FACTORIZATION OF NONCOMMUTATIVE POLYNOMIALS AND NULLSTELLENSÄTZE FOR THE FREE ALGEBRA

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Abstract. This article gives a class of Nullstellensätze for noncommutative polynomials. The singularity set of a noncommutative polynomial $f = f(x_1, \dots, x_g)$ is $\mathscr{Z}(f) = (\mathscr{Z}_n(f))_n$, where $\mathscr{Z}_n(f) = \{X \in \mathrm{M}_n(\mathbb{C})^g \colon \det f(X) = 0\}$. The first main theorem of this article shows that the irreducible factors of f are in a natural bijective correspondence with irreducible components of $\mathscr{Z}_n(f)$ for every sufficiently large n.

With each polynomial h in x and x^* one also associates its real singularity set $\mathscr{Z}^{\mathrm{re}}(h) = \{X \colon \det h(X, X^*) = 0\}$. A polynomial f which depends on x alone (no x^* variables) will be called analytic. The main Nullstellensatz proved here is as follows: for analytic f but for h dependent on possibly both x and x^*, the containment $\mathscr{Z}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ is equivalent to each factor of f being "stably associated" to a factor of h or of h^*.

For perspective, classical Hilbert type Nullstellensätze typically apply only to analytic polynomials f, h, while real Nullstellensätze typically require adjusting the functions by sums of squares of polynomials (sos). Since the above "algebraic certificate" does not involve a sos, it seems justified to think of this as the natural determinantal Hilbert Nullstellensatz. An earlier paper of the authors (Adv. Math. 331 (2018) 589–626) obtained such a theorem for special classes of analytic polynomials f and h. This paper requires few hypotheses and hopefully brings this type of Nullstellensatz to near final form.

Finally, the paper gives a Nullstellensatz for zeros $\mathcal{V}(f) = \{X \colon f(X, X^*) = 0\}$ of a hermitian polynomial f, leading to a strong Positivstellensatz for quadratic free semialgebraic sets by the use of a slack variable.

1. Introduction

Hilbert's Nullstellensatz is a fundamental result in classical algebraic geometry describing polynomials vanishing on a complex algebraic variety. It has been generalized or extended to various settings, including many noncommutative ones. For instance, Amitsur's Nullstellensatz [Ami57] (see also [Pro66]) describes noncommutative polynomials vanishing on the common vanishing set of a given finite set of polynomials in a full matrix algebra. The papers [BK09, KŠ14] discuss vanishing traces of noncommutative polynomials, Cimprič [Cim19] considers Nullstellensatz questions involving polynomial partial differential operators, Salomon, Shalit and Shamovich [SSS18] give free noncommutative analytic Nullstellensätze, Reichstein and Vonessen [RV07] develop a framework for such questions in the context of rings with polynomial identities (see also [vOV81]), etc.

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In a slightly different direction, the real Nullstellensatz and Positivstellensatz are pillars of real algebraic geometry [BCR98]. These too have seen a plethora of noncommutative extensions. For example, the articles [CHMN13, Oza16, Scm09] develop general frameworks, and these ideas yield various applications, see e.g. Positivstellensätze on quantum graphs [NT15], isometries [HMP04], convex semialgebraic sets [HKM12], and algebraic approaches to Connes' embedding conjecture [Oza13].

In this article we prove new complex and real Nullstellensätze for the free algebra and provide a geometric interpretation for factorization of noncommutative polynomials [Coh06, BS15, BHL17]. The motivation for this work comes from the rapidly emerging areas of free analysis [AM16, K-VV14, SSS18, DK+] and free real algebraic geometry [Oza13, NT15, HKMV+] that study function theory on (semi)algebraic sets in the space of matrix tuples of all sizes. For noncommutative polynomials f there are three major types of zeros. These are hard zeros $\{X\colon f(X)=0\}$ [Ami57, SSS18], directional zeros $\mathscr{Z}_{\rm dir}(f)=\{(X,v)\colon f(X)v=0\}$ [HM04, HMP07], and determinantal zeros $\mathscr{Z}(f)=\{X\colon \det f(X)=0\}$ [KV17, HKV18]. For each of these one hopes there will be a Hilbert type Nullstellensatz (treating inclusion of zero sets) and a real Nullstellensatz (treating "real" points in zero sets). For directional zeros, reasonably satisfying theorems along these lines exist; see [HMP07] for a Nullstellensatz and [CHMN13] for a real Nullstellensatz. This article concerns determinantal zeros in Sections 2 and 3, and hard zeros in Section 4.

Our algebraic certificates are a counterpart to Hilbert's Nullstellensatz, and such theorems often carry direct applications to, say, semidefinite optimization and control theory [SIG98, BGM06]. Furthermore, we hope these results may be of interest to researchers in noncommutative algebra, matrix theory or polynomial identities.

1.1. **Main results.** Let $x=(x_1,\dots,x_g)$ be freely noncommuting variables. Elements of
$$\mathbb{C}<x>^{\delta\times\delta}=\mathrm{M}_\delta(\mathbb{C})\otimes\mathbb{C}<x>$$
are **noncommutative matrix polynomials**. If $f=\sum_w f_w w\in\mathbb{C}<x>^{\delta\times\delta}$ and $X\in\mathrm{M}_n(\mathbb{C})^g$, then $f(X)=\sum_w f_w\otimes w(X)\in\mathrm{M}_{\delta n}(\mathbb{C})$ denotes the evaluation of f at X.

1.1.1. *Complex Nullstellensätze.* Our first main result gives a geometric interpretation of irreducibility in free algebras. We say that $f\in\mathbb{C}<x>^{\delta\times\delta}$ is **full** [Coh06, Section 0.1] if it cannot be factored as $f=f_1f_2$ for $f_1\in\mathbb{C}<x>^{\delta\times\varepsilon}$ and $f_2\in\mathbb{C}<x>^{\varepsilon\times\delta}$ with $\varepsilon<\delta$. In the special case $\delta=1$, a polynomial $f\in\mathbb{C}<x>$ is full if and only if $f\neq0$. Following [Coh06] we call f an **atom** if f is not invertible in $\mathbb{C}<x>^{\delta\times\delta}$ and f cannot be written as a product of non-invertible elements in $\mathbb{C}<x>^{\delta\times\delta}$. In particular, every atom is full. Every full matrix admits a complete factorization into atoms [Coh06, Proposition 3.2.9].

Theorem 2.9. *A matrix polynomial f is an atom if and only if $\det f|_{\mathrm{M}_n(\mathbb{C})^g}$ is an irreducible polynomial for all but finitely many $n\in\mathbb{N}$.*

This theorem leads to a geometric description of factorization in free algebras. The geometric objects we use are free singularity sets. For $f\in\mathbb{C}<x>^{\delta\times\delta}$ let
$$\mathscr{Z}(f)=\bigcup_n\mathscr{Z}_n(f),\qquad \mathscr{Z}_n(f)=\{X\in\mathrm{M}_n(\mathbb{C})^g\colon\ \det f(X)=0\}$$
be its **free locus**. In [HKV18, Theorem 4.3] it was shown that components of $\mathscr{Z}(f)$ correspond to a factorization of f in $\mathbb{C}<x>^{\delta\times\delta}$ *if $f(0)$ is an invertible matrix.* Theorem

2.12 below, a free analog of Hilbert's Nullstellensatz, disposes of this assumption. Its proof is based on the aforementioned special case and novel techniques involving point-centered ampliations and representation theory.

To deal with the non-uniqueness of factorization in free algebras, Cohn introduced stable associativity [Coh06, Section 0.5]. Polynomials $f_1 \in \mathbb{C}\mathord{<}x\mathord{>}^{\delta_1\times\delta_1}$ and $f_2 \in \mathbb{C}\mathord{<}x\mathord{>}^{\delta_2\times\delta_2}$ are **stably associated** if there exist $\varepsilon_1, \varepsilon_2 \in \mathbb{N}$ with $\delta_1+\varepsilon_1 = \delta_2+\varepsilon_2$ and invertible $P, Q \in \mathbb{C}\mathord{<}x\mathord{>}^{(\delta_1+\varepsilon_1)\times(\delta_1+\varepsilon_1)}$ such that

$$f_2 \oplus I_{e_2} = P(f_1 \oplus I_{e_1})Q.$$

Actually, if f_1 and f_2 are stably associated, one can always choose $\varepsilon_1 = \delta_2$ and $\varepsilon_2 = \delta_1$ [Coh06, Theorem 0.5.3]. In particular, for $f_1, f_2 \in \mathbb{C}\mathord{<}x\mathord{>}$ stable associativity becomes a question about 2×2 matrices over $\mathbb{C}\mathord{<}x\mathord{>}$. There is a straightforward procedure to generate all stably associated pairs, see [Coh06, Section 2.7].

Theorem 2.12 (Singulärstellensatz)**.** *Let $f_1, \dots, f_s, h$ be full matrix polynomials. Then $\bigcap_j \mathscr{Z}(f_j) \subseteq \mathscr{Z}(h)$ if and only if for some $1 \le j \le s$, every atomic factor of f_j is stably associated to a factor of h.*

See Theorem 2.12 and Proposition 2.14 for the proof.

1.1.2. *Real Nullstellensätze.* We next turn our attention to the "real" setting with an involution. Let $x^* = (x_1^*, \dots, x_g^*)$ be formal adjoints to x. The map $x_j \mapsto x_j^*$ extends to a unique involution $*$ on

$$\mathbb{C}\mathord{<}x, x^*\mathord{>}^{\delta\times\delta} = \mathrm{M}_\delta(\mathbb{C}) \otimes \mathbb{C}\mathord{<}x, x^*\mathord{>}$$

restricting to the conjugate transpose on $\mathrm{M}_\delta(\mathbb{C})$. For $f \in \mathbb{C}\mathord{<}x, x^*\mathord{>}^{\delta\times\delta}$ we define its **real free locus** and the **real free zero set**:

$$\mathscr{Z}^{\mathrm{re}}(f) = \bigcup_{n\in\mathbb{N}} \mathscr{Z}_n^{\mathrm{re}}(f), \qquad \mathscr{Z}_n^{\mathrm{re}}(f) = \{X \in \mathrm{M}_n(\mathbb{C})^g \colon \det f(X, X^*) = 0\},$$
$$\mathcal{V}^{\mathrm{re}}(f) = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n^{\mathrm{re}}(f), \qquad \mathcal{V}_n^{\mathrm{re}}(f) = \{X \in \mathrm{M}_n(\mathbb{C})^g \colon f(X, X^*) = 0\}.$$

A matrix polynomial f depending on x but not on x^* is called **analytic**. For such an f we have $\mathscr{Z}^{\mathrm{re}}(f) = \mathscr{Z}(f)$.

