NONCOMMUTATIVE RATIONAL POSITIVSTELLENSÄTZE WITH OPERATOR COEFFICIENTS

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Abstract. We establish a convex Positivstellensatz for noncommutative rational functions with bounded operator coefficients. A hermitian operator-valued rational function that is positive semidefinite on a matrix convex set described by a linear pencil admits a positivity certificate consisting of rational hermitian squares and a completely positive term involving the defining linear pencil. The certificate comes with an explicit complexity bound independent of the dimension of the coefficient Hilbert space. As a consequence, we solve the operator-valued noncommutative rational analog of Hilbert's 17th problem: every positive semidefinite operator-valued noncommutative rational function is a sum of hermitian squares, and even a single hermitian square when the coefficient Hilbert space is infinite-dimensional. In contrast, the corresponding commutative statement fails for infinite-dimensional operator coefficients.

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1. Introduction

Hilbert's 17th problem asks whether every nonnegative real polynomial is a sum of squares of rational functions. Artin's affirmative solution is a cornerstone of real algebraic geometry [BCR98, Mar08, Sce24]; matrix versions were subsequently obtained by Gondard–Ribenboim [GR74], Procesi–Schacher [PS76], and Djoković [Djo76]. The situation is more rigid when the variables are freely noncommuting and positivity is tested on matrices of all sizes. Helton [Hel02] and McCullough [McC01] proved that a noncommutative polynomial positive semidefinite on all matrix tuples is a sum of hermitian squares of polynomials; see [McC01] also for foundational operator-valued factorization results. Further approaches to noncommutative Positivstellensätze, based on separation arguments, quadratic modules and operator algebras, were developed in [Oza13, NT13, ANT19].

The convex counterpart concerns positivity on matricial solution sets of linear pencils, the so-called free spectrahedra [EPŠ26]. For linear *matrix* pencils, [HKM12] gives a convex Positivstellensatz with optimal degree bounds. Linear *operator* inequalities provide the natural framework for working with arbitrary closed matrix convex sets, under the usual interior normalization, by the matricial Hahn–Banach theorem of Effros–Winkler [EW97] (cf. [HM12]). The interplay between matrix convexity, operator systems, spectrahedral containment and dilation theory has been studied extensively; see, e.g., [FNT17, PSS18] and the recent survey [EPŠ26]. Operator-pencil Positivstellensätze were studied by Zalar [Zal17] (see also [DDOSS17]); more recently, [JKM26a] established operator-valued polynomial factorization and an operator-valued convex Positivstellensatz [JKM26b]. These results motivate the passage from polynomial to rational functions with operator coefficients.

The rational setting is more delicate because evaluations may be undefined, and the singularities are encoded by linear realizations of rational expressions. Klep, Pascoe and Volčič [KPV17] established a sum-of-squares theorem for positive noncommutative rational functions without hermitian singularities. Subsequently, Volčič [Vol21] treated arbitrary rational singularities and developed a rational convex Positivstellensatz using extensions of multiplication operators; see also [Pas18]. We establish a corresponding theorem with operator coefficients and an explicit complexity bound independent of the

coefficient Hilbert space. An additional ingredient is a corrected domain-preserving extension mechanism; we explain its relation to [Vol21] in Subsection 1.3 below.

1.1. **Notation and setting.** Noncommutative (nc) rational functions arise by applying addition, multiplication and inversion to elements of the free algebra $\mathbb{C}\langle x_1, \dots, x_{\mathtt{g}}\rangle$. More precisely, they are equivalence classes of rational expressions under agreement of their matrix evaluations, and form the free skew field, the universal skew field of fractions of the free algebra. Its construction goes back to Amitsur [Ami66], while its algebraic structure was developed systematically by Cohn [Coh95]. The viewpoint of matrix evaluations and realizations, which connects nc rational functions to free analysis, systems theory and free real algebraic geometry, was developed in [HMV06, K-VV12]; see also [Vol18].

A complementary analytic approach to noncommutative rational functions arises from their action as multipliers on the full Fock space. Jury, Martin and Shamovich studied noncommutative rational multipliers and their factorizations [JMS21a, JMS21b], as well as the associated rational Clark measures [JMS23]. More recently, Augat, Martin and Shamovich developed operator realization methods for more general noncommutative analytic functions, including classes of meromorphic functions [AMS25].

We work over $\mathbb{C}$ with freely noncommuting, self-adjoint variables $x = (x_1, \dots, x_{\mathtt{g}})$. Write $\mathbb{C}\langle\!\langle x\rangle\!\rangle$ for the free skew field of noncommutative rational functions and $\mathfrak{R}_{\mathbb{C}}(x)$ for the nondegenerate formal rational expressions. An expression $r \in \mathfrak{R}_{\mathbb{C}}(x)$ determines a rational function $\mathtt{r} \in \mathbb{C}\langle\!\langle x\rangle\!\rangle$, and its *hermitian domain* $\operatorname{hdom} r$ consists of the hermitian matrix tuples at which all the inverses occurring in r can be evaluated. We consider finite algebraic sums

$$\mathbb{F} = \sum_{a=1}^{N} C_a \otimes \mathtt{r}_a \in \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\langle\!\langle x\rangle\!\rangle,$$

where $\mathcal{K}$ is a nonzero complex Hilbert space and $\mathbb{F} = \mathbb{F}^*$. Throughout, positivity of $\mathbb{F}$ refers to all matrix sizes for which the indicated rational expressions are defined.

Let $\mathcal{H}$ be a nonzero complex Hilbert space, and fix a monic hermitian linear operator pencil

$$L = I_{\mathcal{H}} + \sum_{j=1}^{\mathtt{g}} H_j x_j, \qquad H_j = H_j^* \in \mathcal{B}(\mathcal{H}).$$

Its positivity set and its coefficient operator system are

$$\mathcal{D}(L) = \bigcup_{n\geqslant 1} \big\{X \in \mathrm{H}_n(\mathbb{C})^{\mathtt{g}} : L(X) \succeq 0\big\}, \qquad \mathcal{S}_L = \operatorname{span}_{\mathbb{C}}\{I_{\mathcal{H}}, H_1, \dots, H_{\mathtt{g}}\}.$$

The set $\mathcal{D}(L)$ is matrix convex. Allowing an infinite-dimensional $\mathcal{H}$ is essential here: by [HM12, EW97], linear operator inequalities also describe general closed matrix convex sets after the usual interior-point normalization.

For a finite admissible set $R \subset \mathfrak{R}_{\mathbb{C}}(x)$, let $\mathcal{R}$ be the corresponding set of rational functions and put

$$V_0 = \mathbb{C}1, \qquad V_\ell = \operatorname{span}_{\mathbb{C}}\{\mathtt{r}_1 \cdots \mathtt{r}_\ell : \mathtt{r}_i \in \mathcal{R}\} \quad (\ell \geqslant 1).$$

Here admissibility means that R contains $1, x_1, \dots, x_{\mathtt{g}}$, is closed under adjoints and subexpressions, and contains a rational expression having as denominator each irreducible component of a denominator of an expression in R; see Section 4 for details. Every finite

family of rational expressions is contained in such a set. We measure the complexity of expressions by

$$\begin{aligned} \tau(\alpha) &= 0 \quad (\alpha \in \mathbb{C}), & \tau(x_j) &= 1, \\ \tau(r+s) &= \max\{\tau(r), \tau(s)\}, & \tau(rs) &= \tau(r) + \tau(s), \\ \tau(r^{-1}) &= 2\tau(r), & \tau(R) &= \max_{r \in R} \tau(r). \end{aligned}$$

The dependence on expressions, rather than only on the rational functions they represent, is intentional.

1.2. **Main results.** Our main result is a convex rational Positivstellensatz for matrix- and operator-valued rational functions. It has the same complexity bound in the matrix-valued and infinite-dimensional cases.

Theorem A. *Let $R \subset \mathfrak{R}_{\mathbb{C}}(x)$ be finite and admissible, and let $\mathbb{F} = \sum_{a=1}^{N} C_a \otimes \mathbb{r}_a \in \mathcal{B}(\mathcal{K}) \otimes V_1$ be hermitian, where $r_a \in R$ represents $\mathbb{r}_a$. Let $L = I_{\mathcal{H}} + \sum_j H_j x_j$ be a monic hermitian linear operator pencil, and set $\ell_* = 2\tau(R) + 1$.*

Then

$$\mathbb{F}(X) \succeq 0 \qquad \text{for every } X \in \mathcal{D}(L) \cap \bigcap_{r \in R} \operatorname{hdom} r \tag{1.1}$$

if and only if the following certificate exists:

(a) *If $\mathcal{K}$ is infinite-dimensional, there are $\mathbb{r}, \mathbb{q} \in \mathcal{B}(\mathcal{K}) \otimes V_{\ell_*}$ and a unital completely positive map $\pi : \mathcal{S}_L \to \mathcal{B}(\mathcal{K})$ such that*

$$\mathbb{F} = \mathbb{r}^* \mathbb{r} + \mathbb{q}^* \pi(L) \mathbb{q}. \tag{1.2}$$

(b) *If $\mathcal{K} = \mathbb{C}^d$, writing $m = \dim V_{\ell_*}$, there are $\mathbb{r}, \mathbb{q} \in \mathrm{M}_{md \times d}(V_{\ell_*})$ and a unital completely positive map $\pi : \mathcal{S}_L \to \mathrm{M}_{md}(\mathbb{C})$ satisfying*

$$\mathbb{F} = \mathbb{r}^* \mathbb{r} + \mathbb{q}^* \pi(L) \mathbb{q}. \tag{1.3}$$

Here $\pi(L) = I + \sum_j \pi(H_j) x_j$ in the respective coefficient algebra. In particular, the truncation bound ℓ_* is independent of $\dim \mathcal{K}$.

Part (b) is Theorem 8.3, expressed through Proposition 5.2, while part (a) is Theorem 9.1. Notice that neither the size of the matrices on which positivity is tested nor boundedness of $\mathcal{D}(L)$ enters the hypothesis. The same truncation bound is valid for all finite-dimensional compressions of the coefficient operators; this uniformity is crucial for the operator-valued conclusion.

When $\mathcal{H}$ is finite-dimensional, the certificate in Theorem A can be expressed directly in terms of L, without the unital completely positive (UCP) map. This gives the following two corollaries.

Corollary 1.1. *Suppose that $\mathcal{H} = \mathbb{C}^h$ and $\mathcal{K} = \mathbb{C}^d$, and let R, L and $\mathbb{F}$ be as in Theorem A. Write $m = \dim V_{\ell_*}$. Then (1.1) holds if and only if there exist $\mathbb{r} \in \mathrm{M}_{md \times d}(V_{\ell_*})$ and $\mathbb{q}_1, \dots, \mathbb{q}_s \in \mathrm{M}_{h \times d}(V_{\ell_*})$, with $s \leqslant hmd$, such that*

$$\mathbb{F} = \mathbb{r}^* \mathbb{r} + \sum_{\nu=1}^{s} \mathbb{q}_\nu^* L \mathbb{q}_\nu.$$

Proof. By Theorem A, we have $\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*\pi(L)\mathbb{q}$, where $\pi : \mathcal{S}_L \to \mathrm{M}_{md}(\mathbb{C})$ is UCP. By Arveson's extension theorem, π extends to a UCP map on $\mathrm{M}_h(\mathbb{C})$, which admits a Kraus representation

$$\pi(B) = \sum_{\nu=1}^{s} V_\nu^* B V_\nu, \qquad V_\nu \in \mathrm{M}_{h\times md}(\mathbb{C}), \quad s \leqslant hmd.$$

Setting $\mathbb{q}_\nu = V_\nu \mathbb{q}$ gives the required representation. The converse follows from the positivity of $L(X)$ on $\mathcal{D}(L)$. □

Corollary 1.2. *Suppose that $\mathcal{H}$ is finite-dimensional and $\mathcal{K}$ is infinite-dimensional, and let R, L and $\mathbb{F}$ be as in Theorem A. Let $\mathcal{E}$ be a Hilbert space such that $\mathcal{K}$ is unitarily isomorphic to $\mathcal{E} \otimes \mathcal{H}$. Then* (1.1) *holds if and only if there exist $\mathbb{r}, \mathbb{q} \in \mathcal{B}(\mathcal{K}, \mathcal{E} \otimes \mathcal{H}) \otimes V_{\ell_*}$ such that*

$$\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*(I_\mathcal{E} \otimes L)\mathbb{q}.$$

Proof. By Theorem A, we have

$$\mathbb{F} = \mathbb{r}_0^*\mathbb{r}_0 + \mathbb{q}_0^*\pi(L)\mathbb{q}_0,$$

where $\mathbb{r}_0, \mathbb{q}_0 \in \mathcal{B}(\mathcal{K}) \otimes V_{\ell_*}$ and $\pi : \mathcal{S}_L \to \mathcal{B}(\mathcal{K})$ is UCP.

