated with M , that is, $A_j^* A_i e = \delta_i^j e$ if $M = \begin{bmatrix} A_1 & \cdots & A_{g'} \end{bmatrix}$ (cf. Proposition 3.6). By (1), $h^{(1)}(X)$ binds at $e \otimes w$, where w is a binding vector for $I - X^*X$, i.e., $X^*Xw = w$. This concludes the proof since $h(X)(e \otimes w) = h^{(1)}(X)(e \otimes w)$ by (2) and (3). \blacksquare

9. FURTHER ANALYSIS OF DISTINGUISHED ISOMETRIES

We have successfully classified complete isometries, see Theorem 3.3. Distinguished isometries are more challenging and a few sample results are provided below.

Theorem 9.1. *Suppose L is an orthotropic linear pencil in 2 variables. If $\Delta_L(X_1, X_2) \succeq 0$ for all $X_1, X_2 \in \mathbb{C}^{n \times n}$, and clings for all scalar $X_1, X_2 \in \mathbb{C}$, then $\Delta_L(X_1, X_2)$ clings for all $X_1, X_2 \in \mathbb{C}^{n \times n}$.*

Remark 9.2. We conjecture based on inconclusive computer experiments that Theorem 9.1 is false in 3 variables.

9.1. Equations which reformulate the clinging property. Throughout this subsection L will denote an orthotropic $d' \times d$ linear pencil in g variables. We assume it clings for $X \in \mathbb{C}^g$. Let

$$\Delta_L(x) = x^*x - L(x)^*L(x) = x^*Gx$$

be the Gram representation as in Proposition 7.5. Given $G \in (\mathbb{C}^{d \times d})^{g \times g}$ we call the linear subspace of its kernel spanned by all the vectors of the form

$$\begin{bmatrix} \alpha_1 v \\ \vdots \\ \alpha_g v \end{bmatrix} \in \ker G$$

the **scalar binding kernel** \mathcal{N}_0 . (Since L clings for $X \in \mathbb{C}^g$, for every $\alpha_1, \dots, \alpha_g \in \mathbb{C}$ there exists

$$\text{a } 0 \neq v \in \mathbb{C}^d \text{ with } \begin{bmatrix} \alpha_1 v \\ \vdots \\ \alpha_g v \end{bmatrix} \in \ker G.)$$

Fix a basis

$$\left\{ \eta_i := \begin{bmatrix} \alpha_{i,1} v_i \\ \vdots \\ \alpha_{i,g} v_i \end{bmatrix} \mid i = 1, \dots, t+m \right\} \subseteq \mathbb{C}^{gd}$$

for the scalar binding kernel \mathcal{N}_0 of G . We assume that $\{v_1, \dots, v_t\}$ is a maximal linearly independent set and that

$$v_{t+j} = \sum_{i=1}^t \gamma_{ji} v_i$$

for $j = 1, \dots, m$.

Let $X_1, \dots, X_g \in \mathbb{C}^{n \times n}$. We assume that X_1 is invertible and define $Z_i := X_1^{-1} X_i$. (Matrix) binding at X is equivalent to the existence (for all Z_i) of a nontrivial solution to

$$\Delta_L(I_n, Z_2, \dots, Z_g)v = 0.$$

This is implied by the existence of $r_i \in \mathbb{C}^n$ for which there is a nonzero $v \in \mathbb{C}^{dn}$ such that

$$(9.1) \quad \begin{bmatrix} (I_d \otimes I_n)v \\ (I_d \otimes Z_2)v \\ \vdots \\ (I_d \otimes Z_g)v \end{bmatrix} = \sum_{i=1}^{t+m} \eta_i \otimes r_i.$$

In particular,

$$\begin{aligned} v &= \sum_{i=1}^{t+m} v_i \otimes r_i = \sum_{i=1}^t v_i \otimes r_i + \sum_{j=t+1}^{t+m} \sum_{i=1}^t \gamma_{ji} v_i \otimes r_j = \\ &= \sum_{i=1}^t v_i \otimes \underbrace{\left(r_i + \sum_{j=t+1}^{t+m} \gamma_{ji} r_j \right)}_{\Gamma_i(r)} \end{aligned}$$

Similarly, $(I_d \otimes Z_k)v = \sum_{i=1}^t v_i \otimes \Gamma_i(Z_k r)$. Using this in (9.1) yields

$$\sum_{i=1}^t v_i \otimes \Gamma_i(Z_k r) = \sum_{i=1}^t v_i \otimes \Gamma_i(r \operatorname{diag}(\alpha_k)),$$

where $r = \begin{bmatrix} r_1 & \cdots & r_{t+m} \end{bmatrix}$ and $\operatorname{diag}(\alpha_k)$ is the diagonal matrix with $\alpha_{i,k}$ as its (i, i) entry. Linear independence of the v_1, \dots, v_t gives $\Gamma_i(Z_k r - r \operatorname{diag}(\alpha_k)) = 0$ for all $k = 2, \dots, g$ and $i = 1, \dots, t$.