Theorem 3.4 (Analytic Singulärstellensatz)**.** *Let $f_1, \dots, f_s$ be analytic atoms in x and h a full matrix polynomial. Then $\bigcap_j \mathscr{Z}^{\mathrm{re}}(f_j) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ if and only if there is j such that f_j or f_j^* is stably associated to a factor of h.*

A straightforward extension of Theorem 3.4 where both f, h are allowed to contain x^* fails even if $f = f^*$ is hermitian, see Example 3.11. A natural class of f for which the conclusion does hold are unsignatured matrix polynomials f. A hermitian polynomial $f = f^*$ is **unsignatured** if there exist $n \in \mathbb{N}$ and $X, Y \in \mathrm{M}_n(\mathbb{C})^g$ such that $f(X, X^*), f(Y, Y^*)$ are invertible and have different signatures.

Theorem 3.9 (Hermitian Singulärstellensatz)**.** *Let h be a full matrix polynomial and let f be an unsignatured atom. Then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ if and only if f is stably associated to a factor of h.*

A final main result is a Nullstellensatz for real free zero sets of polynomials with a distinguished quadratic term. As with the unsignatured hypothesis in Theorem 3.9 for real free loci, this was done under a certain definiteness assumption.

Theorem 4.5. *For a nonconstant hermitian* $f \in \mathbb{C}{<}x, x^*{>}$ *assume* $\{f \succ 0\} \neq \emptyset$. *Let* $h \in \mathbb{C}{<}x, x^*, y, y^*{>}$. *Then* $\mathcal{V}^{\mathrm{re}}(f - y^*y) \subseteq \mathcal{V}^{\mathrm{re}}(h)$ *if and only if* $h \in (f - y^*y)$.

As a consequence we obtain a necessary and sufficient Positivstellensatz for hereditary quadratic polynomials, by adding a slack variable.

Corollary 4.6. *Let* $f \in \mathbb{C}{<}x, x^*{>}$ *be a nonconstant hermitian hereditary quadratic polynomial with* $\{f \succ 0\} \neq \emptyset$, *and let* y *be an auxiliary variable. If* $h \in \mathbb{C}{<}x, x^*{>}$, *then* $h|_{\{f \succeq 0\}} \succeq 0$ *if and only if*

$$h = f_0 + \sum_j f_j^* f_j$$

for some $f_j \in \mathbb{C}{<}x, x^*, y, y^*{>}$ *with* $f_0 \in (f - y^*y)$.

While previously known necessary and sufficient Positivstellensätze on free semialgebraic sets do not require a slack variable, they only hold if the underlying free semialgebraic set is either convex with a nonempty interior [HKM12] or given by quadratic equations, such as spherical isometries or tuples of unitaries [KVV17]. On the other hand, Corollary 4.6 is an example of a Positivstellensatz on a (possibly) non-convex free semialgebraic set with a nonempty interior.

1.1.3. *Linear Gleichstellensatz.* An affine matrix polynomial is traditionally called a linear pencil. It is indecomposable if it cannot be put in block triangular form with a left and right basis change, cf. Definition 2.5. One can effectively apply the above Nullstellensätze to indecomposable linear pencils L, M to get roughly: $M = PLQ$ for some $P, Q \in \mathrm{GL}_d(\mathbb{C})$ if and only if the the free loci of L and M coincide. This is true for complex zeros (Theorem 2.11), real zeros (Theorem 3.6) and in the context of the Hermitian Nullstellensatz, which requires extra conditions on zeros (Theorem 3.6).

1.1.4. *Zero sets over the reals.* When dealing with real matrix polynomials, it suffices to consider only their evaluations at tuples of real matrices. Namely, for each $n \in \mathbb{N}$ we have a $*$-embedding $\iota : \mathrm{M}_n(\mathbb{C}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{R})$ induced by $\alpha + \beta i \mapsto \left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right)$. Note that $\iota(X)$ is unitarily equivalent to $X \oplus \overline{X}$ for each $X \in \mathrm{M}_n(\mathbb{C})^g$, where $\overline{X}$ is the entry-wise complex conjugate of X. Hence if $f \in \mathbb{R}{<}x, x^*{>}^{\delta \times \delta}$, then $f(X, X^*)$ is singular (resp. zero) if and only if $f(\iota(X), \iota(X)^{\mathrm{t}})$ is singular (resp. zero). Therefore one can replace free loci and free zero sets in the above theorems with their counterparts over $\mathbb{R}$ when applied to real polynomials, which is usually the preferred setting in control theory and optimization.

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2. Factorization in a free algebra and free loci

This section has two main results. In Theorem 2.9 we prove that a matrix polynomial f is an atom if and only if $\mathscr{Z}_n(f)$ is eventually a reduced irreducible hypersurface. Theorem 2.12 shows that f divides h in the sense that every atomic factor of f is stably associated to a factor of h if and only if $\mathscr{Z}(f) \subseteq \mathscr{Z}(h)$. These results are far-reaching generalizations of [HKV18, Theorems A, B] to matrix polynomials that eliminate the assumption of $f(0)$ being invertible. The proofs here rely on representation theory and point-centered ampliations, whereas [HKV18] used invariant theory. For the sake of convenience and later sections we use $\mathbb{C}$ as the base field, but all proofs work for an arbitrary algebraically closed field of characteristic 0.

For $n \in \mathbb{N}$ let $\Omega^n = (\Omega^n_1, \dots, \Omega^n_g)$ be a tuple of $n \times n$ generic matrices. That is, $\Omega^n_j = (\omega_{j\imath\jmath})_{\imath\jmath}$, where $\omega = (\omega_{j\imath\jmath} : \imath, \jmath, j)$ are independent commuting variables. We view the entries of Ω^n as the coordinates of the affine space $\mathrm{M}_n(\mathbb{C})^g$.

By [Coh06, Theorem 5.8.3], a full matrix $f \in \mathbb{C}\langle x\rangle^{\delta\times\delta}$ is stably associated to a linear pencil $L = A_0 + A_1x_1 + \cdots A_gx_g$ of size d such that

$$\begin{pmatrix} A_1 & \cdots & A_g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_1 \\ \vdots \\ A_g \end{pmatrix}$$

have full rank. We call such L an **epic**[1] **pencil**. We also say that L is a **linearization** of f. By the definition of stable associativity there is $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\det f(\Omega^n) = \alpha^n \det L(\Omega^n)$ for all $n \in \mathbb{N}$. Also, f is full if and only if L is full by [Coh06, Theorem 7.5.13]. Furthermore, f is an atom if and only if L is an atom by [Coh06, Proposition 0.5.2, Corollary 0.5.5 and Proposition 3.2.1].

Example 2.1. If $f = x_1x_2 - x_2x_1$, then one can check that the pencil

$$L = A_0 + A_1x_1 + A_2x_2 = \begin{pmatrix} -1 & 0 & x_2 \\ 0 & -1 & x_1 \\ x_1 & -x_2 & 0 \end{pmatrix}$$

is a linearization of f. While L is epic, no linear combination of A_0, A_1, A_2 is invertible, which corresponds to f vanishing on $\mathbb{C}^2$.

We next record facts about full and invertible matrices over $\mathbb{C}\langle x\rangle^{\delta\times\delta}$ that are scattered across the existing literature.

Lemma 2.2. *For $f \in \mathbb{C}\langle x\rangle^{\delta\times\delta}$ the following are equivalent:*

(i) *f is full;*
(ii) *there are $n \in \mathbb{N}$ and $X \in \mathrm{M}_n(\mathbb{C})^g$ such that $\det f(X) \neq 0$;*
(iii) *there exists $n_0 \in \mathbb{N}$ such that $\det f(\Omega^n) \neq 0$ for every $n \geq n_0$.*

Furthermore, a full f is not invertible in $\mathbb{C}\langle x\rangle^{\delta\times\delta}$ if and only if there exists $n_0 \in \mathbb{N}$ such that $\det f(\Omega^n)$ is nonconstant for every $n \geq n_0$.

Proof. By [Coh06, Corollary 7.5.14], f is full if and only if f is invertible over the free skew field of noncommutative rational functions $\mathbb{C}\langle\!\!\langle x\rangle\!\!\rangle$ (see Subsection 4.1 for further information about this skew field). The equivalence of (i) and (ii) now follows from the construction of $\mathbb{C}\langle\!\!\langle x\rangle\!\!\rangle$ described in [K-VV12, Section 2]. On the other hand, (iii) is equivalent to (ii) by linearization and [DM17, Proposition 2.10].

If f is invertible in $\mathbb{C}\langle x\rangle^{\delta\times\delta}$, then $\det f(\Omega^n) \det f^{-1}(\Omega^n) = 1$ is a product of polynomials, so $\det f(\Omega^n)$ is a nonzero constant. If f is not invertible, then either f is not

[1] In [Coh06] such L is called *monic*, which we avoid since monic pencils in control theory and convexity usually refer to pencils with $A_0 = I$.

full or f has an atomic factor h. If $h(0)=0$, then $\det h(\Omega^n)$ is not constant for large enough n by the first part since h is full. If $h(0)\neq 0$, then there exists $n_0\in\mathbb{N}$ such that $h(\Omega^n)$ is nonconstant for all $n\geq n_0$ by [HKV18, Theorem 4.3]. □

2.1. **Point-centered ampliations.** Fix $X\in \mathrm{M}_n(\mathbb{C})^g$ and let

$$y=(y_{j\imath\jmath}\colon 1\leq j\leq g, 1\leq \imath,\jmath\leq n)$$

be gn^2 freely noncommuting variables. For $f\in \mathbb{C}\!<\!x\!>^{\delta\times\delta}$ let

$$f^X=f\left(X_1+(y_{1\imath\jmath})_{\imath,\jmath},\dots,X_g+(y_{g\imath\jmath})_{\imath,\jmath}\right)\in \mathbb{C}\!<\!y\!>^{\delta n\times\delta n}$$

be its **point-centered ampliation** at X. In particular, if $L=A_0+\sum_{j>0}A_jx_j$ and $0_n\in\mathrm{M}_n(\mathbb{C})^g$ is the zero tuple, then L^{0_n} is up to a canonical shuffle equal to

$$(I_n\otimes A_0)+\sum_{j=1}^n\sum_{\imath,\jmath=1}^n (E_{\imath\jmath}\otimes A_j)y_{j\imath\jmath}, \tag{2.1}$$

where $\otimes$ is Kronecker's product and $E_{\imath\jmath}\in\mathrm{M}_n(\mathbb{C})$ are the standard matrix units. For applications of related ideas to noncommutative rational functions see [Vol18, PV].