Since $\mathcal{H}$ is finite-dimensional, Arveson's extension theorem and Stinespring's theorem give an isometry $V : \mathcal{K} \to \mathcal{E} \otimes \mathcal{H}$ such that

$$\pi(B) = V^*(I_\mathcal{E} \otimes B)V, \qquad B \in \mathcal{S}_L.$$

Here we use the infinite-dimensionality of $\mathcal{K}$ to choose $\mathcal{E}$ large enough for the Stinespring representation.

Choose a unitary $U : \mathcal{K} \to \mathcal{E} \otimes \mathcal{H}$ and set $\mathbb{r} = U\mathbb{r}_0$ and $\mathbb{q} = V\mathbb{q}_0$. Then

$$\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*(I_\mathcal{E} \otimes L)\mathbb{q},$$

as required. The converse follows from the positivity of $L(X)$ on $\mathcal{D}(L)$. □

Taking $L = I$ recovers the noncommutative rational Hilbert's 17th problem in its operator-valued form: if $\mathbb{F}$ is positive semidefinite wherever the chosen rational expressions are defined, then it is a finite sum of hermitian rational squares. In fact, if $\dim \mathcal{K} = \infty$, then one square suffices,

$$\mathbb{F} = \mathbb{s}^*\mathbb{s}, \qquad \mathbb{s} \in \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\langle\langle x\rangle\rangle. \tag{1.4}$$

If $\dim \mathcal{K} < \infty$, the number of matrix-valued squares can be bounded in terms of $\dim V_{\ell_*}$; see Corollaries 9.2 and 9.3. This contrasts with the commutative operator-valued case, where there is no analogous rational sum-of-squares theorem in infinite dimensions (Section 9.2).

1.3. **Relation to earlier work.** The rational framework and part of the extension strategy in this paper originate in [Vol21]. In revisiting its proof, we observed that the hypotheses of [Vol21, Proposition 3.2 and Theorem 3.3] do not suffice for the asserted completion conclusions. The issue appears for full linear pencils that are not irreducible: full column rank of an evaluation of the pencil need not give full column rank for its individual irreducible components, although the latter condition is necessary for the proposed completion. A concrete counterexample is given after Proposition 3.2.

We remedy this by requiring the rank conditions for each irreducible component (Proposition 3.2) and by using a more restrictive infinite-dimensional domain that records invertibility of all irreducible denominator components. These changes yield the finite-dimensional hermitian completion result (Proposition 3.4) needed in the multiplication-operator argument. In particular, they supply a valid replacement for the extension step underlying [Vol21, Proposition 3.5] and hence a route to the rational Positivstellensatz and Hilbert's 17th problem. The central approach of [Vol21], that is, linear representations, finite-dimensional extensions, and rational multiplication operators, remains fundamental to the present proof. The new operator-valued conclusions additionally require a Gram–completely positive (Gram–CP) description, finite-dimensional coefficient extraction, and ultraweak closedness of the relevant quadratic modules.

1.4. **Guide to the paper.** Section 2 recalls free skew fields and linear representations. Section 3 establishes the corrected domain-preserving extension results, and Section 4 develops multiplication operators associated with admissible sets. In Section 5 we introduce truncated rational quadratic modules and their Gram–CP representations. Section 6 constructs reproducing tuples and coefficient-extraction functionals; these yield the uniform bounds and product-ultraweak closedness in Section 7. Finally, Section 8 proves the matrix-valued rational Positivstellensatz, while Section 9 passes to general operator coefficients and derives the Hilbert's 17th problem corollaries, including the contrast with the commutative setting.

2. Preliminaries

Let $\mathrm{M}_{m\times n}(\mathbb{C})$ denote the space of complex $m \times n$ matrices, and $\mathrm{M}_n(\mathbb{C}) = \mathrm{M}_{n\times n}(\mathbb{C})$. Let $\mathrm{H}_n(\mathbb{C})$ denote the real space of hermitian $n \times n$ matrices. For $X = (X_1, \dots, X_{\mathtt{g}}) \in \mathrm{M}_{m\times n}(\mathbb{C})^{\mathtt{g}}$, $A \in \mathrm{M}_{p\times m}(\mathbb{C})$ and $B \in \mathrm{M}_{n\times q}(\mathbb{C})$ we write

$$AXB = (AX_1B, \dots, AX_{\mathtt{g}}B) \in \mathrm{M}_{p\times q}(\mathbb{C})^{\mathtt{g}}, \qquad X^* = (X_1^*, \dots, X_{\mathtt{g}}^*) \in \mathrm{M}_{n\times m}(\mathbb{C})^{\mathtt{g}}.$$

2.1. **Free skew field.** We define noncommutative rational functions using formal rational expressions and their matrix evaluations following [HMV06] and [K-VV12]. **Formal rational expressions** are syntactically valid combinations of scalars, freely noncommuting variables $x = (x_1, \dots, x_{\mathtt{g}})$, rational operations and parentheses. A **subexpression** of a formal rational expression r is any formal rational expression which appears in the construction of r.

Given a formal rational expression r and $X \in \mathrm{M}_n(\mathbb{C})^{\mathtt{g}}$, the evaluation $r(X)$ is defined in the natural way if all inverses appearing in r exist at X. The set of all $X \in \mathrm{M}_n(\mathbb{C})^{\mathtt{g}}$ such that r is defined at X is denoted $\operatorname{dom}_n r$. The **(matricial) domain** of r is

$$\operatorname{dom} r = \bigcup_{n\in\mathbb{N}} \operatorname{dom}_n r.$$

Note that $\operatorname{dom}_n r$ is a Zariski open set in $\mathrm{M}_n(\mathbb{C})^{\mathtt{g}}$ for every $n \in \mathbb{N}$. A formal rational expression r is **nondegenerate** if $\operatorname{dom} r \neq \varnothing$; let $\mathfrak{R}_{\mathbb{C}}(x)$ denote the set of all nondegenerate formal rational expressions. On $\mathfrak{R}_{\mathbb{C}}(x)$ we define an equivalence relation $r_1 \sim r_2$ if and only if $r_1(X) = r_2(X)$ for all $X \in \operatorname{dom} r_1 \cap \operatorname{dom} r_2$. Equivalence classes with respect to this relation are called **noncommutative rational functions**. By [K-VV12,

Proposition 2.2] they form a skew field denoted $\mathbb{C}\langle\!\langle x\rangle\!\rangle$, which is the universal skew field of fractions of the free algebra $\mathbb{C}\langle x\rangle$ by [Coh95, Section 4.5]. The equivalence class of $r \in \mathfrak{R}_{\mathbb{C}}(x)$ is denoted $\mathbb{r} \in \mathbb{C}\langle\!\langle x\rangle\!\rangle$.

There is a unique involution $*$ on $\mathbb{C}\langle\!\langle x\rangle\!\rangle$ that is determined by $\alpha^* = \overline{\alpha}$ for $\alpha \in \mathbb{C}$ and $x_j^* = x_j$ for $j = 1, \ldots, \mathtt{g}$. Furthermore, this involution lifts to an involutive map $*$ on the set $\mathfrak{R}_{\mathbb{C}}(x)$. For $r \in \mathfrak{R}_{\mathbb{C}}(x)$ let $\operatorname{hdom}_n r = \operatorname{dom}_n r \cap \mathrm{H}_n(\mathbb{C})^{\mathtt{g}}$. Then

$$\operatorname{hdom} r = \bigcup_{n \in \mathbb{N}} \operatorname{hdom}_n r$$

is the **hermitian domain** of r. Note that $\operatorname{hdom}_n r$ is Zariski dense in $\operatorname{dom}_n r$ because $\mathrm{H}_n(\mathbb{C})$ is Zariski dense in $\mathrm{M}_n(\mathbb{C})$ and $\operatorname{dom}_n r$ is Zariski open in $\mathrm{M}_n(\mathbb{C})^{\mathtt{g}}$. Finally, while a noncommutative rational function $\mathbb{r} \in \mathbb{C}\langle\!\langle x\rangle\!\rangle$ can be represented by various expressions, there exists an expression r representing $\mathbb{r}$ whose hermitian domain contains the hermitian domain of any other expression representing $\mathbb{r}$ [Vol21, Proposition 2.1].

2.2. **Linear representation of a formal rational expression.** Fundamental tools for handling noncommutative rational functions are linear representations [CR99, HMS18]. Let $r \in \mathfrak{R}_{\mathbb{C}}(x)$. By [HMS18, Theorem 4.2 and Algorithm 4.3] there exist $e \in \mathbb{N}$, vectors $u, v \in \mathbb{C}^e$ and an affine matrix pencil $M = M_0 + M_1 x_1 + \cdots + M_{\mathtt{g}} x_{\mathtt{g}}$ with $M_j \in \mathrm{M}_e(\mathbb{C})$ satisfying the following. For every unital $\mathbb{C}$-algebra $\mathcal{A}$ and $a \in \mathcal{A}^{\mathtt{g}}$,

(i) if r can be evaluated at a, then $M(a) = M_0 \otimes I + M_1 \otimes a_1 + \cdots + M_{\mathtt{g}} \otimes a_{\mathtt{g}} \in \mathrm{GL}_e(\mathcal{A})$ and $r(a) = (u \otimes I)^* M(a)^{-1} (v \otimes I)$;
(ii) if $M(a) \in \mathrm{GL}_e(\mathcal{A})$ and $\mathcal{A} = \mathrm{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$, then r can be evaluated at a.

We say that the triple (u, M, v) is a **linear representation** of r of size e, and M is a **denominator** of r. An affine matrix pencil M is **full** [Coh95, Section 1.4] if it is invertible over $\mathbb{C}\langle\!\langle x\rangle\!\rangle$, or equivalently, $\det M(X) \neq 0$ for some matrix tuple X (see, for example, [Vol21, Proposition 2.4]). In particular, full pencils are precisely those that can appear in linear representations of nondegenerate rational expressions.

A pencil is **irreducible** if it cannot be put into a 2×2 block upper triangular form with square diagonal blocks $\left(\begin{smallmatrix} \star & \star \\ 0 & \star \end{smallmatrix}\right)$ by a left and a right basis change. Every nonzero irreducible pencil is full. On the other hand, every full pencil is, up to a left and a right basis change, equal to a block upper-triangular pencil whose diagonal blocks are irreducible pencils. Given a pencil M, we refer to such nonconstant blocks, unique up to basis change, as **irreducible components** of M. Irreducible components of a denominator for r, disregarding multiplicities, are unique up to basis change. Indeed, let M and M' be denominators of r, and suppose M' is an irreducible component that is not a component of M after a basis change. Then, by [HKV22, Theorem 2.12] there is $X \in \mathrm{M}_n(\mathbb{C})^{\mathtt{g}}$ such that $\det M'(X) = 0$ but $\det M(X) \neq 0$, which contradicts the invertibility requirement in the definition of a linear representation.

3. Extensions and retractions

In this section we establish domain-preserving completion results, first for full linear pencils (Proposition 3.2) and then for hermitian tuples in rational domains (Proposition 3.4). Lemma 3.5 provides a variant compatible with changes of inner product.

For the purpose of this section we extend evaluations of linear matrix pencils to tuples of rectangular matrices. If $\Lambda = \sum_{j=1}^{\mathtt{g}} \Lambda_j x_j$ is of size e and $X \in \mathrm{M}_{\ell\times m}(\mathbb{C})^{\mathtt{g}}$ then

$$\Lambda(X) = \sum_{j=1}^{\mathtt{g}} \Lambda_j \otimes X_j \in \mathrm{M}_{e\ell\times em}(\mathbb{C}).$$

We require a technical tool.

