THE TRACIAL HAHN-BANACH THEOREM, POLAR DUALS, MATRIX CONVEX SETS, AND PROJECTIONS OF FREE SPECTRAHEDRA

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Abstract. This article investigates matrix convex sets and introduces their tracial analogs which we call contractively tracial convex sets. In both contexts completely positive (cp) maps play a central role: unital cp maps in the case of matrix convex sets and trace preserving cp (CPTP) maps in the case of contractively tracial convex sets. CPTP maps, also known as quantum channels, are fundamental objects in quantum information theory.

Free convexity is intimately connected with Linear Matrix Inequalities (LMIs) $L(x) = A_0+A_1x_1+\cdots+A_gx_g \succeq 0$ and their matrix convex solution sets $\{X : L(X) \succeq 0\}$, called *free spectrahedra*. The Effros-Winkler Hahn-Banach Separation Theorem for matrix convex sets states that matrix convex sets are solution sets of LMIs with operator coefficients. Motivated in part by cp interpolation problems, we develop the foundations of convex analysis and duality in the tracial setting, including tracial analogs of the Effros-Winkler Theorem.

The projection of a free spectrahedron in $g+h$ variables to g variables is a matrix convex set called a *free spectrahedrop*. As a class, free spectrahedrops are more general than free spectrahedra, but at the same time more tractable than general matrix convex sets. Moreover, many matrix convex sets can be approximated from above by free spectrahedrops. Here a number of fundamental results for spectrahedrops and their polar duals are established. For example, the free polar dual of a free spectrahedrop is again a free spectrahedrop. We also give a Positivstellensatz for free polynomials that are positive on a free spectrahedrop.

1. Introduction

This article investigates matrix convex sets from the perspective of the emerging areas of free real algebraic geometry and free analysis [Voi04, Voi10, KVV14, MS11, Pop10, AM15,

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BB07, dOHMP09, HKM13b, PNA10]. It also introduces **contractively tracial convex sets**, the tracial analogs of matrix convex sets appropriate for the quantum channel and quantum operation interpolation problems. Matrix convex sets arise naturally in a number of contexts, including engineering systems theory, operator spaces, systems and algebras and are inextricably linked to unital completely positive (ucp) maps [SIG97, Arv72, Pau02, Far00, HMPV09]. On the other hand, completely positive trace preserving (CPTP) maps are central to quantum information theory [NC10, JKPP11]. Hence there is an inherent similarity between matrix convex sets and structures naturally occurring in quantum information theory.

Given positive integers g and n, let $\mathbb{S}_n^g$ denote the set of g-tuples $X = (X_1, \dots, X_g)$ of complex $n \times n$ hermitian matrices and let $\mathbb{S}^g$ denote the sequence $(\mathbb{S}_n^g)_n$. We use M_n to denote the algebra of $n \times n$ complex matrices. A subset $\Gamma \subseteq \mathbb{S}^g$ is a sequence $\Gamma = (\Gamma(n))_n$ such that $\Gamma(n) \subseteq \mathbb{S}_n^g$ for each n. A **matrix convex set** is a subset $\Gamma \subseteq \mathbb{S}^g$ that is **closed with respect to direct sums** and (simultaneous) **conjugation by isometries**. Closed under direct sums means if $X \in \Gamma(n)$ and $Y \in \Gamma(m)$, then

$$X \oplus Y := \Big(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \dots, \begin{pmatrix} X_g & 0 \\ 0 & Y_g \end{pmatrix} \Big) \in \Gamma(n+m). \tag{1.1}$$

Likewise, closed under conjugation by isometries means if $X \in \Gamma(n)$ and V is an $n \times m$ isometry, then

$$V^*XV := (V^*X_1V, \dots, V^*X_gV) \in \Gamma(m).$$

The simplest examples of matrix convex sets arise as solution sets of linear matrix inequalities (LMIs). The use of LMIs is a major advance in systems engineering in the past two decades [SIG97]. Furthermore, LMIs underlie the theory of semidefinite programming [BPR13, BN02], itself a recent major innovation in convex optimization [Nem06].