In this subsection we prove that point ampliations preserve atoms. First we require two technical lemmas.

Lemma 2.3. *For sets $S_1,\dots,S_n$ we have*

$$\sum_{i=1}^n\left(|S_i|+\left|S_i\setminus\bigcup_{k\neq i}S_k\right|\right)\geq 2\left|\bigcup_{i=1}^n S_i\right|.$$

Proof. Follows by induction on n using

$$|S_1\cap S_2|+\left|S_1\setminus\bigcup_{k\neq 1}S_k\right|+\left|S_2\setminus\bigcup_{k\neq 2}S_k\right|\geq\left|(S_1\cup S_2)\setminus\bigcup_{k\neq 1,2}S_k\right|.$$ □

Lemma 2.4. *Let $A_0,\dots,A_g\in\mathrm{M}_d(\mathbb{C})$ and $n,d'd''\in\mathbb{N}$ satisfy $d'+d''=nd$. Assume there exist $P_i\in\mathbb{C}^{d'\times d}$, and $Q_i\in\mathbb{C}^{d\times d''}$ for $i=1,\dots,n$ such that*

$$\begin{pmatrix}P_1 & \cdots & P_n\end{pmatrix}\qquad \text{and}\qquad \begin{pmatrix}Q_1\\ \vdots\\ Q_n\end{pmatrix}$$

have full rank and

$$\begin{aligned}&\sum_i P_iA_0Q_i=0\\ &P_iA_jQ_k=0\qquad \text{for all}\quad 1\leq i,k\leq n,\ j>0.\end{aligned}\tag{2.2}$$

Then there exist $U\in\mathbb{C}^{e'\times d}$, $V\in\mathbb{C}^{d\times e''}$ of full rank for some $e',e''\in\mathbb{N}$ satisfying $e'+e''=d$ such that

$$UA_jV=0\qquad \text{for all}\quad j\geq 0. \tag{2.3}$$

Proof. Since the statement is trivial for $n=1$, let $n\geq 2$. By assumption we have $\mathbb{C}^{d'}=\sum_i \operatorname{ran}P_i$. For $1\leq i\leq n$ let $\widehat{P}_i\in\mathrm{M}_{d'}(\mathbb{C})$ be the projection onto the complement of $\operatorname{ran}P_i\cap\sum_{k\neq i}\operatorname{ran}P_k$ in $\operatorname{ran}P_i$ along $\sum_{k\neq i}\operatorname{ran}P_k$. Analogously we define $\widehat{Q}_i\in\mathrm{M}_{d''}(\mathbb{C})$ with respect to $\operatorname{ran}Q_i^{\mathrm{t}}$.

By Lemma 2.3 we have

$$\sum_i (\operatorname{rk} P_i + \operatorname{rk} \widehat{P}_i) \geq 2\dim\left(\sum_i \operatorname{ran} P_i\right) = 2d',$$
$$\sum_i (\operatorname{rk} Q_i + \operatorname{rk} \widehat{Q}_i) \geq 2\dim\left(\sum_i \operatorname{ran} Q_i^{\mathrm{t}}\right) = 2d''.$$

Let $m_i = \operatorname{rk} \widehat{P}_i + \operatorname{rk} Q_i$ and $n_i = \operatorname{rk} P_i + \operatorname{rk} \widehat{Q}_i$. Then

$$\sum_i m_i + \sum_i n_i = \sum_i (\operatorname{rk} P_i + \operatorname{rk} \widehat{P}_i + \operatorname{rk} Q_i + \operatorname{rk} \widehat{Q}_i) \geq 2d' + 2d'' = 2nd.$$

Therefore there exists $1 \leq i \leq n$ such that $m_i \geq d$ or $n_i \geq d$.

Suppose $m_1 \geq d$ (the other cases are treated analogously). Then (2.2) implies

$$\widehat{P_1} P_1 A_j Q_1 = 0 \qquad \text{for all} \quad j \geq 0.$$

Since $\operatorname{rk}(\widehat{P_1} P_1) = \operatorname{rk} \widehat{P_1}$, there exist $e' \leq \operatorname{rk} \widehat{P_1}$ and $e'' \leq \operatorname{rk} Q_1$ satisfying $e' + e'' = d$. By choosing e' linearly independent rows of $\widehat{P_1} P_1$ and e'' linearly independent columns of Q_1 we obtain $U \in \mathbb{C}^{e' \times d}$, $V \in \mathbb{C}^{d \times e''}$ of full rank such that (2.3) holds. □

Definition 2.5. A linear pencil L of size d is **indecomposable** if there are no $P, Q \in \mathrm{GL}_d(\mathbb{C})$ such that

$$PLQ = \begin{pmatrix} \star & 0 \\ \star & \star \end{pmatrix},$$

where the zero block is of size $d' \times d''$ with $d' + d'' = d$.

Remark 2.6. When restricted to monic pencils ($L(0) = I$), Definition 2.5 coincides with the one in [HKMV+] (while in [HKV18] such pencils were called irreducible): namely, a monic pencil is indecomposable if its coefficients admit no nontrivial common invariant subspace (or equivalently, generate the full matrix algebra). If $L(0)$ is invertible, the new definition can be reduced to the old one since L is indecomposable if and only if $L(0)^{-1}L$ is indecomposable, and the latter is a monic pencil.

We will frequently use [Coh06, Theorem 5.8.8] stating that an epic pencil is an atom if and only if it is indecomposable.

Proposition 2.7. *Let f be a matrix polynomial and $X \in \mathrm{M}_n(\mathbb{C})^g$. If f is an atom, then f^X is an atom.*

Proof. Let L be an epic pencil that is stably associated to f. Then L^X is stably associated to f^X for every $X \in \mathrm{M}_n(\mathbb{C})^g$ and $n \in \mathbb{N}$. Since stable associativity preserves atoms, L is an atom and it suffices to show that every point-centered ampliation of L is an atom. Furthermore, an affine change of variables preserves atoms, so it suffices to consider point-centered ampliations at $X = 0_n$ for $n \in \mathbb{N}$, where 0_n is the zero tuple in $\mathrm{M}_n(\mathbb{C})^g$.

Let $L = A_0 + \sum_j A_j x_j$ be an epic pencil of size d. Then L^{0_n} is an epic pencil of size nd and up to a basis change equal to (2.1). Suppose that L^{0_n} is not an atom. Since it is epic, it is not indecomposable by [Coh06, Theorem 5.8.8], so there exist $d', d'' \in \mathbb{N}$ satisfying $d' + d'' = nd$ and $P_0, Q_0 \in \mathrm{GL}_{nd}(\mathbb{C})$ such that

$$P_0 L^{0_n} Q_0 = \begin{pmatrix} \star & 0 \\ \star & \star \end{pmatrix}, \tag{2.4}$$

where the zero block is of size $d' \times d''$. Let $(P_1 \ \cdots \ P_n)$ be the first $d' \times (nd)$ block row of P_0, and let $(Q_1^{\mathrm{t}} \ \cdots \ Q_n^{\mathrm{t}})^{\mathrm{t}}$ be the last $(nd) \times d''$ block column of Q_0. Then the assumptions of Lemma 2.4 are satisfied by (2.4) and (2.1). So there exist $P, Q \in \mathrm{GL}_d(\mathbb{C})$ such that

$$PLQ = \begin{pmatrix} \star & 0 \\ \star & \star \end{pmatrix},$$

where the zero block is of size $e' \times e''$ and $e' + e'' = d$. Therefore L is not indecomposable and hence not an atom by [Coh06, Theorem 5.8.8]. □

2.2. **Irreducibility.** In this section we prove the main irreducibility result on free loci (Theorem 2.9). We start with an observation about the degrees of determinants of a matrix polynomial; for a related rank-stabilizing result see [DM17, Theorem 1.8].

Lemma 2.8. *Let f be a full non-invertible matrix polynomial. Then there exist $n_0, d \in \mathbb{N}$ such that $\deg\det f(\Omega^n) = dn$ for all $n \geq n_0$.*

Proof. For $n \in \mathbb{N}$ let $d_n = \deg\det f(\Omega^n)$. Since $f(\Omega^{n'} \oplus \Omega^{n''}) = f(\Omega^{n'}) \oplus f(\Omega^{n''})$, we have

$$d_{n'} + d_{n''} \leq d_{n'+n''} \tag{2.5}$$

for all $n', n'' \in \mathbb{N}$. By Lemma 2.2 there exist $m \in \mathbb{N}$ and $X^i \in \mathrm{M}_{m+i}(\mathbb{C})^g$ for $i = 0, 1$ such that $\det f^{X^i}(0) = \det f(X^i) \neq 0$. Then there exist $k_0, c_i \in \mathbb{N}$ such that

$$d_{k(m+i)} = \deg\det f^{X^i}(\Omega^k) = c_i k \tag{2.6}$$

for all $k \geq k_0$ by [HKV18, Lemma 3.1] and linearization. Choosing $k = m(m+1)k_0$ yields $mc_1 = (m+1)c_0$. Furthermore, for all $n \in \mathbb{N}$ we have

$$mk_0 d_n \leq d_{mk_0 n} = c_0 k_0 n$$

by (2.5) and (2.6). On the other hand, every $n \geq m(2k_0 + m - 1)$ can be written as $n = k_1 m + k_2(m+1)$ for some $k_1, k_2 \geq k_0$, so (2.5) implies

$$\begin{aligned} (m+1)c_0 n &= (m+1)c_0 k_1 m + mc_1 k_2(m+1) \\ &= m(m+1)(c_0 k_1 + c_1 k_2) \\ &= m(m+1)(d_{k_1 m} + d_{k_2(m+1)}) \\ &\leq m(m+1)d_n. \end{aligned}$$

Hence $d_n = \frac{c_0}{m} n$ for all $n \geq m(2k_0 + m - 1)$, and consequently $\frac{c_0}{m} \in \mathbb{N}$. □

The next irreducibility theorem is one of the central results in this paper. It is the pillar of the Hilbert type Nullstellensätze that follow in Sections 2 and 3.