Lemma 3.1 ([Vol21, Lemma 3.1])**.** *Let $\Lambda = \sum_{j=1}^{\mathtt{g}} \Lambda_j x_j$ be a homogeneous irreducible pencil of size e. Let $\ell \leqslant m$ and denote $n = (m-\ell)(e-1)$. Given $C \in \mathrm{M}_{me\times \ell e}(\mathbb{C})$, consider the pencil $\widetilde{\Lambda}$ of size $(m+n)e$ in $\mathtt{g}(m+n)(n+m-\ell)$ variables z_{jpq},*

$$\widetilde{\Lambda} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + \sum_{j=1}^{\mathtt{g}} \sum_{q=1}^{n+m-\ell} \left(\sum_{p=1}^{m} \begin{pmatrix} 0 & \widehat{E}_{p,q}\otimes \Lambda_j \\ 0 & 0 \end{pmatrix} z_{jpq} + \sum_{p=m+1}^{m+n} \begin{pmatrix} 0 & 0 \\ 0 & \breve{E}_{p-m,q}\otimes \Lambda_j \end{pmatrix} z_{jpq} \right)$$

where $\widehat{E}_{p,q} \in \mathrm{M}_{m\times(n+m-\ell)}(\mathbb{C})$ and $\breve{E}_{p-m,q} \in \mathrm{M}_{n\times(n+m-\ell)}(\mathbb{C})$ are the standard matrix units. If C has full rank, then the pencil $\widetilde{\Lambda}$ is full.

The following proposition strengthens the hypotheses of [Vol21, Proposition 3.2].

Proposition 3.2. *Let Λ be a homogeneous full pencil of size e. Suppose $X \in \mathrm{M}_{m\times\ell}(\mathbb{C})^{\mathtt{g}}$ with $\ell \leqslant m$ is such that $\Lambda'(X)$ has full rank for every irreducible component Λ' of Λ. Then there exist $\widehat{X} \in \mathrm{M}_{m\times(n+m-\ell)}(\mathbb{C})^{\mathtt{g}}$ and $\breve{X} \in \mathrm{M}_{n\times(n+m-\ell)}(\mathbb{C})^{\mathtt{g}}$ for some $n \in \mathbb{N}$ such that*

$$\det \Lambda \begin{pmatrix} X & \widehat{X} \\ 0 & \breve{X} \end{pmatrix} \neq 0. \tag{3.1}$$

Proof. First, observe that if the conclusion holds for some n, it holds for all sufficiently large n. Indeed, for every $k \geqslant e-1$ there is $Z_k \in \mathrm{M}_k(\mathbb{C})^{\mathtt{g}}$ such that $\det \Lambda(Z_k) \neq 0$ by [DM17, Theorem 1.8]. Thus, if (3.1) holds, then also

$$\det \Lambda \begin{pmatrix} X & \widehat{X} & 0 \\ 0 & \breve{X} & 0 \\ 0 & 0 & Z_k \end{pmatrix} \neq 0$$

for every $k \geqslant e-1$.

Second, suppose that the proposition holds for irreducible pencils. Given a pencil Λ, the proposition then holds for its irreducible components $\Lambda_1, \ldots, \Lambda_t$. By the first observation, there is $n \in \mathbb{N}$ such that for every $i = 1, \ldots, t$, the set $\mathcal{O}_i$ of pairs $(\widehat{X}, \breve{X}) \in \mathrm{M}_{m\times(n+m-\ell)}(\mathbb{C})^{\mathtt{g}} \times \mathrm{M}_{n\times(n+m-\ell)}(\mathbb{C})^{\mathtt{g}}$ satisfying (3.1) for Λ_i is Zariski open and nonempty. Hence, $\mathcal{O}_1 \cap \cdots \cap \mathcal{O}_t \neq \varnothing$, so the proposition also holds for full pencils. Thus, we can without loss of generality assume that Λ is irreducible.

If $m = \ell$, full column rank makes $\Lambda(X)$ invertible. A block-diagonal extension with any invertible evaluation of the full pencil Λ then proves the assertion. Hence in the following construction we may assume $m > \ell$.

Let $n_1 = (m-\ell)(e-1)$ and $\widetilde{e} = (m+n_1)e$. Apply Lemma 3.1 to $C = \sum_{j=1}^{\mathtt{g}} X_j \otimes \Lambda_j$; since the resulting pencil $\widetilde{\Lambda}$ is full, there exists $Z \in \mathrm{M}_{\widetilde{e}-1}(\mathbb{C})^{\mathtt{g}(m+n_1)(n_1+m-\ell)}$ such that $\widetilde{\Lambda}(Z)$ is invertible. Therefore the matrix

$$\sum_{j=1}^{\mathtt{g}} \Lambda_j \otimes \left(\left(\begin{smallmatrix} I\otimes X_j & 0 \\ 0 & 0 \end{smallmatrix}\right) + \sum_{q=1}^{n_1+m-\ell} \left(\sum_{p=1}^{m} \left(\begin{smallmatrix} 0 & Z_{jpq}\otimes \widehat{E}_{p,q} \\ 0 & 0 \end{smallmatrix}\right) + \sum_{p=m+1}^{m+n_1} \left(\begin{smallmatrix} 0 & 0 \\ 0 & Z_{jpq}\otimes \breve{E}_{p-m,q} \end{smallmatrix}\right) \right) \right)$$

is invertible since it is similar to $\widetilde{\Lambda}(Z)$ (via a permutation matrix). Thus there are $\widehat{X} \in \mathrm{M}_{m\times(n+m-\ell)}(\mathbb{C})^{\mathtt{g}}$ and $\check{X} \in \mathrm{M}_{n\times(n+m-\ell)}(\mathbb{C})^{\mathtt{g}}$ such that

$$\det \Lambda \begin{pmatrix} X & \widehat{X} \\ 0 & \check{X} \end{pmatrix} \neq 0,$$

where $n = (\widetilde{e}-2)m + n_1(\widetilde{e}-1)$. □

Remark 3.3. In [Vol21, Proposition 3.2], only the full rank of $\Lambda(X)$ is imposed, instead of the full rank for all irreducible components. However, taking

$$\Lambda = \begin{pmatrix} x_1 & x_2 \\ 0 & x_3 \end{pmatrix}, \quad X = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

shows that while $\Lambda(X)$ has full rank, there are no $\widehat{X}, \check{X}$ satisfying (3.1).

Let $\mathrm{M}_\infty(\mathbb{C})$ be the algebra of $\mathbb{N}\times\mathbb{N}$ matrices over $\mathbb{C}$ that have only finitely many nonzero entries in each column; that is, elements of $\mathrm{M}_\infty(\mathbb{C})$ can be viewed as linear operators on $\oplus^{\mathbb{N}}\mathbb{C}$. Given $r \in \mathfrak{R}_{\mathbb{C}}(x)$ with a denominator M, let $\mathrm{dom}_\infty r$ denote the set of tuples $X \in \mathrm{M}_\infty(\mathbb{C})^{\mathtt{g}}$ such that $M'(X)$ is invertible for every irreducible component M' of M. Note that for $X \in \mathrm{dom}_\infty r$, both $r(X)$ and $M(X)$ are well-defined, and $r(X) = (u^*\otimes I)M(X)^{-1}(v\otimes I)$ where (u, M, v) is a linear representation of r. While a denominator of r is not unique, the set $\mathrm{dom}_\infty r$ is independent of the choice of the denominator (by uniqueness of irreducible components of denominators). Note that this definition of $\mathrm{dom}_\infty r$ is slightly more restrictive than the one in [Vol21, Section 3].

Proposition 3.4. *Let $r_1, \dots, r_t \in \mathfrak{R}_{\mathbb{C}}(x)$. If $X \in \mathrm{H}_\ell(\mathbb{C})^{\mathtt{g}}$ and $Y \in \mathrm{M}_{m\times\ell}(\mathbb{C})^{\mathtt{g}}$ are such that*

$$\begin{pmatrix} X & Y^* & 0 \\ Y & & W \\ 0 & & \end{pmatrix} \in \bigcap_{i=1}^{t} \mathrm{dom}_\infty r_i$$

for some $W \in \mathrm{M}_\infty(\mathbb{C})^{\mathtt{g}}$, then there exist $n \geqslant m$, $E \in \mathrm{GL}_n(\mathbb{C})$ and $Z \in \mathrm{H}_n(\mathbb{C})^{\mathtt{g}}$ such that

$$\begin{pmatrix} X & \begin{pmatrix} Y^* & 0 \end{pmatrix} E^* \\ E \begin{pmatrix} Y \\ 0 \end{pmatrix} & Z \end{pmatrix} \in \bigcap_{i=1}^{t} \mathrm{hdom}\, r_i.$$

Proof. Let M be the direct sum of denominators of $r_1, \dots, r_t$, of size e. By assumption,

$$M' \begin{pmatrix} X & Y^* & 0 \\ Y & & W \\ 0 & & \end{pmatrix}$$

is an invertible matrix over $\mathrm{M}_\infty(\mathbb{C})$ for every irreducible component M' of M. If $M' = M'_0 + M'_1 x_1 + \cdots + M'_{\mathtt{g}} x_{\mathtt{g}}$, then the matrices

$$M'_0 \otimes \begin{pmatrix} I \\ 0 \end{pmatrix} + \sum_{j=1}^{\mathtt{g}} M'_j \otimes \begin{pmatrix} X_j \\ Y_j \end{pmatrix}, \qquad M'_0 \otimes \begin{pmatrix} I & 0 \end{pmatrix} + \sum_{j=1}^{\mathtt{g}} M'_j \otimes \begin{pmatrix} X_j & Y_j^* \end{pmatrix}$$

have full rank.

Let Λ be the homogenization of M with x_0. By Proposition 3.2 and its transpose there exist $k \in \mathbb{N}$ and

$$A' \in \mathrm{M}_{\ell\times(k+m)}(\mathbb{C})^{1+\mathtt{g}}, \quad B' \in \mathrm{M}_{m\times(k+m)}(\mathbb{C})^{1+\mathtt{g}}, \quad C' \in \mathrm{M}_{k\times(k+m)}(\mathbb{C})^{1+\mathtt{g}},$$
$$A'' \in \mathrm{M}_{(k+m)\times\ell}(\mathbb{C})^{1+\mathtt{g}}, \quad B'' \in \mathrm{M}_{(k+m)\times m}(\mathbb{C})^{1+\mathtt{g}}, \quad C'' \in \mathrm{M}_{(k+m)\times k}(\mathbb{C})^{1+\mathtt{g}}$$

such that

$$\Lambda\begin{pmatrix} X & A' \\ Y & B' \\ 0 & C' \end{pmatrix}, \qquad \Lambda\begin{pmatrix} X & Y^* & 0 \\ A'' & B'' & C'' \end{pmatrix}$$

are invertible; here, $X_0 = I$ and $Y_0 = 0$. Consequently, there exists $\varepsilon \in \mathbb{C}\backslash\{0\}$ such that

$$\begin{pmatrix} \Lambda(X) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \Lambda\begin{pmatrix} X & Y^* & 0 \\ A'' & B'' & C'' \end{pmatrix} \\ \Lambda\begin{pmatrix} X & A' \\ Y & B' \\ 0 & C' \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

is invertible; this matrix is similar to

$$\Lambda\begin{pmatrix} X & 0 & \varepsilon X & \varepsilon Y^* & 0 \\ 0 & 0 & \varepsilon A'' & \varepsilon B'' & \varepsilon C'' \\ \varepsilon X & \varepsilon A' & 0 & 0 & 0 \\ \varepsilon Y & \varepsilon B' & 0 & 0 & 0 \\ 0 & \varepsilon C' & 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

Thus the matrix (3.2) is invertible; its block structure and the linearity of Λ imply that (3.2) is invertible for every $\varepsilon \neq 0$, so we can choose $\varepsilon = 1$. After performing elementary row and column operations on (3.2) we conclude that

$$\Lambda\begin{pmatrix} X & Y^* & 0 & 0 & 0 \\ Y & 0 & -Y & B' & 0 \\ 0 & -Y^* & -X & A' & 0 \\ 0 & B'' & A'' & 0 & C'' \\ 0 & 0 & 0 & C' & 0 \end{pmatrix}$$

is invertible.