Matrix convex sets determined by LMIs are based on a free analog of an affine linear functional, often called a **linear pencil**. Given a positive integer d and g hermitian $d \times d$ matrices A_j, let

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j. \tag{1.2}$$

This linear pencil is often denoted by L_A to emphasize the dependence on A. In the case that $A_0 = I_d$, we call L **monic**. Replacing $x \in \mathbb{R}^g$ with a tuple $X = (X_1, \dots, X_g)$ of $n \times n$ hermitian matrices and letting $W \otimes Z$ denote the Kronecker product of matrices leads to the evaluation of the free affine linear functional,

$$L(X) = A_0 \otimes I_n + \sum A_j \otimes X_j. \tag{1.3}$$

The inequality $L(X) \succeq 0$ is a **free linear matrix inequality (free LMI)**. The solution set Γ of this free LMI is the sequence of sets

$$\Gamma(n) = \{X \in \mathbb{S}_n^g : L(X) \succeq 0\}$$

and is known as a **free spectrahedron** (or **free LMI domain**). It is easy to see that Γ is a matrix convex set.

By the Effros-Winkler matricial Hahn-Banach Separation Theorem [EW97], (up to a technical hypothesis) every matrix convex set is the solution set of $L(X) \succeq 0$, as in equation (1.3), of some monic linear pencil *provided* the A_j are allowed to be hermitian operators on a (common) Hilbert space. More precisely, every matrix convex set is a (perhaps infinite) intersection of free spectrahedra. Thus, being a spectrahedron imposes a strict finiteness condition on a matrix convex set.

In between (closed) matrix convex sets and spectrahedra lie the class of domains we call **spectrahedrops**. Namely coordinate projections of free spectrahedra. A subset $\Delta \subseteq \mathbb{S}^g$ is a **free spectrahedrop** if there exists a pencil

$$L(x, y) = A_0 + \sum_{j=1}^{g} A_j x_j + \sum_{k=1}^{h} B_k y_k,$$

in $g + h$ variables such that

$$\Delta(n) = \{X \in \mathbb{S}_n^g : \exists Y \in \mathbb{S}_n^h \text{ such that } L(X, Y) \succeq 0\}. \tag{1.4}$$

In applications, presented with a convex set, one would like, for optimization purposes say, to know if it is a spectrahedron or a spectrahedrop. Alternately, presented with an algebraically defined set $\Gamma \subseteq \mathbb{S}^g$ that is not necessarily convex, it is natural to consider the relaxation obtained by replacing Γ with its matrix convex hull or an approximation thereof. Thus, it is of interest to know when the convex hull of a set is a spectrahedron or perhaps a spectrahedrop. An approach to these problems via approximating from above by spectrahedrops was pursued in the article [HKM16]. Here we develop the duality approach. Typically the second polar dual of a set is its closed matrix convex hull.

1.1. **Results on Polar Duals and Free Spectrahedrops.**

We list here our main results on free spectrahedrops and polar duals. For the reader unfamiliar with the terminology, the definitions not already introduced can be found in Section 2 with the exception of *free polar dual* whose definition appears in Subsection 4.2.

(1) A perfect free Positivstellensatz (Theorem 5.1) for any symmetric free polynomial p on a free spectrahedrop Δ as in (1.4). It says that $p(X)$ is positive semidefinite for

all $X \in \Delta$ if and only if p has the form

$$p(x) = f(x)^* f(x) + \sum_{\ell} q_\ell(x)^* L(x,y) q_\ell(x)$$

where f and and q_ℓ are vectors with polynomial entries. If the degree of p is less than or equal to $2r+1$, then f and q_ℓ have degree no greater than r;

(2) The free polar dual of a free spectrahedrop is a free spectrahedrop (Theorem 4.11 and Corollary 4.17);

(3) The matrix convex hull of a union of finitely many bounded free spectrahedrops is a bounded free spectrahedrop (Proposition 4.18);

(4) A matrix convex set is, in a canonical sense, generated by a finite set (equivalently a single point) if and only if it is the polar dual of a free spectrahedron (Theorem 4.6).