Theorem 2.9. *Let f be a matrix polynomial. Then f is an atom if and only if there is $n_0 \in \mathbb{N}$ such that $\det f(\Omega^n)$ is an irreducible polynomial for every $n \geq n_0$.*

Proof. Assume f is not an atom. If f is not full, then $\det f(\Omega^n) = 0$ for all n. If f is invertible over $\mathbb{C}\langle x \rangle$, then $\det f(\Omega^n)$ is a nonzero constant for every n. If f is full and not invertible, then $f = f_1 f_2$ for some full non-invertible matrix polynomials f_1, f_2. Then $\det f(\Omega^n) = \det f_1(\Omega^n) \det f_2(\Omega^n)$ is a proper factorization for all sufficiently large n by Lemma 2.2. This proves one implication of the theorem.

Now let f be an atom. By Lemma 2.8 there exist $m, d \in \mathbb{N}$ such that $\deg\det f(\Omega^n) = dn$ for all $n \geq m$. In particular, there is $X \in \mathrm{M}_m(\mathbb{C})^g$ such that $\det f(X) \neq 0$. Then f^X is an atom by Proposition 2.7. Since $\det f^X(0) = \det f(X) \neq 0$, there exists $k_0 \in \mathbb{N}$

such $\det f^X(\Omega^k)$ is an irreducible polynomial for every $k \geq k_0$ by [HKV18, Theorem 4.3]. Since an affine change of variables does not affect irreducibility, $\det f^{0_m}(\Omega^k)$ is also irreducible for every $k \geq k_0$. By the definition of f^{0_m} we then conclude that $\det f(\Omega^{km})$ is irreducible for all $k \geq k_0$.

Now let $n \geq (\max\{2d, k_0\} + 1)m$. Then $n = km + r$ for $k \geq \max\{2d, k_0\}$ and $m \leq r < 2m$. Suppose that $\det f(\Omega^n) = pq$ for $p, q \in \mathbb{C}[\omega]$. By the choice of m in the previous paragraph there is $X \in \mathrm{M}_r(\mathbb{C})^g$ such that $\det f(X) \neq 0$. Since the polynomial $\det f(X \oplus \Omega^{km}) = \det f(X) \det f(\Omega^{km})$ is irreducible, we can without loss of generality assume that p evaluated at $X \oplus \Omega^{km}$ equals 1. Thus $\deg q \geq dkm$, and so $\deg p \leq dn - dkm = dr < 2dm$.

By [HKV18, Lemma 2.1], p is given by a pure trace polynomial; that is, there is a formal polynomial t in traces of words over x,

$$t = \sum_{i=1}^{k} \alpha_i \prod_{j=1}^{\ell_i} \mathrm{tr}(w_{ij}), \tag{2.7}$$

such that the degree of t equals the degree of p and $p = t(\Omega^n)$. We also consider another pure trace polynomial

$$u = \sum_{i=1}^{k} \alpha_i \prod_{j=1}^{\ell_i} \big(\mathrm{tr}(w_{ij}) + \mathrm{tr}(w_{ij}(X))\big). \tag{2.8}$$

Note that $\deg u = \deg t$ and $u(\Omega^{(km)}) = p(X \oplus \Omega^{km}) = 1$. Hence $u - 1$ is a pure trace identity for $(km) \times (km)$ matrices which has degree $\deg p < 2dm \leq km + 1$. Therefore $u - 1$ is the zero polynomial by [Pro76, Theorem 4.5]. Since $\deg u = \deg t$, the trace polynomial t is also zero, so p is constant. Hence $\det f(\Omega^n)$ is irreducible for every $n \geq (\max\{2d, k_0\} + 1)m$. □

2.3. **Complex Nullstellensatz.** In this subsection we show that indecomposable pencils are determined by their free loci (Theorem 2.11), which then leads to the geometric reformulation of factorization in the free algebra (Theorem 2.12).

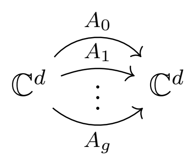
Consider the actions of $\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_d(\mathbb{C})$ and $\mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C})$ on $\mathrm{M}_d(\mathbb{C})^{g+1}$ given by $(P, Q) \cdot X = PXQ^{-1}$. The following fact is probably well-known to specialists in invariant theory. We include a proof for the sake of completeness.

Lemma 2.10. *Let* $L = A_0 + \sum_{j>0} A_j x_j$ *be of size* d. *If* L *is indecomposable, then the* $\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_d(\mathbb{C})$*-orbit of* $A = (A_0, \dots, A_g)$ *in* $\mathrm{M}_d(\mathbb{C})^{g+1}$ *is Zariski closed, and the* $\mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C})$*-stabilizer of* A *is* $\{(\alpha I, \alpha^{-1} I) \colon \alpha \in \mathbb{C} \setminus \{0\}\}$.

Proof. Since L is indecomposable, we have

$$\dim \left(\sum_{j=0}^{g} A_j V \right) - \dim V > 0$$

for every proper subspace V in $\mathbb{C}^d$. If we view A as a (d, d)-dimensional representation



of the $(g+1)$-Kronecker quiver, then the previous observation implies that A is $(1,-1)$-stable according to [Kin94, Definition 1.1 and Section 3]. By [Kin94, Proposition 3.1], stability of A as a quiver representation is equivalent to stability of A as a point in $\mathrm{M}_d(\mathbb{C})^{g+1}$ with the action of $\mathrm{GL}_d(\mathbb{C})\times\mathrm{GL}_d(\mathbb{C})$ according to [Kin94, Definition 2.1]. Note that $\Delta=\{(\alpha I,\alpha^{-1}I)\colon \alpha\in\mathbb{C}\setminus\{0\}\}$ are precisely elements in $\mathrm{GL}_d(\mathbb{C})\times\mathrm{GL}_d(\mathbb{C})$ that act trivially on $\mathrm{M}_d(\mathbb{C})^{g+1}$. Therefore stability of A implies

$$(\mathrm{GL}_d(\mathbb{C})\times\mathrm{GL}_d(\mathbb{C}))\cdot A=\dim(\mathrm{GL}_d(\mathbb{C})\times\mathrm{GL}_d(\mathbb{C}))/\Delta,$$

so the stabilizer of A equals Δ. Furthermore, since $\mathrm{SL}_d(\mathbb{C})\times\mathrm{SL}_d(\mathbb{C})$ is the commutator subgroup of $\mathrm{GL}_d(\mathbb{C})\times\mathrm{GL}_d(\mathbb{C})$, the $\mathrm{SL}_d(\mathbb{C})\times\mathrm{SL}_d(\mathbb{C})$-orbit of A in $\mathrm{M}_d(\mathbb{C})^{g+1}$ is Zariski closed by [Shm07, Theorem 1(i)⇒(iii)]. □

Theorem 2.11 (Linear Gleichstellensatz)**.** *Let L and M be indecomposable linear pencils of sizes d and e, respectively. Then $\mathscr{Z}(L)=\mathscr{Z}(M)$ if and only if $d=e$ and $M=PLQ$ for some $P,Q\in\mathrm{GL}_d(\mathbb{C})$.*

Proof. By Theorem 2.9 there exists $n_0\in\mathbb{N}$ such that $\det L(\Omega^n)$ and $\det M(\Omega^n)$ are irreducible for all $n\ge n_0$. By $\mathscr{Z}(L)=\mathscr{Z}(M)$ and irreducibility we see that $\det L(\Omega^n)$ and $\det M(\Omega^n)$ are equal up to a multiplicative constant for every $n\ge n_0$. Thus there exists $\alpha\in\mathbb{C}\setminus\{0\}$ such that $\det L(\Omega^n)=\alpha^n\det M(\Omega^n)$ for all $n\in\mathbb{N}$. After multiplying L by α^{-1} we can therefore assume that

$$\det L(\Omega^n)=\det M(\Omega^n)\qquad \text{for all } n\in\mathbb{N}. \tag{2.9}$$

Let $L=A_0+\sum_{j>0}A_jx_j$ and $M=B_0+\sum_{j>0}B_jx_j$. Then

$$\det\left(A_0\otimes Y+\sum_{j>0}A_j\otimes X_j\right)=(\det Y)^d\det L(Y^{-1}X) \tag{2.10}$$

for every $X\in\mathrm{M}_n(\mathbb{C})^g$ and $Y\in\mathrm{GL}_n(\mathbb{C})$, and similarly for M. By (2.9), the polynomials

$$(\det Y)^d\det L(Y^{-1}X),\qquad (\det Y)^e\det M(Y^{-1}X) \tag{2.11}$$

agree up to a factor of $\det Y$. However, since L is indecomposable, the left-hand side of (2.10) is an irreducible polynomial in X for large enough n. Analogous conclusion holds for M, so the polynomials in (2.11) are equal, and thus $d=e$. Consequently, (2.9) and (2.10) imply

$$\forall n\in\mathbb{N},\ \forall X\in\mathrm{M}_n(\mathbb{C})^{g+1}\colon\qquad \det\left(\sum_{j=0}^g A_j\otimes X_j\right)=\det\left(\sum_{j=0}^g B_j\otimes X_j\right). \tag{2.12}$$

Let us view $A=(A_0,\dots,A_g)$ and $B=(B_0,\dots,B_g)$ as elements of $\mathrm{M}_d(\mathbb{C})^{g+1}$ with the action of $\mathrm{SL}_d(\mathbb{C})\times\mathrm{SL}_d(\mathbb{C})$. Then $p(A)=p(B)$ for every $\mathrm{SL}_d(\mathbb{C})\times\mathrm{SL}_d(\mathbb{C})$-invariant polynomial function $p:\mathrm{M}_d(\mathbb{C})^{g+1}\to\mathbb{C}$ by (2.12) and Theorem [SvdB01, Theorem 2.3] or [DM17, Theorem 1.4]. Therefore the Zariski closures of $\mathrm{SL}_d(\mathbb{C})\times\mathrm{SL}_d(\mathbb{C})$-orbits of A and B coincide. But the orbits of A and B are closed by Lemma 2.10, so A and B lie in the same orbit. □

Theorem 2.12 (Singulärstellensatz)**.** *Let f and h be full matrix polynomials. Then $\mathscr{Z}(f)\subseteq\mathscr{Z}(h)$ if and only if every atomic factor of f is stably associated to a factor of h.*

Proof. It suffices to assume that f is an atom. Let $h = h_1 \dots h_\ell$ be a factorization of h into atoms.