We have seen that for $n = 2(m+k)+\ell$ there exists $Z' \in \mathrm{M}_n(\mathbb{C})^{1+\mathtt{g}}$ such that

$$\det\left(M_0 \otimes \begin{pmatrix} I & 0 & 0 \\ 0 & Z'_0 \end{pmatrix} + \sum_{j=1}^{\mathtt{g}} M_j \otimes \begin{pmatrix} X_j & Y_j^* & 0 \\ Y_j & Z'_j \\ 0 \end{pmatrix} \right) \neq 0. \tag{3.3}$$

The set of all $Z' \in \mathrm{M}_n(\mathbb{C})^{1+\mathtt{g}}$ satisfying (3.3) is thus a nonempty Zariski open set in $\mathrm{M}_n(\mathbb{C})^{1+\mathtt{g}}$. Since the set of positive definite $n \times n$ matrices is Zariski dense in $\mathrm{M}_n(\mathbb{C})$, there exists $Z' \in \mathrm{H}_n(\mathbb{C})^{1+\mathtt{g}}$ with $Z'_0 \succ 0$ such that (3.3) holds. If $Z'_0 = E^{-1}E^{-*}$, let

$Z_j = EZ'_jE^*$ for $1 \leqslant j \leqslant \mathtt{g}$. Then

$$M \begin{pmatrix} X & \begin{pmatrix} Y^* & 0 \end{pmatrix} E^* \\ E \begin{pmatrix} Y \\ 0 \end{pmatrix} & Z \end{pmatrix}$$

is invertible, so

$$\begin{pmatrix} X & \begin{pmatrix} Y^* & 0 \end{pmatrix} E^* \\ E \begin{pmatrix} Y \\ 0 \end{pmatrix} & Z \end{pmatrix} \in \bigcap_{i=1}^{t} \operatorname{hdom} r_i$$

by the definition of linear representations. □

The following lemma extends the result of Proposition 3.4 to tuples that are not necessarily hermitian but are simultaneously similar to a hermitian tuple in the prescribed rational domains.

Lemma 3.5. *Let $r_1, \dots, r_t \in \mathfrak{R}_{\mathbb{C}}(x)$, and let S, U_0, W_0 be finite-dimensional Hilbert spaces. Suppose that $T = (T_1, \dots, T_{\mathtt{g}})$ is a tuple of operators on $S \oplus U_0 \oplus W_0$ of the form*

$$T_j = \begin{pmatrix} X_j & * & * \\ Y_j & * & * \\ 0 & * & * \end{pmatrix}, \qquad X_j = X_j^*, \quad 1 \leqslant j \leqslant \mathtt{g}.$$

Assume that T is simultaneously similar to a hermitian tuple $\widehat{X} \in \bigcap_{i=1}^{t} \operatorname{hdom} r_i$. Then there exist $k \in \mathbb{N}$, an invertible operator E on $U_0 \oplus \mathbb{C}^k$, and a tuple of hermitian operators $Z = (Z_1, \dots, Z_{\mathtt{g}})$ on $U_0 \oplus \mathbb{C}^k$ such that

$$\widetilde{X}_j = \begin{pmatrix} X_j & \begin{pmatrix} Y_j^* & 0 \end{pmatrix} E^* \\ E \begin{pmatrix} Y_j \\ 0 \end{pmatrix} & Z_j \end{pmatrix}, \qquad 1 \leqslant j \leqslant \mathtt{g},$$

satisfies

$$\widetilde{X} \in \bigcap_{i=1}^{t} \operatorname{hdom} r_i.$$

Proof. Let $M' = M'_0 + \sum_{j=1}^{\mathtt{g}} M'_j x_j$ be an irreducible component of a denominator of one of the expressions $r_1, \dots, r_t$ of size e. Since T is simultaneously similar to $\widehat{X}$, the matrix $M'(T)$ is invertible. Thus, its restriction to $\mathbb{C}^e \otimes S$ has full column rank. This restriction is represented by

$$M'_0 \otimes \begin{pmatrix} I_S \\ 0 \\ 0 \end{pmatrix} + \sum_{j=1}^{\mathtt{g}} M'_j \otimes \begin{pmatrix} X_j \\ Y_j \\ 0 \end{pmatrix}.$$

Moreover, T^* is also simultaneously similar to $\widehat{X}$. Indeed, if $T_j = F^{-1}\widehat{X}_j F$, then $T_j^* = F^*\widehat{X}_j(F^*)^{-1}$. Thus $M'(T^*)$ is invertible as well. Since $X_j = X_j^*$, its first block row is

$$M'_0 \otimes \begin{pmatrix} I_S & 0 & 0 \end{pmatrix} + \sum_{j=1}^{\mathtt{g}} M'_j \otimes \begin{pmatrix} X_j & Y_j^* & 0 \end{pmatrix},$$

which therefore has full row rank.

The rest of the proof is the same as in the proof of Proposition 3.4. □

4. Multiplication operators of admissible sets

In this section we introduce admissible sets of rational expressions and the associated multiplication operators. Lemma 4.1 shows that these operators lie in the prescribed rational domains, while Proposition 4.2 relates their finite-dimensional extensions to evaluations of rational expressions.

We say that a finite set $R \subset \mathfrak{R}_{\mathbb{C}}(x)$ is **admissible** if

(1) R contains $1, x_1, \dots, x_{\mathtt{g}}$;
(2) R is closed under subexpressions and $*$;
(3) if $r \in R$ has a denominator M with an irreducible component M', then there is $r' \in R$ with M' as a denominator.

Note that every $r \in \mathfrak{R}_{\mathbb{C}}(x)$ belongs to a finite admissible set because only finitely many (up to basis change) irreducible pencils are components of denominators of r. Indeed, by the inclusion of domains and [HKV22, Theorem 2.12], taking subexpressions does not introduce new irreducible denominator components, and taking adjoints only produces their adjoints.

Let $\mathcal{R} \subset \mathbb{C}\langle\langle x\rangle\rangle$ be the set of noncommutative rational functions represented by an admissible R. For $\ell \in \mathbb{N}$ we define an increasing sequence of finite-dimensional vector subspaces

$$V_\ell = \operatorname{span}_{\mathbb{C}} \overbrace{\mathcal{R} \cdots \mathcal{R}}^{\ell} \subset \mathbb{C}\langle\langle x\rangle\rangle.$$

Then, $V := \bigcup_{\ell\in\mathbb{N}} V_\ell$ is a finitely generated $*$-subalgebra of $\mathbb{C}\langle\langle x\rangle\rangle$. For $j = 1, \dots, \mathtt{g}$ we define left multiplication operators

$$\mathfrak{X}_j : V \to V, \qquad \mathfrak{X}_j \mathbb{r} = x_j \mathbb{r}.$$

By [Vol21, Lemma 4.1], there is a linear functional $\phi : V \to \mathbb{C}$ such that $\phi(\mathbb{r}^*) = \overline{\phi(\mathbb{r})}$ and $\phi(\mathbb{r}\mathbb{r}^*) > 0$ for all $\mathbb{r} \in V\backslash\{0\}$. This functional gives rise to an inner product on V,

$$(\mathbb{r}_1, \mathbb{r}_2) = \phi(\mathbb{r}_2^* \mathbb{r}_1). \tag{4.1}$$

With respect to this inner product we can inductively build an ordered orthogonal basis $\mathscr{B}$ of V with the property that $\mathscr{B} \cap V_\ell$ is a basis of V_ℓ for every $\ell \in \mathbb{N}$. The following lemma gives rise to tuples $\mathfrak{X} \in \operatorname{dom}_\infty r$ to which Proposition 3.4 will later be applied.

Lemma 4.1. *With respect to the inner product* (4.1) *and the ordered basis $\mathscr{B}$ as above, the operators $\mathfrak{X}_1, \dots, \mathfrak{X}_g$ are represented by hermitian matrices in $\mathrm{M}_\infty(\mathbb{C})$, and $\mathfrak{X} \in \operatorname{dom}_\infty r$ for every $r \in R$.*

Proof. Since

$$\big(\mathfrak{X}_j \mathbb{r}_1, \mathbb{r}_2\big) = \phi\big(\mathbb{r}_2^* x_j \mathbb{r}_1\big) = \big(\mathbb{r}_1, \mathfrak{X}_j \mathbb{r}_2\big)$$

for all $\mathbb{r}_1, \mathbb{r}_2 \in V$ and $\mathfrak{X}_j(V_\ell) \subseteq V_{\ell+1}$ for all $\ell \in \mathbb{N}$, it follows that the matrix representation of $\mathfrak{X}_j$ with respect to $\mathscr{B}$ is hermitian and has only finitely many nonzero entries in each column and row. The rest follows by induction on the construction of r since $\mathfrak{X}_j$ are the left multiplication operators on V and R is admissible. □

We use a complexity measure $\tau : \mathfrak{R}_{\mathbb{C}}(x) \to \mathbb{N} \cup \{0\}$ defined recursively as follows:

(i) $\tau(\alpha) = 0$ for $\alpha \in \mathbb{C}$ and $\tau(x_j) = 1$ for $1 \leqslant j \leqslant \mathtt{g}$;
(ii) $\tau(r_1 + r_2) = \max\{\tau(r_1), \tau(r_2)\}$ for $r_1, r_2 \in \mathfrak{R}_{\mathbb{C}}(x)$;
(iii) $\tau(r_1 r_2) = \tau(r_1) + \tau(r_2)$ for $r_1, r_2 \in \mathfrak{R}_{\mathbb{C}}(x)$;
(iv) $\tau(r^{-1}) = 2\tau(r)$ for $r, r^{-1} \in \mathfrak{R}_{\mathbb{C}}(x)$.

For a finite set $R \subset \mathfrak{R}_{\mathbb{C}}(x)$ let $\tau(R) = \max_{r \in R} \tau(r)$. If R is admissible, then $x_1 \in R$, so $\tau(R) \geqslant 1$.

Proposition 4.2 ([Vol21, Proposition 4.3])**.** *Let the notation be as above, and $\ell \in \mathbb{N}$. Let $r \in R$ and $s \in R \cdots R$ (ℓ times) satisfy $2\tau(r) + \tau(s) \leqslant \ell + 2$. If U is a finite-dimensional Hilbert space containing $V_{\ell+1}$ and X is a $\mathtt{g}$-tuple of hermitian operators on U such that $X \in \operatorname{hdom} r$ and*

$$X_j|_{V_\ell} = \mathfrak{X}_j|_{V_\ell} \quad \text{for } j = 1, \ldots, \mathtt{g},$$

then

$$r(X)\mathbb{s} = \mathbb{r}\mathbb{s}.$$

The proposition also holds with V_k replaced by $V_k^{1\times d}$, with $\mathfrak{X}_j$ acting componentwise and s replaced by a standard unit row. Indeed, the proof of [Vol21, Proposition 4.3] applies componentwise and uses only the stated multiplication identities.

5. Truncated rational quadratic module

We introduce truncated rational quadratic modules with matrix coefficients. Their Gram–CP description in Proposition 5.2 establishes convexity and provides the foundation for the later operator-valued extension.

Let $\mathcal{H}$ be a nonzero complex Hilbert space and let

$$L = I_{\mathcal{H}} + \sum_{j=1}^{\mathtt{g}} H_j x_j, \qquad H_j = H_j^* \in \mathcal{B}(\mathcal{H}).$$

Set $\mathcal{S}_L = \operatorname{span}_{\mathbb{C}}\{I_{\mathcal{H}}, H_1, \ldots, H_{\mathtt{g}}\}$ and $\mathcal{D}_n(L) = \{X \in \mathrm{H}_n(\mathbb{C})^{\mathtt{g}} : L(X) \succeq 0\}$. We write $\mathcal{D}(L) = \bigcup_{n \geqslant 1} \mathcal{D}_n(L)$. Fix a finite admissible set $R \subset \mathfrak{R}_{\mathbb{C}}(x)$ and let $\mathcal{R} \subset \mathbb{C}\langle\!\langle x \rangle\!\rangle$ be the corresponding set of noncommutative rational functions. Let $V_k = V_k(\mathcal{R})$ be the vector spaces from Section 4. For $\ell \geqslant 0$ put $m = m_\ell = \dim_{\mathbb{C}} V_\ell$ (with $V_0 = \mathbb{C}1$).