1.2. **Results on Interpolation of cp Maps and Quantum Channels.** A completely positive (**cp**) map $M_n \to M_m$ that is trace preserving is called a **quantum channel**, and a cp map that is trace non-increasing for positive semidefinite arguments is a **quantum operation**. These maps figure prominently in quantum information theory [NC10].

The cp interpolation problem is formulated as follows. Given $A \in \mathbb{S}_n^g$ and $B \in \mathbb{S}_m^g$, does there exists a cp map $\Phi : M_n \to M_m$ such that, for $1 \le \ell \le g$,

$$\Phi(A_\ell) = B_\ell?$$

One can require further that Φ be unital, a quantum channel or a quantum operation. Imposing either of the latter two constraints pertains to quantum information theory [Ha11, Kle07, NCSB98], where one is interested in quantum channels (resp., quantum operations) that send a prescribed set of quantum states into another set of quantum states.

1.2.1. *Algorithmic Aspects.* A byproduct of the methods used in this paper and in [HKM13a] produces solutions to these cp interpolation problems in the form of an algorithm, Theorem 3.4 in Subsection 3.2. Ambrozie and Gheondea [AG15] solved these interpolation problems with LMI algorithms. While equivalent to theirs, our solutions are formulated as concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver. These interpolation results are a basis for proofs of results outlined in Section 1.1.

1.3. **Free Tracial Hahn-Banach Theorem.** Matrix convex sets are closely connected with ranges of unital cp maps. Indeed, given a tuple $A \in \mathbb{S}_m^g$, the matrix convex hull of the set $\{A\}$ is the sequence of sets

$$\big(\{B \in \mathbb{S}_n^g : B_j = \Phi(A_j) \text{ for some ucp map } \Phi : M_m \to M_n\}\big)_n.$$

From the point of view of quantum information theory it is natural to consider hulls of ranges of quantum operations. We say that $\mathcal{Y} \subseteq \mathbb{S}^g$ is **contractively tracial** if for all positive integers m, n, elements $Y \in \mathcal{Y}(m)$, and finite collections $\{C_\ell\}$ of $n \times m$ matrices such that

$$\sum C_\ell^* C_\ell \preceq I_m,$$

it follows that $\sum C_j Y C_j^* \in \mathcal{Y}(n)$. It is clear that an intersections of contractively tracial sets is again contractively tracial, giving rise, in the usual way, to the notion of the **contractive tracial hull**, denoted **cthull**. For a tuple A,

$$\operatorname{cthull}(A) = \{B : \Phi(A) = B \ \text{ for some quantum operation } \Phi\}.$$

While the unital and quantum interpolation problems have very similar formulations, contractive tracial hulls possess far less structure than matrix convex hulls. A subset $\mathscr{Y} \subseteq \mathbb{S}^g$ is **levelwise convex** if each $\mathscr{Y}(m)$ is convex (as a subset of $\mathbb{S}^g_m$). (Generally **levelwise** refers to a property holding for each $\mathscr{Y}(m) \subseteq \mathbb{S}^g_m$.) As is easily seen, contractive tracial hulls need not be levelwise convex nor closed with respect to direct sums. However they do have a few good properties. These we develop in Section 6.

Section 7 contains notions of free spectrahedra and corresponding Hahn-Banach type separation theorems tailored to the tracial setting. To understand *convex* contractively tracial sets, given $B \in \mathbb{S}^g_k$, let $\mathfrak{H}_B = (\mathfrak{H}_B(m))_m$ denote the sequence of sets

$$\mathfrak{H}_B(m) = \Big\{Y \in \mathbb{S}^g_m : \exists T \succeq 0,\ \operatorname{tr}(T) \le 1,\ \ I \otimes T - \sum B_j \otimes Y_j \succeq 0\Big\}.$$

We call $\mathfrak{H}_B$ a **tracial spectrahedron**. (Note that $\mathfrak{H}_B$ is not closed under direct sums, and thus it is not a matrix convex set.) These $\mathfrak{H}_B$ are all contractively tracial and levelwise convex. Indeed for such structural reasons, and in view of the tracial Hahn-Banach separation theorem immediately below, we believe these to be the natural analogs of free spectrahedra in the tracial context.