($\Leftarrow$) If f is stably associated to h_i, then there is $\alpha \neq 0$ such that $\det f(\Omega^n) = \alpha^n \det h_i(\Omega^n)$ for all n, so $\mathscr{Z}(f) = \mathscr{Z}(h_i) \subseteq \mathscr{Z}(h)$.

($\Rightarrow$) Since $h_1, \dots, h_\ell$ are atoms, there exists $n_0 \in \mathbb{N}$ such that $\mathscr{Z}_n(f)$ and $\mathscr{Z}_n(h_i)$ for $1 \leq i \leq \ell$ are irreducible algebraic sets for every $n \geq n_0$. Since

$$\mathscr{Z}(f) \subseteq \mathscr{Z}(h) = \mathscr{Z}(h_1) \cup \cdots \cup \mathscr{Z}(h_\ell),$$

we conclude that for every $n \geq n_0$, $\mathscr{Z}_n(f) = \mathscr{Z}_n(h_i)$ for some i. Hence there exists i such that $\mathscr{Z}_n(f) = \mathscr{Z}_n(h_i)$ for infinitely many n. Since $\mathscr{Z}_k(p)$ embeds into $\mathscr{Z}_{km}(p)$ via $X \mapsto X^{\oplus m}$ for every $k, m \in \mathbb{N}$ and any matrix polynomial p, we have $\mathscr{Z}_n(f) = \mathscr{Z}_n(h_i)$ for all n. Let L and M be the epic pencils that are stably associated to f and h_i. Then L, M are atoms and thus indecomposable by [Coh06, Theorem 5.8.8]. Since $\mathscr{Z}(L) = \mathscr{Z}(M)$, L and M are stably associated by Theorem 2.11. Therefore f and h_i are stably associated. □

Corollary 2.13. *Let f be an atom and h a full matrix polynomial. Then $\mathscr{Z}(f) = \mathscr{Z}(h)$ if and only if h is a product of matrix polynomials each of which is stably associated to f.*

Finally, let us record the following observation about free loci, which implies the introductory version of Theorem 2.12 above, and will be used several times later in the text. While we usually think of free loci as analogs of hypersurfaces, their intersections do not behave as lower-dimensional varieties.

Proposition 2.14. *Let $f_1, \dots, f_s, h$ be matrix polynomials. If $\bigcap_j \mathscr{Z}(f_j) \subseteq \mathscr{Z}(h)$, then there exists $j \geq 1$ such that $\mathscr{Z}(f_j) \subseteq \mathscr{Z}(h)$.*

Proof. Suppose $\mathscr{Z}(f_j) \not\subseteq \mathscr{Z}(h)$ for $j \geq 2$. Hence there exist matrix tuples $X^2, \dots, X^s$ such that for $j \geq 2$,

$$\det f_j(X^j) = 0 \qquad \text{and} \qquad \det h(X^j) \neq 0.$$

If $\det f_1(X) = 0$ for some X, then

$$X \oplus \bigoplus_{j \geq 2} X^j \in \bigcap_{j=1}^{s} \mathscr{Z}(f_j)$$

and so $\det h(X) = 0$. Therefore $\mathscr{Z}(f_1) \subseteq \mathscr{Z}(h)$. □

3. Real Nullstellensätze

In this section we prove two new real Nullstellensätze for the free algebra. In Theorem 3.4 we give a geometric condition for an analytic (no x^* variables) matrix polynomial f to be a factor of an arbitrary matrix polynomial h. This result is under a natural assumption extended to arbitrary f in Theorem 3.9. The proofs rely on preceding results in this paper and real algebraic geometry applied to the real structure on matrix tuples.

3.1. **Real structure.** For $f \in \mathbb{C}\langle x, x^*\rangle^{\delta\times\delta}$ and $(X,Y) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$ let $f(X,Y)$ denote the involution-free evaluation of f at (X,Y) given by $x_j \mapsto X_j$ and $x_j^* \mapsto Y_j$, and let $f(X,X^*)$ denote the $*$-evaluation at X, where $X^* = (X_1^*,\dots,X_g^*)$.

Fix $n \in \mathbb{N}$. The map

$$\mathscr{J} : \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g \to \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g, \qquad (X,Y) \mapsto (Y^*, X^*) \tag{3.1}$$

is conjugate-linear and involutive. Thus $\mathscr{J}$ is a **real structure** on the complex space $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$. Let $\mathcal{W}$ be a complex algebraic set in $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$. If $\mathscr{J}$ preserves $\mathcal{W}$, then let $\mathcal{W}^{\mathrm{re}}$ denote the set of points in $\mathcal{W}$ fixed by $\mathscr{J}$,

$$\mathcal{W}^{\mathrm{re}} = \mathcal{W} \cap \bigcup_{n\in\mathbb{N}} \{(X,X^*) \colon X \in \mathrm{M}_n(\mathbb{C})^g\}.$$

Then $\mathcal{W}^{\mathrm{re}}$ is a real algebraic set, also called the set of **real points** on $\mathcal{W}$.

Proposition 3.1. *Let $\mathcal{W}$ be a complex algebraic set in $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$. If $\mathcal{W}$ is irreducible, $\mathscr{J}$ preserves $\mathcal{W}$ and $\mathcal{W}^{\mathrm{re}}$ contains a smooth point of $\mathcal{W}$, then $\mathcal{W}^{\mathrm{re}}$ is Zariski dense in $\mathcal{W}$.*

Proof. This is a special case of a more general statement about real points on a complex variety with a real structure, see e.g. [Bec82, Lemma 1.5] or [DE70, Theorem 4.9]. □

Recall the definition of the real free locus of $f \in \mathbb{C}\langle x, x^*\rangle^{\delta\times\delta}$ from the introduction. To derive results about $\mathscr{Z}^{\mathrm{re}}(f)$, we consider the (non-real) free locus of f throughout this section in an involution-free way; that is, we "forget" the involutive relation between the variables x and x^*, and thus

$$\mathscr{Z}_n(f) = \{(X,Y) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g : \det f(X,Y) = 0\}$$

as a complex algebraic set in $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$. If $f = f^*$ is hermitian, then $\mathscr{Z}_n(f)$ is preserved by the real structure $\mathscr{J}$, and the real points of the free locus of f are related to the real free locus of f as follows:

$$\mathscr{Z}_n(f)^{\mathrm{re}} = \{(X,X^*) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g \colon X \in \mathscr{Z}_n^{\mathrm{re}}(f)\}.$$

3.2. **Analytic Nullstellensatz.** Let $f \in \mathbb{C}\langle x\rangle^{\delta\times\delta} \subset \mathbb{C}\langle x, x^*\rangle^{\delta\times\delta}$; such polynomials are called **analytic**. Although f contains no x^*, it is convenient to view it as a matrix over $\mathbb{C}\langle x, x^*\rangle$, and thus $\mathscr{Z}(f) = \{(X,Y) : \det f(X) = 0\}$. Observe that the real structure $\mathscr{J}$ on $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$ preserves

$$\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*) = \{(X,Y) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g \colon \det f(X) = 0 = \det f^*(Y)\}$$

since $f(X^*) = f^*(X)^*$. The set of real points of this algebraic set is then

$$(\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*))^{\mathrm{re}} = \{(X,X^*) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g \colon X \in \mathscr{Z}_n^{\mathrm{re}}(f)\}.$$

Proposition 3.2. *Let $f \in \mathbb{C}\langle x\rangle^{\delta\times\delta}$ be an atom. There exists n_0 such that for every $n \ge n_0$, we have that $(\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*))^{\mathrm{re}}$ is Zariski dense in $\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*)$.*

The proof uses smooth points, hence involves derivatives and their properties. Let $\Upsilon^n = (\Upsilon_1^n,\dots,\Upsilon_g^n)$ be another tuple of $n\times n$ generic matrices. We view the entries of (Ω^n, Υ^n) as the coordinates of the affine space $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$.

Lemma 3.3. *For all $X \in \mathrm{M}_n(\mathbb{C})^g$,*

$$\left(\frac{\partial}{\partial \omega_{\imath\jmath j}} \det f(\Omega^n)\right)\bigg|_{\Omega^n=X} = \overline{\left(\frac{\partial}{\partial \upsilon_{\jmath\imath j}} \det f^*(\Upsilon^n)\right)\bigg|_{\Upsilon^n=X^*}}.$$

Proof. A consequence of the identity $\det f(X) = \overline{\det f^*(X^*)}$. □

Proof of Proposition 3.2. By Theorem 2.9 there exists n_0 such that $\det f(\Omega^n)$ is irreducible for all $n \geq n_0$. Now fix $n \geq n_0$. Since $\det f(\Omega^n)$ is irreducible, there exists $X \in \mathrm{M}_n(\mathbb{C})^g$ such that

$$(\mathrm{grad}_{\Omega^n} \det f(\Omega^n))|_{\Omega^n=X} \neq 0, \tag{3.2}$$

where grad_{Ω^n} denotes the gradient with respect to the gn^2 variables $\omega_{\jmath\imath j}$. By Lemma 3.3 we also have

$$(\mathrm{grad}_{\Upsilon^n} \det f^*(\Upsilon^n))|_{\Upsilon^n=X^*} \neq 0. \tag{3.3}$$

The algebraic set $\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*)$ is defined by $\det f(\Omega^n) = 0$ and $\det f^*(\Upsilon^n) = 0$, and the $2 \times (gn^2 + gn^2)$ Jacobian matrix of this pair has the form

$$\mathrm{J}(\Omega^n, \Upsilon^n) = \begin{pmatrix} \mathrm{grad}_{\Omega^n} \det f(\Omega^n) & 0 \\ 0 & \mathrm{grad}_{\Upsilon^n} \det f^*(\Upsilon^n) \end{pmatrix}.$$

Then $\mathrm{J}(X, X^*)$ has rank 2 by (3.2) and (3.3), so (X, X^*) is a smooth point of $\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*)$. Finally, $\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*)$ is irreducible since it is a product of two irreducible hypersurfaces in $\mathrm{M}_n(\mathbb{C})^g$,

$$\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*) = \{X\colon \det f(X) = 0\} \times \{Y\colon \det f^*(Y) = 0\}.$$

The statement then follows by Proposition 3.1. □

Theorem 3.4 (Analytic Singulärstellensatz)**.** *Let $f \in \mathbb{C}\langle x\rangle^{\delta\times\delta}$ be an atom and $h \in \mathbb{C}\langle x, x^*\rangle^{\varepsilon\times\varepsilon}$ a full matrix. Then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ if and only if f or f^* is stably associated to a factor of h.*

Proof. Only ($\Rightarrow$) is non-trivial. If $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$, then $(\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*))^{\mathrm{re}} \subseteq \mathscr{Z}_n(h)$ for all n. By Proposition 3.2, $(\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*))^{\mathrm{re}}$ is Zariski dense in $\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*)$ for all n large enough. Therefore $\mathscr{Z}_n(f) \cap \mathscr{Z}_n(f^*) \subseteq \mathscr{Z}_n(h)$ for all n large enough, and consequently for all n. Therefore $\mathscr{Z}(f) \subseteq \mathscr{Z}(h)$ or $\mathscr{Z}(f^*) \subseteq \mathscr{Z}(h)$ by Proposition 2.14, so the conclusion follows by Theorem 2.12. □

As a corollary we obtain the following somewhat unexpected statement.