Definition 5.1. For a positive integer d define

$$Q_\ell^{(d)}(L; \mathcal{R}) = \{\mathbb{r}^*\mathbb{r} + \mathbb{q}^*\pi(L)\mathbb{q} \ : \ \mathbb{r}, \mathbb{q} \in \mathrm{M}_{md\times d}(V_\ell),\ \pi \in \mathrm{UCP}(\mathcal{S}_L, \mathrm{M}_{md}(\mathbb{C}))\}, \tag{5.1}$$

where $\pi(L) = I_{md} + \sum_j \pi(H_j)x_j$. This is a subset of

$$\mathrm{M}_d(V_{2\ell+1})_{\mathrm{sa}} = \{\mathbb{F} \in \mathrm{M}_d(V_{2\ell+1}) : \mathbb{F} = \mathbb{F}^*\}.$$

Proposition 5.2. *Let $\mathbb{e}_1, \ldots, \mathbb{e}_m$ be a basis of V_ℓ, and set $\mathbf{e}_\ell = \operatorname{col}(\mathbb{e}_1 I_d, \ldots, \mathbb{e}_m I_d)$. Then $\mathbb{F} \in \mathrm{M}_d(V_{2\ell+1})_{\mathrm{sa}}$ is in $Q_\ell^{(d)}(L; \mathcal{R})$ if and only if there exist positive semidefinite $G \in \mathrm{M}_m(\mathrm{M}_d(\mathbb{C}))_+$ and a completely positive map $\psi : \mathcal{S}_L \to \mathrm{M}_m(\mathrm{M}_d(\mathbb{C}))$ satisfying*

$$\mathbb{F} = \mathbf{e}_\ell^*\big(G + \psi(L)\big)\mathbf{e}_\ell. \tag{5.2}$$

In particular, $Q_\ell^{(d)}(L; \mathcal{R})$ is a convex cone.

Proof. If $\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*\pi(L)\mathbb{q}$, write $\mathbb{r} = \mathsf{R}\,\mathbf{e}_\ell$ and $\mathbb{q} = \mathsf{Q}\,\mathbf{e}_\ell$ for $\mathsf{R}, \mathsf{Q} \in \mathrm{M}_m(\mathrm{M}_d(\mathbb{C}))$. Then $G = \mathsf{R}^*\mathsf{R}$ and $\psi(B) = \mathsf{Q}^*\pi(B)\mathsf{Q}$ give (5.2).

Conversely, suppose that (5.2) holds. Let $\mathsf{R} = G^{1/2}$ and $D = \psi(I_{\mathcal{H}})^{1/2}$. By Arveson's extension theorem and Stinespring's theorem [Pau02], there is a representation ρ of $\mathcal{B}(\mathcal{H})$ and an operator Γ with $\psi(B) = \Gamma^*\rho(B)\Gamma$. Writing $\Gamma = \Theta D$ in the polar decomposition, we have $\Theta^*\Theta = P$, where P is the support projection of D. Fix a state ω on $\mathcal{S}_L$. Then

$$\pi(B) = \Theta^*\rho(B)\Theta + \omega(B)(I_{md} - P)$$

is a unital completely positive map and satisfies $\psi(B) = D\pi(B)D$ for $B \in \mathcal{S}_L$. Taking $\mathbb{r} = \mathsf{R}\mathbf{e}_\ell$ and $\mathbb{q} = D\mathbf{e}_\ell$ gives $\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*\pi(L)\mathbb{q}$, as required. □

6. Finite-dimensional reproducing tuples and coefficient extraction

In this section we construct finite-dimensional hermitian tuples that reproduce prescribed rational functions (Proposition 6.2), using the domain-preserving extensions of Proposition 6.1. These tuples yield coefficient-extraction functionals (Lemma 6.3), which will be used to control the truncated quadratic modules.

We continue with the notation of Section 5.

Proposition 6.1. *For every $\ell \in \mathbb{N}$, there exist an integer $n_\ell \geqslant 0$ and a $\boldsymbol{g}$-tuple of hermitian operators $\widetilde{X}^{(\ell)}$ on $V_{\ell+1} \oplus \mathbb{C}^{n_\ell}$ such that*

$$\widetilde{X}^{(\ell)} \in \bigcap_{r\in R} \mathrm{hdom}\, r \qquad \text{and} \qquad \widetilde{X}^{(\ell)}_j|_{V_\ell} = \mathfrak{X}_j|_{V_\ell}, \quad j = 1, \dots, \boldsymbol{g}.$$

Proof. Fix $\ell \in \mathbb{N}$, and let U_0 be the orthogonal complement of $V_{\ell+1}$ in $V_{\ell+2}$. With respect to the orthogonal decomposition $V = V_{\ell+1} \oplus U_0 \oplus V_{\ell+2}^\perp$, the self-adjoint multiplication tuple has the block form

$$\mathfrak{X} = \begin{pmatrix} X & \begin{pmatrix} Y^* & 0 \end{pmatrix} \\ \begin{pmatrix} Y \\ 0 \end{pmatrix} & W \end{pmatrix}.$$

By Lemma 4.1, the tuple $\mathfrak{X}$ belongs to $\bigcap_{r\in R} \mathrm{dom}_\infty\, r$. Applying Proposition 3.4 to the finitely many expressions in R, we obtain E, Z such that

$$\widetilde{X}^{(\ell)} = \begin{pmatrix} X & \begin{pmatrix} Y^* & 0 \end{pmatrix} E^* \\ E \begin{pmatrix} Y \\ 0 \end{pmatrix} & Z \end{pmatrix} \in \bigcap_{r\in R} \mathrm{hdom}\, r.$$

Since $\mathfrak{X}_j(V_\ell) \subseteq V_{\ell+1}$, $\widetilde{X}^{(\ell)}_j|_{V_\ell} = \mathfrak{X}_j|_{V_\ell}$. □

Proposition 6.2. *For every $\ell \in \mathbb{N}$, there exist integers $K_\ell \geqslant \ell$ and $n_{K_\ell} \geqslant 0$, and a $\boldsymbol{g}$-tuple of hermitian operators $\mathfrak{A}^{(\ell)}$ acting on $U^{(\ell)} = V_{K_\ell+1} \oplus \mathbb{C}^{n_{K_\ell}}$, such that $\mathfrak{A}^{(\ell)} \in \bigcap_{r\in R} \mathrm{hdom}\, r$ and*

$$\mathbb{q}(\mathfrak{A}^{(\ell)})\mathbb{1} = \mathbb{q}, \qquad \mathbb{q} \in V_\ell. \tag{6.1}$$

Proof. Fix $\ell \in \mathbb{N}$. Choose an integer K_ℓ such that $K_\ell \geqslant \ell$ and $K_\ell + 2 \geqslant (\ell+1)\tau(R)$. By Proposition 6.1, the tuple $\mathfrak{A}^{(\ell)} := \widetilde{X}^{(K_\ell)}$ acts on $V_{K_\ell+1} \oplus \mathbb{C}^{n_{K_\ell}}$ and satisfies $\mathfrak{A}^{(\ell)} \in \bigcap_{r\in R} \mathrm{hdom}\, r$.

Now we shall prove (6.1). Since V_ℓ is spanned by products of at most ℓ elements of $\mathcal{R}$, it suffices by linearity to consider

$$\mathfrak{q} = \mathfrak{q}_1 \cdots \mathfrak{q}_k, \qquad 1 \leqslant k \leqslant \ell,$$

where each $\mathfrak{q}_j \in \mathcal{R}$. Choose representatives $q_j \in R$ of $\mathfrak{q}_j$. For $1 \leqslant j \leqslant k$, put $s_j := q_{j+1} \cdots q_k$, with $s_k = 1$. Then $\tau(s_j) \leqslant (k-j)\tau(R)$, and therefore

$$2\tau(q_j) + \tau(s_j) \leqslant 2\tau(R) + (k-j)\tau(R) \leqslant (\ell+1)\tau(R) \leqslant K_\ell + 2.$$

Because $K_\ell \geqslant \ell$, each s_j may be viewed as a product of K_ℓ elements of R by inserting copies of 1. Therefore, a successive application of Proposition 4.2 gives

$$(q_1 \cdots q_k)(\mathfrak{A}^{(\ell)})\mathbb{1} = \mathfrak{q}_1 \cdots \mathfrak{q}_k.$$

Thus (6.1) holds on a spanning set of V_ℓ, and hence on all of V_ℓ. □

Lemma 6.3. *Let $k \in \mathbb{N}$, $\mathcal{W} \subseteq V_k$ be a finite-dimensional subspace, and let $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_w\}$ be an orthonormal basis of $\mathcal{W}$. For each $1 \leqslant j \leqslant w$ define a linear functional*

$$\Omega_j : \mathcal{B}(U^{(k)}) \longrightarrow \mathbb{C}, \qquad T \longmapsto \langle T\mathbb{1}, \mathfrak{e}_j \rangle.$$

Let $\mathcal{K}$ be a complex Hilbert space. If $\mathbb{F} = \sum_{j=1}^{w} C_j \otimes \mathfrak{e}_j \in \mathcal{B}(\mathcal{K}) \otimes \mathcal{W}$, then

$$(\mathrm{id}_{\mathcal{B}(\mathcal{K})} \otimes \Omega_j)\mathbb{F}(\mathfrak{A}^{(k)}) = C_j, \qquad 1 \leqslant j \leqslant w, \tag{6.2}$$

where the tuple $\mathfrak{A}^{(k)}$ is from Proposition 6.2.

Proof. By Proposition 6.2, $\mathfrak{e}_i(\mathfrak{A}^{(k)})\mathbb{1} = \mathfrak{e}_i$ for every i. Hence

$$(\mathrm{id}_{\mathcal{B}(\mathcal{K})} \otimes \Omega_j)\mathbb{F}(\mathfrak{A}^{(k)}) = \sum_{i=1}^{w} C_i \, \Omega_j(\mathfrak{e}_i(\mathfrak{A}^{(k)})) = \sum_{i=1}^{w} C_i \, (\mathfrak{e}_i, \mathfrak{e}_j) = C_j.$$ □

7. Ultraweak closedness of the truncated rational quadratic module

We prove that the truncated rational quadratic modules are closed in the product ultraweak topology (Theorem 7.3). The key ingredient is a uniform bound on their Gram–CP representations (Lemma 7.2), obtained using the reproducing tuples constructed in Section 6.

7.1. **Product ultraweak topology and intrinsic quadratic modules.** Let $\mathcal{K}$ be a complex Hilbert space. For $n \geqslant 0$, let $\mathcal{A}_n = \mathcal{B}(\mathcal{K}) \otimes V_n$, with the algebraic tensor product. Choose a basis $\mathbb{b}_1, \ldots, \mathbb{b}_s$ of V_n. A net $\sum_a C_{\alpha,a} \otimes \mathbb{b}_a$ converges in the *product ultraweak topology* if every coefficient $C_{\alpha,a}$ converges ultraweakly in $\mathcal{B}(\mathcal{K})$. This topology is independent of the choice of basis. Identifying $\mathcal{A}_n$ with $\mathcal{B}(\mathcal{K})^s$, it is the weak-$*$ topology arising from the predual consisting of s-tuples of trace-class operators $\mathcal{T}(\mathcal{K})^s$, with its ℓ^1-norm. We write

$$\left\| \sum_a C_a \otimes \mathbb{b}_a \right\|_{\mathrm{c}} := \max_a \|C_a\|.$$

Let $\mathcal{H}, \mathcal{K}$ be nonzero complex Hilbert spaces. The coefficients of L act on $\mathcal{H}$, while those of the rational function act on $\mathcal{K}$. We keep the admissible set R fixed. For a basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_m$ of V_ℓ define the *intrinsic rational quadratic module* $Q_\ell(\mathcal{K}, L; \mathcal{R})$ by

$$Q_\ell(\mathcal{K}, L; \mathcal{R}) \coloneqq \{\mathbf{e}_\ell^*(G + \psi(L))\mathbf{e}_\ell \,:\, G \in \mathcal{B}(\mathcal{K}^m)_+,\ \psi \in \mathrm{CP}(\mathcal{S}_L, \mathcal{B}(\mathcal{K}^m))\},$$

where $\mathbf{e}_\ell = \operatorname{col}(\mathbb{e}_1 I_{\mathcal{K}}, \dots, \mathbb{e}_m I_{\mathcal{K}})$. Thus $Q_\ell(\mathbb{C}^d, L; \mathcal{R}) = Q_\ell^{(d)}(L; \mathcal{R})$ by Proposition 5.2. A change of basis in V_ℓ only changes the Gram–CP data by congruence, so the cone is independent of the basis. Also, $Q_\ell(\mathcal{K}, L; \mathcal{R}) \subseteq (\mathcal{B}(\mathcal{K}) \otimes V_{2\ell+1})_{\mathrm{sa}}$.