Thus for all these i, k :

$$(9.2) \quad (Z_k - \alpha_{i,k})r_i + \sum_{j=t+1}^{t+m} \gamma_{ji}(Z_k - \alpha_{j,k})r_j = 0.$$

Hence if all the $Z_k - \alpha_{i,k}$ are invertible,

$$(9.3) \quad r_i = - \sum_{j=t+1}^{t+m} b_{ij}(\operatorname{diag}(\alpha_k), Z_k)r_j,$$

where

$$b_{ij}(\text{diag}(\alpha_k), Z) := \gamma_{ji}(Z - \alpha_{i,k})^{-1}(Z - \alpha_{j,k}).$$

Equations derived so far reformulate then clinging property and we say how precisely in the following lemma.

Lemma 9.3. *Consider the following conditions:*

- (i) *For each Z_2, \dots, Z_g the system of equations (9.2) has a solution r_1, \dots, r_{t+m} ;*
- (ii) *Δ_L clings.*

Then (i) \Rightarrow (ii) and if $\mathcal{N}_0 = \ker G$, then (ii) \Rightarrow (i).

Proof. Follows from the computations given above. ■

9.2. The general case and the proof of Theorem 9.1. Now we give a theorem more general than Theorem 9.1 that implies Theorem 9.1.

Theorem 9.4. *Suppose $t(g-2) < m$. If $\Delta_L(X) \succeq 0$ for all $X \in (\mathbb{C}^{n \times n})^g$ and clings for all scalar $X \in \mathbb{C}^g$, then $\Delta_L(X)$ clings for all $X \in (\mathbb{C}^{n \times n})^g$.*

Proof. We assume all $b_{ij}(\text{diag}(\alpha_k), Z_k)$ exists, i.e., all $Z_k - \alpha_{i,k}$ are invertible. This causes no loss of generality: the set of all matrix g -tuples that make Δ_L cling is closed and our condition implies clinging on a dense subset.

Equation (9.3) gives $r_i = r_i(k)$ as a function of k . By Lemma 9.3 we need to show that for every choice of Z_i the system (9.2) has a solution, i.e., $r_i(2) = r_i(3) = \dots = r_i(g)$ for all $i = 1, \dots, t$. This yields $tn(g-2)$ homogeneous equations in mn unknowns. Thus if $m > t(g-2)$ this system will always have a nontrivial solution. ■

Proof of Theorem 9.1. Fix a basis

$$\left\{ \left[\begin{array}{c} \alpha_i v_i \\ \beta_i v_i \end{array} \right] \mid i = 1, \dots, t+m \right\}$$

for the scalar binding kernel \mathcal{N}_0 , where $\{v_1, \dots, v_t\}$ is a maximal linearly independent set. In view of Theorem 9.4 it suffices to show that $m > 0$.

Suppose $m = 0$ and choose α, β with $\frac{\alpha}{\beta} \neq \frac{\alpha_i}{\beta_i}$ for all i . By scalar binding, there is a nonzero vector u with $\begin{bmatrix} \alpha_i u \\ \beta_i u \end{bmatrix} \in \mathcal{N}_0$, i.e., for some λ_i :

$$\begin{bmatrix} \alpha u \\ \beta u \end{bmatrix} = \sum_{i=1}^t \lambda_i \eta_i = \sum_{i=1}^t \lambda_i \begin{bmatrix} \alpha_i v_i \\ \beta_i v_i \end{bmatrix}$$

Hence

$$\beta \sum_{i=1}^t \lambda_i \alpha_i v_i = \alpha \sum_{i=1}^t \lambda_i \beta_i v_i$$

and thus by the linear independence of the v_i , $\beta \lambda_i \alpha_i = \alpha \lambda_i \beta_i$ for all $i = 1, \dots, t$. As at least one λ_j is nonzero, this implies

$$\frac{\alpha}{\beta} = \frac{\alpha_j}{\beta_j},$$

contrary to our assumption. Thus $m > 0$, as desired. ■

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO

E-mail address: helton@math.ucsd.edu

UNIVERZA V LJUBLJANI, FAKULTETA ZA MATEMATIKO IN FIZIKO, AND UNIVERZA V MARIBORU, FAKULTETA ZA NARAVOSLOVJE IN MATEMATIKO

E-mail address: igor.klep@fmf.uni-lj.si

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA

E-mail address: sam@math.ufl.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO

E-mail address: nsling@euclid.ucsd.edu

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