Corollary 3.5. *Let $h \in \mathbb{C}\langle x, x^*\rangle^{\varepsilon\times\varepsilon}$ and $f_j \in \mathbb{C}\langle x\rangle^{\delta_j\times\delta_j}$ for $j = 1, \dots, s$ be full matrices. Then*

$$\mathscr{Z}^{\mathrm{re}}(f_1) \cap \cdots \cap \mathscr{Z}^{\mathrm{re}}(f_s) \subseteq \mathscr{Z}^{\mathrm{re}}(h) \tag{3.4}$$

if and only if there exists j such that each atomic factor of f_j is stably associated to a factor of either h or h^*.

Proof. If (3.4) holds, then as in the proof of Proposition 2.14 we see that $\mathscr{Z}^{\mathrm{re}}(f_j) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ for some j. The rest is an immediate consequence of Theorem 3.4. □

Restricting Theorem 3.4 to linear pencils yields the following Gleichstellensatz.

Corollary 3.6. *Let L, M be indecomposable linear pencils. If L is analytic, then $\mathscr{Z}^{\mathrm{re}}(L) = \mathscr{Z}^{\mathrm{re}}(M)$ if and only if L, M are of the same size d and there exist $P, Q \in \mathrm{GL}_d(\mathbb{C})$ such that $M = PLQ$ or $M = PL^*Q$.*

Proof. Combine Theorems 2.11 and 3.4. □

3.3. **Hermitian Nullstellensatz.** In real algebraic geometry, an ideal $J \subset \mathbb{R}[\omega]$ is called real if J consists precisely of polynomials vanishing on the real zero set of J. Theorem 3.9 below is inspired by the characterization of real principal ideals [BCR98, Theorem 4.5.1]. Namely, if $p \in \mathbb{R}[\omega]$ is irreducible, then (p) is a real ideal if and only if p changes sign. Recall that a hermitian matrix polynomial $f = f^*$ is called unsignatured if there exist $n \in \mathbb{N}$ and $X, Y \in \mathrm{M}_n(\mathbb{C})^g$ such that $f(X, X^*), f(Y, Y^*)$ are invertible and have different signatures.

Remark 3.7. If $f(X, X^*)$ is positive (resp. negative) definite for some $X \in \mathrm{M}_n(\mathbb{C})^g$, then f is unsignatured if and only if f (resp. $-f$) is not a sum of hermitian squares. There are also unsignatured polynomials that are never definite, for instance $f = x_1x_1^* - x_1^*x_1$ (because its trace is constantly 0). Another example of a non-unsignatured atom is

$$f = \begin{pmatrix} 1 + xx^* & x \\ x^* & -1 - x^*x \end{pmatrix}. \tag{3.5}$$

Proposition 3.8. *Let f be a hermitian polynomial, $n \in \mathbb{N}$ and $X, Y \in \mathrm{M}_n(\mathbb{C})^g$ such that $f(X, X^*)$ and $f(Y, Y^*)$ are invertible with different signatures. If $\mathscr{Z}_n(f)$ is irreducible, then $\mathscr{Z}_n(f)^{\mathrm{re}}$ is Zariski dense in $\mathscr{Z}_n(f)$.*

Proof. As $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$ is endowed with the real structure $\mathscr{J}$, we can view

$$\mathbb{A} = \{(Z, Z^*) \colon Z \in \mathrm{M}_n(\mathbb{C})^g\}$$

as the corresponding real affine space. There exists $\eta > 0$ such that for every $Z \in \mathrm{M}_n(\mathbb{C})^g$ with $\|Z - Y\| < \eta$, the matrix $f(Z, Z^*)$ is invertible with the same signature as $f(Y, Y^*)$. Let D be an open ball of radius η about Y, intersected by the affine subspace through Y that is perpendicular to $X - Y$, i.e.,

$$D = \left\{ (Z, Z^*) \in \mathrm{M}_n(\mathbb{C})^g \colon \|Z - Y\| < \eta \text{ and } \sum_j \mathrm{tr}((Z_j - Y_j)^*(X_j - Y_j)) = 0 \right\}.$$

Then D is a semialgebraic set in $\mathbb{A}$ of (real) dimension $gn^2 - 1$. Let $\mathcal{C}$ be the convex hull of $\{(X, X^*)\} \cup D$. If $(Z, Z^*) \in D$, then $f(Z, Z^*)$ and $f(X, X^*)$ have different signatures, so $\mathscr{Z}_n(f)^{\mathrm{re}}$ intersects the interior of the line segment between (X, X^*) and (Z, Z^*). Moreover, by the choice of D, every line through (X, X^*) intersects D at most once. Therefore we have a surjective map

$$\varphi : \mathscr{Z}_n(f)^{\mathrm{re}} \cap \mathcal{C} \to D$$

given by projections onto D along the lines through (X, X^*). Then φ is clearly semialgebraic, so $\dim(\mathscr{Z}_n(f)^{\mathrm{re}} \cap \mathcal{C}) = gn^2 - 1$ by [BCR98, Theorem 2.8.8]. Therefore its Zariski closure in $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$ (with real structure $\mathscr{J}$) is a hypersurface by [BCR98, Proposition 2.8.2]. Since the latter is contained in the irreducible $\mathscr{Z}_n(f)$, we conclude that $\mathscr{Z}_n(f)^{\mathrm{re}} \cap \mathcal{C}$ is Zariski dense in $\mathscr{Z}_n(f)$. □

Theorem 3.9 (Hermitian Singulärstellensatz)**.** *Let $h \in \mathbb{C}\langle x, x^*\rangle^{\varepsilon \times \varepsilon}$ be a full matrix and let $f \in \mathbb{C}\langle x, x^*\rangle^{\delta \times \delta}$ be an unsignatured hermitian atom. Then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ if and only if f is stably associated to a factor of h.*

Proof. Again only ($\Rightarrow$) is non-trivial. By $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ we have $\mathscr{Z}_n(f)^{\mathrm{re}} \subseteq \mathscr{Z}_n(h)$ for all n. Since f is an unsignatured atom, $\mathscr{Z}_n(f)$ is irreducible and $\mathscr{Z}_n(f)^{\mathrm{re}}$ is Zariski dense in $\mathscr{Z}_n(f)$ for infinitely many n by Proposition 3.8 and Theorem 2.9. Consequently $\mathscr{Z}_n(f) \subseteq \mathscr{Z}_n(h)$ for infinitely many n, so $\mathscr{Z}(f) \subseteq \mathscr{Z}(h)$ and Theorem 2.12 applies. □

Similarly to Corollary 3.5, we can use a modified proof of Proposition 2.14 to obtain the following.

Corollary 3.10. *For $j = 1, \dots, s$ let $f_j \in \mathbb{C}\langle x, x^* \rangle^{\delta_j \times \delta_j}$ be unsignatured hermitian atoms and $h \in \mathbb{C}\langle x, x^* \rangle^{\varepsilon \times \varepsilon}$ a full matrix. Then*

$$\mathscr{Z}^{\mathrm{re}}(f_1) \cap \cdots \cap \mathscr{Z}^{\mathrm{re}}(f_s) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$$

if and only if for some j, f_j is stably associated to a factor of h.

Example 3.11. Theorem 3.9 does not hold for arbitrary hermitian atoms f. For example, if $f = x_1 x_1^* + x_2 x_2^*$ and $h = x_1$, then $\mathscr{Z}^{\mathrm{re}}(f) \subseteq \mathscr{Z}^{\mathrm{re}}(h)$ but f is not stably associated to h. For another example, let f be as in (3.5). Then $f(0) = 1 \oplus -1$ and $f(X, X^*)$ is invertible for every X, so f has a constant signature on $\mathrm{M}_n(\mathbb{C})$ for every n. Moreover, f is not invertible in $\mathbb{C}\langle x, x^* \rangle^{2\times 2}$, and is an atom. Hence f and $h = 1$ satisfy $\mathscr{Z}^{\mathrm{re}}(f) = \emptyset = \mathscr{Z}^{\mathrm{re}}(h)$ but f is not stably associated to h. For an algorithm checking whether the free real locus of a polynomial is empty, see [KPV17].

We conclude this section with a Linear Gleichstellensatz for hermitian indecomposable pencils. Since every hermitian monic pencil is unsignatured, the following corollary generalizes [KV17, Corollary 5.5] and preceding versions with operator-algebraic proofs [HKM13, Zal17, DDOSS17] to non-monic pencils.

Corollary 3.12 (Hermitian Linear Gleichstellensatz)**.** *Let L, M be hermitian indecomposable linear pencils. If L is unsignatured, then $\mathscr{Z}^{\mathrm{re}}(L) = \mathscr{Z}^{\mathrm{re}}(M)$ if and only if L, M are of the same size d and there exists $P \in \mathrm{GL}_d(\mathbb{C})$ such that $M = \pm PLP^*$.*

Proof. The implication ($\Leftarrow$) is obvious, so we consider ($\Rightarrow$). Since $\mathscr{Z}^{\mathrm{re}}(L) = \mathscr{Z}^{\mathrm{re}}(M)$ and L, M are atoms, L is stably associated to M by Theorem 3.9. Therefore L, M are of the same size d and $M = PLQ$ for $P, Q \in \mathrm{GL}_d(\mathbb{C})$ by Theorem 2.11. Since L, M are hermitian, we also have $M = Q^* L P^*$. Therefore $(P^{-1}Q^*, P^*Q^{-1})$ stabilizes L, so $P^{-1}Q^* = \alpha I$ by Lemma 2.10. Since $M = \alpha PLP^*$ and L, M are hermitian, we have $\alpha \in \mathbb{R} \setminus \{0\}$, so after rescaling P we can choose $\alpha = \pm 1$. □

4. Null- and Positivstellensatz with hard zeros

In this section we present a new real Nullstellensatz for hard zeros $\mathcal{V}^{\mathrm{re}}$ as opposed to determinantal zeros $\mathscr{Z}^{\mathrm{re}}$ discussed above. While the unsignatured condition above is too weak (at least for our techniques), a stronger unsignatured condition succeeds as is seen in Theorem 4.5. Its proof depends on basic commutative algebra and the technique of rational resolvable ideals developed in [KVV17]. Then, we use this in Corollary 4.6 to prove a Positivstellensatz (by using a slack variable) for domains defined by quadratic polynomials.