7.2. **Uniform bounds and ultraweak closedness.**

Lemma 7.1. *Let $k \geqslant 0$ and let $\mathcal{W} \subseteq V_k$ have basis $\mathbb{e}_1, \dots, \mathbb{e}_w$. There are $n \geqslant 1$, a hermitian tuple $X \in \bigcap_{r \in R} \operatorname{hdom}_n r$, and vectors $\eta, \zeta_1, \dots, \zeta_w \in \mathbb{C}^n$ such that*

$$L(X) \succeq \tfrac{1}{2} I, \qquad \langle \mathbb{e}_i(X)\eta, \zeta_j \rangle = \delta_{ij}.$$

In particular, the functionals $\Omega_j(T) = \langle T\eta, \zeta_j \rangle$ extract the coefficients of every element of $\mathcal{B}(\mathcal{K}) \otimes \mathcal{W}$, for any Hilbert space $\mathcal{K}$.

Proof. Let $\mathfrak{A} = \mathfrak{A}^{(\max\{1,k\})}$ be the tuple furnished by Proposition 6.2 and put $\eta = \mathbb{1}$. The reproducing property shows that the vectors $\mathbb{e}_1(\mathfrak{A})\eta, \dots, \mathbb{e}_w(\mathfrak{A})\eta$ are linearly independent. We claim that this remains true after replacing $\mathfrak{A}$ by a sufficiently small positive scalar multiple, provided one avoids finitely many scalars.

Indeed, along the line $t \mapsto t\mathfrak{A}$, each determinant needed to evaluate an expression in R is a rational function of t. These determinants do not vanish at $t = 1$, and hence have only finitely many zeros. The same reasoning applies to a nonzero minor of the matrix with columns $\mathbb{e}_i(t\mathfrak{A})\eta$. Thus the expressions in R remain defined and the indicated vectors remain independent for arbitrarily small $t > 0$. As L is monic, we may also arrange that

$$L(t\mathfrak{A}) = I + t \sum_{j=1}^{\mathrm{g}} H_j \otimes \mathfrak{A}_j \succeq \tfrac{1}{2} I.$$

Set $X = t\mathfrak{A}$ and choose ζ_j dual to the independent vectors $\mathbb{e}_i(X)\eta$. For $\mathbb{F} = \sum_i C_i \otimes \mathbb{e}_i$, we then have

$$(\mathrm{id} \otimes \Omega_j)\mathbb{F}(X) = \sum_i C_i \langle \mathbb{e}_i(X)\eta, \zeta_j \rangle = C_j. \qquad \square$$

Lemma 7.2. *Fix $\ell \geqslant 0$ and a basis $\mathbb{e}_1, \dots, \mathbb{e}_m$ of V_ℓ. There is a constant $\kappa > 0$, depending only on L, R, and the chosen bases, such that every representation*

$$\mathbb{F} = \mathbf{e}_\ell^*(G + \psi(L))\mathbf{e}_\ell, \qquad G \succeq 0, \quad \psi \in \mathrm{CP}(\mathcal{S}_L, \mathcal{B}(\mathcal{K}^m)),$$

satisfies

$$\|G\| + \|\psi(I_{\mathcal{H}})\| \leqslant \kappa \|\mathbb{F}\|_{\mathrm{c}}. \tag{7.1}$$

Here the coefficient norm is taken in a fixed basis of $V_{2\ell+1}$. In particular, κ is independent of $\mathcal{K}$.

Proof. Choose X, η from Lemma 7.1 with $\mathcal{W} = V_\ell$. The vectors $\mathbb{e}_i(X)\eta$ are linearly independent. Thus there exists $\beta > 0$ such that

$$\Big\| \sum_i v_i \otimes \mathbb{e}_i(X)\eta \Big\|^2 \geqslant \beta \sum_i \|v_i\|^2 \tag{7.2}$$

for vectors v_i in any Hilbert space. The same β works for every such space.

Factor $G = \mathsf{R}^*\mathsf{R}$, with block entries R_{ji}. By Arveson's extension theorem and Stinespring's theorem, write $\psi(B) = \Gamma^*\rho(B)\Gamma$ for $B \in \mathcal{S}_L$, where ρ is a unital representation of $\mathcal{B}(\mathcal{H})$ and $\Gamma = (\Gamma_1 \ \cdots \ \Gamma_m)$. Set $\widetilde{\mathfrak{q}} = \sum_i \Gamma_i \otimes \mathfrak{e}_i$. Then

$$\mathbf{e}_\ell^*\psi(L)\mathbf{e}_\ell = \widetilde{\mathfrak{q}}^*\rho(L)\widetilde{\mathfrak{q}}.$$

As $L(X) \succeq \frac{1}{2}I$, positivity is preserved by $\rho \otimes \mathrm{id}$, and hence $\rho(L)(X) \succeq \frac{1}{2}I$. For $\xi \in \mathcal{K}$,

$$\begin{aligned}&\big\langle(\mathbf{e}_\ell^* G\mathbf{e}_\ell)(X)(\xi\otimes\eta), \xi\otimes\eta\big\rangle \\ &\qquad = \sum_j \Big\| \sum_i \mathsf{R}_{ji}\xi \otimes \mathfrak{e}_i(X)\eta\Big\|^2 \geqslant \beta \sum_{i,j} \|\mathsf{R}_{ji}\xi\|^2 = \beta\Big\langle \sum_i G_{ii}\xi, \xi\Big\rangle,\end{aligned}$$

while

$$\begin{aligned}&\big\langle(\mathbf{e}_\ell^*\psi(L)\mathbf{e}_\ell)(X)(\xi\otimes\eta), \xi\otimes\eta\big\rangle \\ &\qquad \geqslant \frac{1}{2}\Big\| \sum_i \Gamma_i\xi \otimes \mathfrak{e}_i(X)\eta\Big\|^2 \geqslant \frac{\beta}{2} \sum_i \|\Gamma_i\xi\|^2 = \frac{\beta}{2}\Big\langle \sum_i \psi(I_\mathcal{H})_{ii}\xi, \xi\Big\rangle.\end{aligned}$$

Adding these estimates yields

$$\beta\Big\langle \Big(\sum_i G_{ii} + \tfrac{1}{2}\sum_i \psi(I_\mathcal{H})_{ii}\Big)\xi, \xi\Big\rangle \leqslant \|\eta\|^2\|\mathbb{F}(X)\|\|\xi\|^2.$$

For a positive block matrix, its norm is at most the sum of the norms of its diagonal blocks. We therefore obtain $\|G\| + \|\psi(I_\mathcal{H})\| \leqslant \kappa_1\|\mathbb{F}(X)\|$. Since X is fixed, evaluation at X is bounded in the coefficient norm, independently of $\mathcal{K}$. Thus $\|\mathbb{F}(X)\| \leqslant \kappa_2\|\mathbb{F}\|_{\mathrm{c}}$, which proves the assertion. □

Theorem 7.3. *For every $\ell \geqslant 0$ and every Hilbert space $\mathcal{K}$, $Q_\ell(\mathcal{K}, L; \mathcal{R})$ is closed in the product ultraweak topology of $(\mathcal{B}(\mathcal{K}) \otimes V_{2\ell+1})_{\mathrm{sa}}$.*

Proof. Choose a basis $\mathbb{b}_1, \ldots, \mathbb{b}_s$ of $V_{2\ell+1}$. It suffices, by the Krein–Šmulian theorem, to show that the intersection of the cone with every closed ball of the coefficient norm is product-ultraweakly closed.

Let $(\mathbb{F}_\alpha)$ be a coefficient-norm bounded net in $Q_\ell(\mathcal{K}, L; \mathcal{R})$ converging product-ultraweakly to $\mathbb{F}$. Choose Gram–CP data (G_α, ψ_α) for each $\mathbb{F}_\alpha$. Lemma 7.2 gives uniform bounds on G_α and $\psi_\alpha(I_\mathcal{H})$. Since ψ_α is completely positive, $\|\psi_\alpha\| = \|\psi_\alpha(I_\mathcal{H})\|$.

Let $B_1, \ldots, B_p$ be a basis of $\mathcal{S}_L$. The Banach–Alaoglu theorem and the finiteness of this basis give a subnet, still indexed by α, for which

$$G_\alpha \longrightarrow G, \qquad \psi_\alpha(B_i) \longrightarrow \Psi_i \quad (1 \leqslant i \leqslant p)$$

ultraweakly. The operators Ψ_i determine a linear map $\psi : \mathcal{S}_L \to \mathcal{B}(\mathcal{K}^m)$; by linearity, $\psi_\alpha(B) \to \psi(B)$ ultraweakly for all $B \in \mathcal{S}_L$. The ultraweak closedness of the positive cone gives $G \succeq 0$. Applied to positive matrices over $\mathcal{S}_L$, the same argument gives $[\psi(B_{ij})] \succeq 0$ whenever $[B_{ij}] \succeq 0$. Thus ψ is completely positive.

To identify the limit, expand

$$\mathbb{F}_\alpha = \sum_{i,j}\big((G_\alpha)_{ij} + \psi_\alpha(I_\mathcal{H})_{ij}\big) \otimes \mathfrak{e}_i^*\mathfrak{e}_j + \sum_{k=1}^{\mathfrak{g}} \sum_{i,j} \psi_\alpha(H_k)_{ij} \otimes \mathfrak{e}_i^* x_k \mathfrak{e}_j.$$

Every rational function on the right belongs to $V_{2\ell+1}$ and is independent of α. Upon expressing these functions in the fixed basis $\mathbb{b}_1, \ldots, \mathbb{b}_s$, each coefficient is a finite linear

combination of blocks of G_α and of $\psi_\alpha(I_{\mathcal{H}}), \psi_\alpha(H_1), \dots, \psi_\alpha(H_{\mathtt{g}})$. Their ultraweak limits are therefore the coefficients of $\mathbf{e}_\ell^*(G + \psi(L))\mathbf{e}_\ell$. By uniqueness of the product-ultraweak limit,

$$\mathbb{F} = \mathbf{e}_\ell^*(G + \psi(L))\mathbf{e}_\ell.$$

Hence $\mathbb{F}$ belongs to the cone. The assertion now follows from the Krein–Šmulian theorem, since the cone is convex by Proposition 5.2 and its intrinsic Gram–CP description. □

8. Finite-dimensional rational Positivstellensatz

In this section we prove the matrix-valued rational Positivstellensatz with an explicit truncation bound (Theorem 8.3). The proof combines separation of the truncated quadratic module with the construction of feasible finite-dimensional extensions of multiplication operators (Lemma 8.2).

Lemma 8.1. *Let $\ell \geqslant 2$, and let $\lambda : \mathrm{M}_d(V_{2\ell+1}) \to \mathbb{C}$ be a linear functional which is strictly positive on $Q_\ell^{(d)}(L;\mathcal{R})\backslash\{0\}$. Then*

$$\langle \mathbb{u}, \mathbb{v} \rangle = \lambda(\mathbb{v}^* \mathbb{u})$$

defines an inner product on $U = V_\ell^{1\times d}$. The multiplication operators $\mathfrak{X}_j : S = V_{\ell-1}^{1\times d} \to U$ are symmetric on S. If $A_j = P_S \mathfrak{X}_j|_S$, then

$$L(A) = I_{\mathcal{H}} \otimes I_S + \sum_{j=1}^{g} H_j \otimes A_j \succeq \delta I$$

for some $\delta > 0$.

Proof. If $\mathbb{u} \neq 0$, then $\mathbb{u}^*\mathbb{u}$ is a nonzero element of $Q_\ell^{(d)}(L;\mathcal{R})$, so $\lambda(\mathbb{u}^*\mathbb{u}) > 0$. Moreover, for $\mathbb{u}, \mathbb{v} \in S$,

$$\langle \mathfrak{X}_j \mathbb{u}, \mathbb{v} \rangle = \lambda(\mathbb{v}^* x_j \mathbb{u}) = \langle \mathbb{u}, \mathfrak{X}_j \mathbb{v} \rangle.$$

Given $0 \neq \zeta \in \mathcal{H} \otimes S$, write

$$\zeta = \sum_{a=1}^{k} h_a \otimes \mathbb{u}_a,$$

where the h_a are orthonormal and $\mathbb{u}_a \in S$. We may assume $k < md$, where $m = \dim V_\ell$. Let $\Theta : \mathbb{C}^k \to \mathcal{H}$ be the isometry defined by $\Theta e_a = h_a$. Fix a state ω on $\mathcal{S}_L$ and define

$$\pi(B) = \Theta^* B \Theta \oplus \omega(B) I_{md-k}, \qquad B \in \mathcal{S}_L.$$

Then $\pi \in \mathrm{UCP}(\mathcal{S}_L, \mathrm{M}_{md}(\mathbb{C}))$. Let $\mathbb{q} \in \mathrm{M}_{md\times d}(V_{\ell-1})$ have rows $\mathbb{u}_1, \dots, \mathbb{u}_k$, followed by zero rows. Then

$$\begin{aligned}
\langle L(A)\zeta, \zeta \rangle &= \sum_{a=1}^{k} \langle \mathbb{u}_a, \mathbb{u}_a \rangle + \sum_{j=1}^{\mathtt{g}} \sum_{a,b=1}^{k} \langle H_j h_b, h_a \rangle \langle A_j \mathbb{u}_b, \mathbb{u}_a \rangle \\
&= \sum_{a=1}^{k} \lambda(\mathbb{u}_a^* \mathbb{u}_a) + \sum_{j=1}^{\mathtt{g}} \sum_{a,b=1}^{k} \langle H_j h_b, h_a \rangle \lambda(\mathbb{u}_a^* x_j \mathbb{u}_b) \\
&= \lambda \left(\sum_{a,b=1}^{k} \pi(I_{\mathcal{H}})_{ab} \mathbb{u}_a^* \mathbb{u}_b + \sum_{j=1}^{\mathtt{g}} \sum_{a,b=1}^{k} \pi(H_j)_{ab} \mathbb{u}_a^* x_j \mathbb{u}_b \right) \\
&= \lambda\big(\mathbb{q}^* \pi(L) \mathbb{q}\big) > 0.
\end{aligned}$$

The inequality is strict because $\mathfrak{q}^*\pi(L)\mathfrak{q}$ is a nonzero element of $Q_\ell^{(d)}(L;\mathcal{R})$. Indeed, since $\zeta \neq 0$, we have $\mathfrak{q} \neq 0$. By Lemma 7.1, there exists a hermitian tuple $X \in \bigcap_{r\in R} \operatorname{hdom}_n r$ such that $L(X) \succeq \frac{1}{2}I$ and evaluation at X is injective on V_ℓ. In particular, $\mathfrak{q}(X) \neq 0$. Since π is UCP, $\pi(L)(X) \succeq \frac{1}{2}I$, and hence

$$(\mathfrak{q}^*\pi(L)\mathfrak{q})(X) = \mathfrak{q}(X)^*\pi(L)(X)\mathfrak{q}(X) \succeq \frac{1}{2}\mathfrak{q}(X)^*\mathfrak{q}(X) \neq 0.$$

Thus $\mathfrak{q}^*\pi(L)\mathfrak{q} \neq 0$, and the strict positivity of λ on $Q_\ell^{(d)}(L;\mathcal{R})\backslash\{0\}$ proves the claim.

Since the unit sphere of $\mathrm{M}_{md\times d}(V_{\ell-1})$ and $\mathrm{UCP}(\mathcal{S}_L, \mathrm{M}_{md}(\mathbb{C}))$ are compact, the strict positivity above is uniform for $\|\zeta\| = 1$. Thus $L(A) \succeq \delta I$ for some $\delta > 0$. □

Lemma 8.2. *Let $\ell \geqslant 2$, and equip $U = V_\ell^{1\times d}$ with an inner product for which the compressions*

$$A_j = P_S\mathfrak{X}_j|_S, \qquad S = V_{\ell-1}^{1\times d},$$

are hermitian. Suppose that $L(A) \succeq \delta I$ for some $\delta > 0$. Then there exists a finite-dimensional hermitian tuple

$$X \in \mathcal{D}(L) \cap \bigcap_{r\in R} \operatorname{hdom} r$$

such that

$$X_j|_{V_{\ell-2}^{1\times d}} = \mathfrak{X}_j|_{V_{\ell-2}^{1\times d}} \qquad (1 \leqslant j \leqslant \mathsf{g}). \tag{8.1}$$

Proof. We first work with the inner product induced by ϕ in Section 4. By Lemma 4.1, the multiplication operators are hermitian and lie in $\bigcap_{r\in R} \operatorname{dom}_\infty r$.

Write $U = S \oplus_\phi U_0$, where U_0 is the orthogonal complement of S in U with respect to the inner product induced by ϕ, and let $\kappa = \dim U_0$. By Proposition 3.4, there exist $n \geqslant \kappa$, an invertible $E_0 \in \mathrm{M}_n(\mathbb{C})$, and a hermitian tuple

$$\widehat{X}_j = \begin{pmatrix} A_j^\phi & (Y_j^* \;\; 0)E_0^* \\ E_0\binom{Y_j}{0} & \widetilde{Z}_j \end{pmatrix}, \qquad 1 \leqslant j \leqslant \mathsf{g}, \tag{8.2}$$

on $S \oplus \mathbb{C}^n$ such that $\widehat{X} \in \bigcap_{r\in R} \operatorname{hdom} r$. Here

$$\mathfrak{X}_j|_S = \begin{pmatrix} A_j^\phi \\ Y_j \end{pmatrix}$$

is written relative to $U = S \oplus_\phi U_0$, and A_j^ϕ denotes the compression with respect to the ϕ-inner product.

Identifying $\mathbb{C}^n$ with $U_0 \oplus \mathbb{C}^{n-\kappa}$, define

$$J : U \longrightarrow S \oplus \mathbb{C}^n, \qquad J(s+u) = s \oplus E_0\begin{pmatrix} u \\ 0 \end{pmatrix}, \quad s \in S, \quad u \in U_0.$$

Since E_0 is invertible, J is injective. Moreover, for $s \in S$,

$$\widehat{X}_j Js = \begin{pmatrix} A_j^\phi s \\ E_0\binom{Y_j s}{0} \end{pmatrix} = J\mathfrak{X}_j s.$$

Thus,

$$\widehat{X}_j J|_S = J\mathfrak{X}_j|_S, \qquad 1 \leqslant j \leqslant \mathsf{g}.$$

Choose an orthonormal basis of S and extend it to an orthonormal basis of U, with respect to the given inner product (not the one induced by ϕ). Applying J to these

basis vectors and extending them gives a basis of $S \oplus \mathbb{C}^n$, which is not required to be orthonormal. The matrices T_j of $\widehat{X}_j$ in this basis are simultaneously similar to $\widehat{X}_j$ in (8.2) and have the form

$$T_j = \begin{pmatrix} A_j & * & * \\ Y_j & * & * \\ 0 & * & * \end{pmatrix}, \qquad A_j = A_j^*,$$

where

$$\mathfrak{X}_j|_S = \begin{pmatrix} A_j \\ Y_j \end{pmatrix}$$

is written relative to the orthogonal decomposition $U = S \oplus S^\perp$ for the given inner product.

By Lemma 3.5, there exist an invertible matrix E and a hermitian tuple Z such that

$$\widetilde{X}_j = \begin{pmatrix} A_j & (Y_j^* \ \ 0)E^* \\ E\binom{Y_j}{0} & Z_j \end{pmatrix}, \qquad 1 \leqslant j \leqslant \mathtt{g},$$

belongs to $\bigcap_{r\in R} \operatorname{hdom} r$.

For $\varepsilon_0, \varepsilon_1 > 0$, let

$$X_j(\varepsilon_0, \varepsilon_1) = \begin{pmatrix} A_j & \varepsilon_0(Y_j^* \ \ 0)E^* \\ \varepsilon_0 E\binom{Y_j}{0} & \varepsilon_1 Z_j \end{pmatrix}.$$

Since $X(1,1) = \widetilde{X}$ belongs to $\bigcap_{r\in R} \operatorname{hdom} r$, the product of the denominator determinants of the expressions in R, evaluated at $X(\varepsilon_0, \varepsilon_1)$, is a polynomial that does not vanish at $(1,1)$. Thus, it is not identically zero and cannot vanish entirely on any open subset of $\mathbb{R}^2$. Hence, for arbitrarily small positive $\varepsilon_0, \varepsilon_1$, we have

$$X(\varepsilon_0, \varepsilon_1) \in \bigcap_{r\in R} \operatorname{hdom} r.$$

Since $L(A) \succeq \delta I$ for some $\delta > 0$, we have

$$L(X(0,0)) = L(A) \oplus I \succeq \min\{\delta, 1\} I.$$

By continuity, $L(X(\varepsilon_0, \varepsilon_1))$ remains positive definite for sufficiently small $\varepsilon_0, \varepsilon_1 > 0$. Thus, we can choose these parameters so that

$$X := X(\varepsilon_0, \varepsilon_1) \in \mathcal{D}(L) \cap \bigcap_{r\in R} \operatorname{hdom} r.$$

Finally, $\mathfrak{X}_j(V_{\ell-2}^{1\times d}) \subseteq S$, so Y_j vanishes on $V_{\ell-2}^{1\times d}$. This proves (8.1). □

Theorem 8.3. *Let $\mathbb{F} = \sum_{a=1}^N C_a \otimes \mathtt{r}_a \in \mathrm{M}_d(\mathbb{C}) \otimes V_1$ be hermitian, where $C_a \in \mathrm{M}_d(\mathbb{C})$ and $r_a \in R$ represents $\mathtt{r}_a$ for each a. Set $\ell_* = 2\tau(R) + 1$. Then $\mathbb{F}(X) \succeq 0$ for every $X \in \mathcal{D}(L) \cap \bigcap_{r\in R} \operatorname{hdom} r$ if and only if $\mathbb{F} \in Q_{\ell_*}^{(d)}(L; \mathcal{R})$.*

Proof. The reverse implication follows from complete positivity. For the forward implication, suppose that $\mathbb{F} \notin Q_{\ell_*}^{(d)}(L; \mathcal{R})$. By Theorem 7.3, the cone $Q_{\ell_*}^{(d)}(L; \mathcal{R})$ is closed. It is also pointed. Indeed, Lemma 7.1, applied to $V_{2\ell_*+1}$, gives a hermitian tuple X such that $L(X) \succ 0$ and evaluation at X is injective on $V_{2\ell_*+1}$. If both $\mathbb{G}$ and $-\mathbb{G}$ belong to $Q_{\ell_*}^{(d)}(L; \mathcal{R})$, then

$$\mathbb{G}(X) \succeq 0 \quad \text{and} \quad -\mathbb{G}(X) \succeq 0.$$

Hence $\mathbb{G}(X) = 0$, and injectivity gives $\mathbb{G} = 0$.

Since $Q_{\ell_*}^{(d)}(L;\mathcal{R})$ is a closed pointed convex cone in a finite-dimensional real vector space, the Hahn–Banach separation theorem (see [Bar02, Theorem III.1.3]) gives a real linear functional $\lambda : \mathrm{M}_d(V_{2\ell_*+1})_{\mathrm{sa}} \to \mathbb{R}$ such that

$$\lambda(\mathbb{F}) < 0, \qquad \lambda(\mathbb{G}) > 0 \quad (0 \neq \mathbb{G} \in Q_{\ell_*}^{(d)}(L;\mathcal{R})).$$

We extend λ complex linearly to $\lambda : \mathrm{M}_d(V_{2\ell_*+1}) \to \mathbb{C}$.