4.1. **Background on rationally resolvable ideals.** For an ideal $J \subset \mathbb{C}\langle x\rangle$ let

$$\mathcal{V}(J) = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n(J), \qquad \mathcal{V}_n(J) = \{X \in \mathrm{M}_n(\mathbb{C})^g : f(X) = 0 \ \forall f \in J\}$$

be its **free zero set**. If $f \in \mathbb{C}\langle x\rangle$, then $\mathcal{V}(f)$ denotes the free zero set of the ideal generated by f. We say that J has the **Nullstellensatz property** if

$$f|_{\mathcal{V}(J)} = 0 \quad \Longleftrightarrow \quad f \in J$$

for all $f \in \mathbb{C}\langle x\rangle$. This is a noncommutative analog of a radical ideal in classical algebraic geometry [Eis95, Section 1.6].

We recall rationally resolvable ideals from [KVV17]. The free algebra $\mathbb{C}\langle x\rangle$ admits the universal skew field of fractions $\mathbb{C}\,(\!\!<x>\!\!)$, the **free skew field**, whose elements are called **noncommutative rational functions** [Coh06, BR11, K-VV12]. For $1 \le h < g$ let $\tilde{x} = (x_1, \dots, x_h)$, and fix a tuple $\mathbb{r} = (\mathbb{r}_{h+1}, \dots, \mathbb{r}_g)$ with $\mathbb{r}_i \in \mathbb{C}\,(\!\!<\tilde{x}>\!\!)$. The **graph** of $\mathbb{r}$ is

$$\Gamma(\mathbb{r}) = \bigcup_{n\in\mathbb{N}} \left\{ (\widetilde{X}, \mathbb{r}(\widetilde{X})) \in \mathrm{M}_n(\mathbb{C})^g : \widetilde{X} \in \bigcap_i \operatorname{dom} \mathbb{r}_i \right\}.$$

An ideal $J \subset \mathbb{C}\langle x\rangle$ is

(1) **formally rationally resolvable** with rational resolvent $\mathbb{r}$ if $J \cap \mathbb{C}\langle \tilde{x}\rangle = \{0\}$ and the sets $\{x_{h+1} - \mathbb{r}_{h+1}, \dots, x_g - \mathbb{r}_g\}$ and J generate the same ideal in the amalgamated product
$$\mathbb{C}\langle x\rangle *_{\mathbb{C}\langle \tilde{x}\rangle} \mathbb{C}\,(\!\!<\tilde{x}>\!\!),$$
the subring of $\mathbb{C}\,(\!\!<x>\!\!)$ generated by $\mathbb{C}\langle x\rangle$ and $\mathbb{C}\,(\!\!<\tilde{x}>\!\!)$.

(2) **geometrically rationally resolvable** with rational resolvent $\mathbb{r}$ if $\Gamma(\mathbb{r}) \subseteq \mathcal{V}(J)$ and for every $h \in \mathbb{C}\langle x\rangle$, $h|_{\Gamma(\mathbb{r})} = 0$ implies $h|_{\mathcal{V}(J)} = 0$.

Intuitively, (1) and (2) allude to relations that can be resolved in a rational manner, either in algebraic or geometric sense. The Nullstellensatz property and rational resolvability are related as follows.

Theorem 4.1 ([KVV17, Theorem 2.5 and Proposition 2.6]). *Let $J \subset \mathbb{C}\langle x\rangle$ be an ideal. If $\mathbb{C}\langle x\rangle/J$ embeds into a skew field and J is formally rationally resolvable with a rational resolvent containing no nested inverses, then J is geometrically rationally resolvable and has the Nullstellensatz property.*

Here, nested inverses refer to presentations of noncommutative rational functions; for example, $(x_1 - x_3 x_2^{-1} x_4)^{-1}$ cannot be presented without an inverse inside of an inverse, while $(x_1 - x_2^{-1})^{-1} = x_2(x_1x_2 - 1)^{-1}$ admits a presentation without nested inverses. Finally, as in Subsection 3.1, we see that if $f = f^* \in \mathbb{C}\langle x, x^*\rangle$ is hermitian, the real structure $\mathscr{J}$ preserves $\mathcal{V}_n(f) \subset \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})^g$, and its real points are related to the real free zero set of f from the introduction, $\mathcal{V}_n(f)^{\mathrm{re}} = \{(X, X^*) : X \in \mathcal{V}_n^{\mathrm{re}}(f)\}$.

4.2. **A real Nullstellensatz for some free zero sets.** The proof of Theorem 4.5 requires a lemma from commutative algebra and an embedding into a free skew field.

Lemma 4.2. *Let $M, L, R \in \mathrm{M}_n(\mathbb{C}[\omega])$ be such that $\det(MLR) \neq 0$, and consider*

$$\mathcal{W} = \big\{(X, Y) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C}) : M(X) - L(X)YR(X) = 0\big\}.$$

Let $\mathcal{W}_1 \subseteq \mathcal{W}$ be the union of irreducible components of $\mathcal{W}$ for which $\det M|_{\mathcal{W}\setminus\mathcal{W}_1} = 0$ *and* $\det M$ *is not identically zero on any component of $\mathcal{W}_1$. Similarly, let $\mathcal{W}_2 \subseteq \mathcal{W}$ be the union of irreducible components of $\mathcal{W}$ such that* $\det(LR)|_{\mathcal{W}\setminus\mathcal{W}_2} = 0$ *and* $\det(LR)$ *is not identically zero on any component of $\mathcal{W}_2$. Then $\mathcal{W}_1 = \mathcal{W}_2$ and $\mathcal{W}_1$ is irreducible.*

Proof. By assumption we have $\mathcal{W}_1 \neq \emptyset$. It is clear that $\mathcal{W}_1 \subseteq \mathcal{W}_2$. Now let $\mathcal{Z}$ be a component in $\mathcal{W}\setminus\mathcal{W}_1$ and suppose $\mathcal{Z} \not\subseteq \mathcal{W}\setminus\mathcal{W}_2$. Then $\mathcal{Z}$ contains a Zariski dense subset of points (X, Y) with $\det(L(X)R(X)) \neq 0$. Fix such a point and let $\varepsilon > 0$ be arbitrary. Since $Y = L(X)^{-1}M(X)R(X)^{-1}$, there clearly exists $(X_\varepsilon, Y_\varepsilon) \in \mathrm{M}_n(\mathbb{C})^g \times \mathrm{M}_n(\mathbb{C})$ such that

$$\det(M(X_\varepsilon)L(X_\varepsilon)R(X_\varepsilon)) \neq 0, \quad Y_\varepsilon = L(X_\varepsilon)^{-1}M(X_\varepsilon)R(X_\varepsilon)^{-1}, \quad \|X-X_\varepsilon\|, \|Y-Y_\varepsilon\| \leq \varepsilon.$$

Therefore $(X_\varepsilon, Y_\varepsilon) \in \mathcal{W}_1$ for every $\varepsilon > 0$. Since algebraic sets in a complex affine space are closed with respect to the Euclidean topology, we get $(X, Y) \in \mathcal{W}_1$. Thus $\mathcal{Z} \subseteq \mathcal{W}_1$, a contradiction. Hence $\mathcal{W} \setminus \mathcal{W}_1 \subseteq \mathcal{W} \setminus \mathcal{W}_2$ and therefore $\mathcal{W}_2 \subseteq \mathcal{W}_1$.

Let A be the coordinate ring of $\mathcal{W}_1$. Then $\det M$ is not a zero divisor in A by the definition of $\mathcal{W}_1$, and neither is $\det(LR)$ by the previous paragraph. Therefore A embeds into $A_\mathcal{S}$, where $\mathcal{S}$ is the multiplicative set generated by $\{\det M, \det L, \det R\}$. Let J be the ideal in $\mathbb{C}[\omega, \upsilon]$ generated by the entries of $M - L\Upsilon R$. Then $\mathbb{C}[\omega, \upsilon]/\sqrt{J}$ is the coordinate ring of $\mathcal{W}$ and by the previous paragraph,

$$A_\mathcal{S} = \left(\mathbb{C}[\omega, \upsilon]/\sqrt{J}\right)_\mathcal{S} = \mathbb{C}[\omega, \upsilon]_\mathcal{S}/(\sqrt{J})_\mathcal{S} = \mathbb{C}[\omega, \upsilon]_\mathcal{S}/\sqrt{J_\mathcal{S}} \tag{4.1}$$

since localization is exact [Eis95, Proposition 2.5] and commutes with the radical [Eis95, Proposition 2.2, Corollary 2.6 and Corollary 2.12]. Note that matrices L, R are invertible over $\mathbb{C}[\omega, \upsilon]_\mathcal{S}$. Therefore the ideal $J_\mathcal{S}$ in

$$\mathbb{C}[\omega, \upsilon]_\mathcal{S} = \mathbb{C}[\omega]_\mathcal{S} \otimes_\mathbb{C} \mathbb{C}[\upsilon]$$

is generated by the entries of $\Upsilon - L^{-1}MR^{-1}$. By (4.1) we thus have $A_\mathcal{S} \cong \mathbb{C}[\omega]_\mathcal{S}$, so A is an integral domain and $\mathcal{W}_1$ is irreducible. □

Let y and y' be two freely noncommuting variables.