Equip $U = V_{\ell_*}^{1\times d}$ with the inner product defined in Lemma 8.1. By that lemma, the compressed multiplication tuple A is hermitian and satisfies $L(A) \succeq \delta I$ for some $\delta > 0$. By Lemma 8.2, we obtain a finite-dimensional hermitian tuple $X \in \mathcal{D}(L) \cap \bigcap_{r\in R} \mathrm{hdom}\, r$ such that

$$X_j|_{V_{\ell_*-2}^{1\times d}} = \mathfrak{X}_j|_{V_{\ell_*-2}^{1\times d}}, \qquad j = 1,\dots,\mathrm{g}.$$

Since

$$2\tau(r_a) \leqslant 2\tau(R) \leqslant \ell_* = (\ell_* - 2) + 2,$$

Proposition 4.2, applied with $\ell = \ell_* - 2$ and $s = 1$, gives

$$r_a(X)\mathbb{v}_i = \mathbb{r}_a \mathbb{v}_i, \qquad i = 1,\dots,d,$$

where $\mathbb{v}_i \in V_0^{1\times d}$ is the row vector with 1 in the i-th position and zeros elsewhere.

Let $e_1,\dots,e_d$ be the standard basis of $\mathbb{C}^d$ and set $\xi = \sum_{i=1}^d e_i \otimes \mathbb{v}_i$. Using the preceding identities and the definition of the inner product, we obtain

$$\begin{aligned}
\langle \mathbb{F}(X)\xi, \xi\rangle &= \sum_{a=1}^{N} \sum_{i,j=1}^{d} (C_a)_{ij} \langle r_a(X)\mathbb{v}_j, \mathbb{v}_i\rangle \\
&= \sum_{a=1}^{N} \sum_{i,j=1}^{d} (C_a)_{ij} \lambda(\mathbb{v}_i^* \mathbb{r}_a \mathbb{v}_j) \\
&= \lambda\left(\sum_{a=1}^{N} C_a \otimes \mathbb{r}_a \right) \\
&= \lambda(\mathbb{F}) < 0.
\end{aligned}$$

This contradicts the assumed positivity of $\mathbb{F}(X)$. Therefore $\mathbb{F} \in Q_{\ell_*}^{(d)}(L;\mathcal{R})$. □

9. From finite-dimensional to operator-valued rational functions

We extend the finite-dimensional Positivstellensatz (Theorem 8.3) to rational functions with coefficients in $\mathcal{B}(\mathcal{K})$ (Theorem 9.1). The passage to infinite-dimensional coefficient spaces relies on the uniform truncation bound of Theorem 8.3 and the ultraweak closedness established in Theorem 7.3.

Theorem 9.1. *Let $\mathcal{K}$ be an infinite-dimensional Hilbert space, and let $\mathbb{F} = \sum_{a=1}^N C_a \otimes \mathbb{r}_a \in \mathcal{B}(\mathcal{K}) \otimes V_1$ be hermitian, where $C_a \in \mathcal{B}(\mathcal{K})$ and $r_a \in R$ represents $\mathbb{r}_a$. Set $\ell_* = 2\tau(R) + 1$. If $\mathbb{F}(X) \succeq 0$ for every $X \in \mathcal{D}(L) \cap \bigcap_{r\in R} \mathrm{hdom}\, r$, then $\mathbb{F} \in Q_{\ell_*}(\mathcal{K}, L;\mathcal{R})$. Equivalently, there exist $\mathbb{r}, \mathbb{q} \in \mathcal{B}(\mathcal{K}) \otimes V_{\ell_*}$ and $\pi \in \mathrm{UCP}(\mathcal{S}_L, \mathcal{B}(\mathcal{K}))$ such that*

$$\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*\pi(L)\mathbb{q}. \tag{9.1}$$

The converse also holds.

Proof. Let (P_α) be the net of finite-rank orthogonal projections on $\mathcal{K}$, ordered by inclusion of their ranges. Set $\mathcal{K}_\alpha = P_\alpha\mathcal{K}$ and $d_\alpha = \dim \mathcal{K}_\alpha$, and consider

$$\mathbb{F}_\alpha = \sum_{a=1}^{N} (P_\alpha C_a P_\alpha)|_{\mathcal{K}_\alpha} \otimes \mathbb{r}_a.$$

Since compression preserves positivity, $\mathbb{F}_\alpha(X) \succeq 0$ for every $X \in \mathcal{D}(L) \cap \bigcap_{r\in R} \operatorname{hdom} r$. The expressions r_a and the admissible set R remain unchanged, so the bound ℓ_* is the same for every α. By Theorem 8.3, $\mathbb{F}_\alpha \in Q_{\ell_*}^{(d_\alpha)}(L;\mathcal{R})$.

Let $m = \dim V_{\ell_*}$ and let $\iota_\alpha : \mathcal{K}_\alpha^m \to \mathcal{K}^m$ denote the natural inclusion. Choose Gram–CP data (G_α, ψ_α) for $\mathbb{F}_\alpha$ and define

$$\widetilde{G}_\alpha = \iota_\alpha G_\alpha \iota_\alpha^*, \qquad \widetilde{\psi}_\alpha(B) = \iota_\alpha \psi_\alpha(B) \iota_\alpha^*, \quad B \in \mathcal{S}_L.$$

Then $\widetilde{G}_\alpha \succeq 0$ and $\widetilde{\psi}_\alpha$ is completely positive. Moreover,

$$\sum_{a=1}^{N} P_\alpha C_a P_\alpha \otimes \mathbb{r}_a = \mathbf{e}_{\ell_*}^* \big(\widetilde{G}_\alpha + \widetilde{\psi}_\alpha(L)\big)\mathbf{e}_{\ell_*} \in Q_{\ell_*}(\mathcal{K}, L;\mathcal{R}).$$

Since $P_\alpha \to I_\mathcal{K}$ strongly, $P_\alpha C_a P_\alpha \to C_a$ ultraweakly for each a. Thus the expression on the left converges to $\mathbb{F}$ in the product ultraweak topology. By Theorem 7.3,

$$\mathbb{F} \in Q_{\ell_*}(\mathcal{K}, L;\mathcal{R}).$$

Write $\mathbb{F} = \mathbf{e}_{\ell_*}^*(G + \psi(L))\mathbf{e}_{\ell_*}$, where $G \succeq 0$ and ψ is completely positive. Set $\mathsf{R} = G^{1/2}$ and $D = \psi(I_\mathcal{H})^{1/2}$. As in the proof of Proposition 5.2, there is a UCP map $\widehat{\pi} : \mathcal{S}_L \to \mathcal{B}(\mathcal{K}^m)$ such that $\psi(B) = D\widehat{\pi}(B)D$ for every $B \in \mathcal{S}_L$.

Since $\mathcal{K}$ is infinite-dimensional, we may choose a unitary $J : \mathcal{K}^m \to \mathcal{K}$. Using the basis of V_{ℓ_*}, the block entries of $J\mathsf{R}$ and JD determine elements $\mathbb{r}, \mathbb{q} \in \mathcal{B}(\mathcal{K}) \otimes V_{\ell_*}$. Taking $\pi(B) = J\widehat{\pi}(B)J^*$, we obtain

$$\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*\pi(L)\mathbb{q},$$

as required. □

9.1. **Operator-valued noncommutative rational Hilbert's 17th problem.** Taking $L = I$ in the preceding Positivstellensätze yields operator-valued versions of noncommutative rational Hilbert's 17th problem. We obtain a single-square representation for infinite-dimensional coefficient spaces (Corollary 9.2) and a finite sum-of-squares representation in the matrix-valued case (Corollary 9.3).

Corollary 9.2. *Let $\mathcal{K}$ be an infinite-dimensional Hilbert space and let*

$$\mathbb{F} = \sum_{a=1}^{N} C_a \otimes \mathbb{r}_a \in \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\langle\langle x \rangle\rangle$$

be hermitian, where $r_a \in \mathfrak{R}_\mathbb{C}(x)$ represents $\mathbb{r}_a$. If $\mathbb{F}(X) \succeq 0$ for every

$$X \in \bigcap_{a=1}^{N} \operatorname{hdom} r_a,$$

then there exists $\mathbb{s} \in \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\langle\langle x \rangle\rangle$ such that

$$\mathbb{F} = \mathbb{s}^*\mathbb{s}.$$

Proof. Choose a finite admissible set R containing $r_1, \dots, r_N$. Applying Theorem 9.1 with $L = I$ gives

$$\mathbb{F} = \mathbb{r}^*\mathbb{r} + \mathbb{q}^*\mathbb{q}$$

for some $\mathbb{r}, \mathbb{q} \in \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\langle\!\langle x \rangle\!\rangle$. Since $\mathcal{K}$ is infinite-dimensional, there exist isometries $U, V \in \mathcal{B}(\mathcal{K})$ with $U^*V = 0$. Thus $\mathbb{s} = U\mathbb{r} + V\mathbb{q}$ satisfies $\mathbb{F} = \mathbb{s}^*\mathbb{s}$. □

Corollary 9.3. *Let $\mathcal{K}$ be a nonzero finite-dimensional Hilbert space and let*

$$\mathbb{F} = \sum_{a=1}^{N} C_a \otimes \mathbb{r}_a \in \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\langle\!\langle x \rangle\!\rangle$$

be hermitian, where $r_a \in \mathfrak{R}_{\mathbb{C}}(x)$ represents $\mathbb{r}_a$. If $\mathbb{F}(X) \succeq 0$ for every

$$X \in \bigcap_{a=1}^{N} \operatorname{hdom} r_a,$$

then there exist finitely many $\mathbb{s}_j \in \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\langle\!\langle x \rangle\!\rangle$ such that

$$\mathbb{F} = \sum_j \mathbb{s}_j^* \mathbb{s}_j. \tag{9.2}$$

Moreover, for any finite admissible set R containing $r_1, \dots, r_N$, one may take $\mathbb{s}_j \in \mathcal{B}(\mathcal{K}) \otimes V_{\ell_*}$, with at most $\dim V_{\ell_*}$ summands, where $\ell_* = 2\tau(R) + 1$.

Proof. Fix a finite admissible set R containing $r_1, \dots, r_N$. Identify $\mathcal{B}(\mathcal{K})$ with $\mathrm{M}_d(\mathbb{C})$, where $d = \dim \mathcal{K}$. Applying Theorem 8.3 with $L = I$ gives $\mathbb{F} \in Q_{\ell_*}^{(d)}(I; \mathcal{R})$. By Proposition 5.2,

$$\mathbb{F} = \mathbf{e}_{\ell_*}^* H \mathbf{e}_{\ell_*}, \qquad H \succeq 0.$$

Writing $H = T^*T$ and taking the block rows $\mathbb{s}_j$ of $T\mathbf{e}_{\ell_*}$ yields the required (9.2). □

9.2. **The commutative case.** The operator-valued analog of Hilbert's 17th problem fails in the commutative setting, even for polynomials.

Indeed, let $\mathcal{P}$ be the cone of nonnegative ternary sextic forms, and choose a sequence (p_n) dense in its unit-norm section. Then

$$F = \operatorname{diag}(p_1, p_2, \dots) \in \mathcal{B}(\ell^2) \otimes \mathbb{C}[x_1, x_2, x_3]$$

is positive semidefinite everywhere. If $F = \sum_j q_j^* q_j$ for finitely many $q_j \in \mathcal{B}(\ell^2) \otimes \mathbb{C}(x_1, x_2, x_3)$, clearing their finitely many scalar denominators gives a nonzero $h \in \mathbb{R}[x_1, x_2, x_3]$ such that

$$h^2 p_n \in \sum \mathbb{R}[x_1, x_2, x_3]^2 \qquad (n \geqslant 1).$$

By density and closedness of the sum-of-squares cone in fixed degree, the same holds for every $p \in \mathcal{P}$. But this contradicts Reznick's theorem on the absence of uniform denominators [Rez05].

In contrast, the matrix-valued commutative Hilbert's 17th problem has an affirmative answer. For every d, a hermitian matrix $F \in \mathrm{M}_d(\mathbb{C}(x_1, \dots, x_{\mathtt{g}}))$ that is positive semidefinite wherever defined on $\mathbb{R}^{\mathtt{g}}$ is a finite sum of hermitian squares of rational matrices. The real symmetric version was established by Gondard and Ribenboim [GR74] and independently by Procesi and Schacher [PS76]. The complex hermitian version is due to Djoković

[Djo76]; see also the constructive proof of Hillar and Nie [HN08]. Indeed, rational congruence diagonalization reduces the assertion to Artin's scalar theorem, applied to the nonnegative diagonal entries.

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