Lemma 4.3. *Let $f \in \mathbb{C}{<}x{>}$ be nonconstant. Then the ideal $(f - y'y)$ in $\mathbb{C}{<}x, y, y'{>}$ is formally rationally resolvable (by pairing y' with the resolvent fy^{-1}), and the quotient algebra $\mathbb{C}{<}x, y, y'{>}/(f - y'y)$ embeds into a free skew field.*

Proof. First we claim that $(f - y'y) \cap \mathbb{C}{<}x, y{>} = \{0\}$. For every element $p \in (f - y'y)$ we have

$$p(X, Y, f(X)Y^{-1}) = 0$$

for $X \in \mathrm{M}_n(\mathbb{C})^g$ and $Y \in \mathrm{GL}_n(\mathbb{C})$. Hence if $h \in (f - y'y) \cap \mathbb{C}{<}x, y{>}$, then h is zero on $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{GL}_n(\mathbb{C})$ for every $n \in \mathbb{N}$. Since $\mathrm{M}_n(\mathbb{C})^g \times \mathrm{GL}_n(\mathbb{C})$ is Zariski dense in $\mathrm{M}_n(\mathbb{C})^{g+1}$, we conclude that $h = 0$. Moreover, $f - y'y$ and $fy^{-1} - y'$ clearly generate the same ideal in $\mathbb{C}{<}x, y, y^{-1}{>}$. Therefore the ideal $(f - y'y)$ is formally rationally resolvable with the rational resolvent fy^{-1}. Consider the homomorphism

$$\phi : \mathbb{C}{<}x, y, y'{>}/(f - y'y) \to \mathbb{C}{<}x, y, y^{-1}{>}$$

defined by $y' \mapsto fy^{-1}$. Let W be the set of words not containing $y'y$ as a sub-word and let $V = \operatorname{span} W \subset \mathbb{C}{<}x, y, y'{>}$. If $\pi : \mathbb{C}{<}x, y, y'{>} \to \mathbb{C}{<}x, y, y'{>}/(f - y'y)$ is the canonical projection, then it is easy to see that $\pi|_V : V \to \mathbb{C}{<}x, y, y'{>}/(f - y'y)$ is an isomorphism of vector spaces. On W we define a degree function $\tilde{d}: W \to \mathbb{Z}^3_{\geq 0}$ by setting $\tilde{d}(w) = (d_1, d_2, d_3)$, where d_1 is the number of y's in w, d_2 is the number of y''s in w, and d_3 is the length of w. We extend $\tilde{d}$ to a function $V \to \mathbb{Z}^3_{\geq 0}$ by

$$\tilde{d}\left(\sum_{w\in W} \alpha_w w\right) = \max_{w\colon \alpha_w \neq 0} \tilde{d}(w),$$

where $\mathbb{Z}^3_{\geq 0}$ is ordered lexicographically.

As a subalgebra of a free group algebra, $\mathbb{C}<x, y, y^{-1}>$ admits a natural basis consisting of reduced words in x, y, y^{-1}. For each such word w we define $\hat{d}(w) = (d_1, d_2, d_3)$ analogously as above, where d_2 is now the number of y^{-1} in w. Similarly as above we obtain the extension $\hat{d}: \mathbb{C}<x, y, y^{-1}> \to \mathbb{Z}^3_{\geq 0}$.

Since the elements of $\mathbb{C}<x> \setminus \mathbb{C}$ are freely independent of y and y^{-1}, by looking at the highest terms with respect to $\tilde{d}$ and $\hat{d}$ one observes that

$$\hat{d}\big((\phi \circ \pi|_V)(h)\big) = \Big(\tilde{d}(h)_1, \tilde{d}(h)_2, \tilde{d}(h)_3 + (\deg f)\tilde{d}(h)_2\Big)$$

for every $h \in V$. Therefore $\phi \circ \pi|_V$ is injective, so ϕ is an embedding. Hence we are done because the free group algebra embeds into a free skew field (see e.g. [Coh06, Corollary 7.11.8]). □

Proposition 4.4. *Let $f \in \mathbb{C}<x>$ be nonconstant and let $h \in \mathbb{C}<x, y, y'>$. Then there exists $N \in \mathbb{N}$ such that*

$$h\left(X, Y, f(X)Y^{-1}\right) = 0$$

for all $(X, Y) \in \mathrm{M}_N(\mathbb{C})^g \times \mathrm{GL}_N(\mathbb{C})$ implies $h \in (f - y'y)$. In particular, $(f - y'y)$ has the Nullstellensatz property.

Proof. By Lemma 4.3 and Theorem 4.1, the ideal $(f - y'y)$ is geometrically rationally resolvable has the Nullstellensatz property. The existence of the bound N then follows by [KVV17, Theorem 3.9]. □

Theorem 4.5. *For a nonconstant $f = f^* \in \mathbb{C}<x, x^*>$ assume that $f(X_0, X_0^*) \succ 0$ for some X_0. Let $h \in \mathbb{C}<x, x^*, y, y^*>$. Then $\mathcal{V}^{\mathrm{re}}(f - y^*y) \subseteq \mathcal{V}^{\mathrm{re}}(h)$ if and only if $h \in (f - y^*y)$.*

Proof. Let $n \in \mathbb{N}$ be such that $f(X_0, X_0^*) = Y_0^* Y_0$ for some $X_0 \in \mathrm{M}_n(\mathbb{C})^g$ and $Y_0 \in \mathrm{GL}_n(\mathbb{C})$. Let Ω be a $2g$-tuple of $n \times n$ generic matrices, and Υ, Υ' two additional $n \times n$ generic matrices. Then $\det f(\Omega) \neq 0$ since $f(X_0, X_0^*)$ is invertible. By Lemma 4.2 there exists a unique irreducible component $\mathcal{W} \subset \mathrm{M}_n(\mathbb{C})^{2g+2}$ of $\mathcal{V}_n(f - y^*y)$ such that $\det f(\Omega)$ and $\det \Upsilon$ are not identically 0 on $\mathcal{W}$.

Since f is hermitian, $\mathcal{W}$ inherits the real structure $\mathscr{J}$ from $\mathrm{M}_n(\mathbb{C})^{g+1} \times \mathrm{M}_n(\mathbb{C})^{g+1}$. Note that the derivative of $f(\Omega) - \Upsilon'\Upsilon$ at $(X_0, X_0^*, Y_0, Y_0^*) \in \mathcal{W}^{\mathrm{re}}$ with respect to $\upsilon'_{\imath\jmath}$ equals $E_{\imath\jmath} Y_0$, where $E_{\imath\jmath} \in \mathrm{M}_n(\mathbb{C})$ are the standard matrix units. The Jacobian matrix of the system of equations $f(\Omega) - \Upsilon'\Upsilon = 0$ at (X_0, X_0^*, Y_0, Y_0^*) is then equal to

$$\big(\overbrace{\star \quad \cdots \quad \star}^{(2g+1)n^2} \quad -I_n \otimes Y_0^{\mathrm{t}}\big) \in \mathbb{C}^{n^2 \times (gn^2 + gn^2 + n^2 + n^2)}.$$

Therefore $(X_0, X_0^*, Y_0, Y_0^*) \in \mathcal{W}^{\mathrm{re}}$ is a nonsingular point of $\mathcal{W}$, so $\mathcal{W}^{\mathrm{re}}$ is Zariski dense in $\mathcal{W}$ by Proposition 3.1. Because h vanishes on $\mathcal{W}^{\mathrm{re}}$, it also vanishes on $\mathcal{W}$. Since $\mathcal{W}$ is the unique component of $\mathcal{V}_n(f - y'y)$ on which $\det \Upsilon$ does not constantly vanish, we have

$$h\left(X, Y, f(X)Y^{-1}\right) = 0$$

for all $(X, Y) \in \mathrm{M}_n(\mathbb{C})^{2g} \times \mathrm{GL}_n(\mathbb{C})$. Since n can be taken arbitrarily large, we have $h \in (f - y^*y)$ by Proposition 4.4. □

4.3. **A Positivstellensatz for hereditary quadratic polynomials.** Let $f \in \mathbb{C}<x, x^*>$ be a hermitian hereditary quadratic polynomial. That is,

$$f = \alpha + \vec{x}^*v + v^*\vec{x} + \vec{x}^*H\vec{x},$$

where H is a hermitian $g \times g$ matrix, $\vec{x}$ is the column vector consisting of the variables x_j, $v \in \mathbb{C}^g$ and $\alpha \in \mathbb{R}$. Then $\{f \succ 0\} \neq \emptyset$ if and only if $-f$ is not a sum of (hermitian) squares (this is not true for more general polynomials, e.g., $x_1x_1^* - x_1^*x_1$).

Corollary 4.6. *Let $f \in \mathbb{C}<x, x^*>$ be a nonconstant hermitian hereditary quadratic polynomial with $\{f \succ 0\} \neq \emptyset$, and let y be an auxiliary noncommuting variable. If $h \in \mathbb{C}<x, x^*>$, then $h|_{\{f\succeq 0\}} \succeq 0$ if and only if*

$$h = f_0 + \sum_j f_j^* f_j \tag{4.2}$$

for some $f_j \in \mathbb{C}<x, x^*, y, y^*>$ with $f_0 \in (f - y^*y)$.

Proof. ($\Rightarrow$) If $h|_{\{f\succeq 0\}} \succeq 0$, then $h|_{\mathcal{V}(f-y^*y)} \succeq 0$, where h is viewed as an element of $\mathbb{C}<x, x^*, y, y^*>$. Since $f - y^*y$ is also a quadratic hereditary polynomial, there exist $f_j \in \mathbb{C}<x, x^*, y, y^*>$ with $f_0|_{\mathcal{V}(f-y^*y)} = 0$ such that

$$h = f_0 + \sum_j f_j^* f_j$$

by [HMP04, Theorem 4.1 and Section 4.2.c] (or rather its version over $\mathbb{C}$). Moreover, $f_0 \in (h - y^*y)$ by Theorem 4.5.

($\Leftarrow$) Suppose (4.2) holds. Let X be such that $f(X, X^*) \succeq 0$. If $Y^*Y = f(X, X^*)$ then $f_0(X, X^*, Y, Y^*) = 0$ and hence $h(X, X^*) \succeq 0$. □

Remark 4.7. In general, $h|_{\{f\succeq 0\}} \succeq 0$ does not imply that h is of the form

$$\sum_j f_j^* f_j + \sum_k g_k^* f g_k,$$

i.e., h does not necessarily belong to the quadratic module generated by f. An example with $f = 1 - x^*x$ and $h = 1 - \frac{4}{3}x^*x + \alpha x^*xx^*x$ for $\frac{1}{3} < \alpha < \frac{4}{9}$ is a consequence of [D'AP09, Section 3.1]. Thus the slack variable y of Corollary 4.6 is necessary.

Remark 4.8. Corollary 4.6 is a rare example of an "if and only if" noncommutative Positivstellensatz. Another one appears in [HKM12] and applies to quadratic f whose positivity set is convex. For such f our Corollary 4.6 can be readily proved as a consequence of [HKM12, Theorem 1.1(1)].

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