Theorem 1.1 (cf. Theorem 7.6)**.** *If $\mathcal{Y} \subseteq \mathbb{S}^g$ is contractively tracial, levelwise convex and closed, and if $Z \in \mathbb{S}^g_m$ is not in $\mathcal{Y}(m)$, then there exists a $B \in \mathbb{S}^g_m$ such that $\mathcal{Y} \subseteq \mathfrak{H}_B$, but $Z \notin \mathfrak{H}_B$.*

Because of the asymmetry between B and Y in the definition of $\mathfrak{H}_B$, there is a second type of tracial spectrahedron. Given $Y \in \mathbb{S}^g_k$, we define the **opp-tracial spectrahedron** as the sequence $\mathfrak{H}^{\mathrm{opp}}_Y = (\mathfrak{H}^{\mathrm{opp}}_Y(m))_m$

$$\mathfrak{H}^{\mathrm{opp}}_Y(m) = \{B \in \mathbb{S}^g_m : \exists T \succeq 0,\ \operatorname{tr}(T) \le 1,\ \ I \otimes T - \sum B_j \otimes Y_j \succeq 0\Big\}. \tag{1.5}$$

Proposition 7.11 computes the hulls resulting from the two different double duals determined by the two notions of tracial spectrahedron.

1.4. **Reader's guide.** The paper is organized as follows. Section 2 introduces terminology and notation used throughout the paper. Section 3 solves the cp interpolation problems, and includes a background section on cp maps. Section 4 contains our main results on polar duals, free spectrahedra and free spectrahedrops. It uses the results of Section 3. In particular, we show that a matrix convex set is finitely generated if and only if it is the polar dual of a free spectrahedron (Theorem 4.6). Furthermore, we prove that the polar dual of a free spectrahedrop is again a free spectrahedrop (Theorem 4.11). Section 5 contains the "perfect" Convex Positivstellensatz for free polynomials positive semidefinite on free spectrahedrops. The proof depends upon the results of Section 4. In Section 6 we introduce tracial sets and hulls and discuss their connections with the quantum interpolation problems from Section 3. Finally, Section 7 introduces tracial spectrahedra and proves a Hahn-Banach separation theorem in the tracial context, see Theorem 7.6. This theorem is then used to suggest corresponding notions of duality. Section 8 contains examples.

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2. Preliminaries

This section introduces terminology and presents preliminaries on free polynomials, free sets and free convexity needed in the sequel.

2.1. **Free Sets.** A set $\Gamma \subseteq \mathbb{S}^g$ is closed with respect to **(simultaneous) unitary conjugation** if for each n, each $A \in \Gamma(n)$ and each $n \times n$ unitary matrix U,

$$U^*AU = (U^*A_1U, \dots, U^*A_gU) \in \Gamma(n).$$

The set Γ is a **free set** if it is closed with respect to direct sums (see equation (1.1)) and simultaneous unitary conjugation. In particular, a matrix convex set is a free set. We refer the reader to [Voi04, Voi10, KVV14, MS11, Pop10, AM15, BB07] for a systematic study of free sets and free function theory. The set Γ is **(uniformly) bounded** if there is a $C \in \mathbb{R}_{>0}$ such that $C - \sum X_j^2 \succeq 0$ for all $X \in \Gamma$.

2.2. **Free Polynomials.** One natural way free sets arise is as the nonnegativity set of a free polynomial. Given a positive integers ℓ and ν, let $\mathbb{C}^{\ell\times\nu}$ denote the collection of $\ell \times \nu$ matrices. An expression of the form

$$P = \sum_w B_w w \in \mathbb{C}^{\ell\times\nu}\langle x\rangle,$$

where $B_w \in \mathbb{C}^{\ell\times\nu}$, and the sum is a finite sum over the words in the variables x, is a **free (noncommutative) matrix-valued polynomial**. The collection of all $\ell \times \nu$-valued free polynomials is denoted $\mathbb{C}^{\ell\times\nu}\langle x\rangle$ and $\mathbb{C}\langle x\rangle$ denotes the set of scalar-valued free polynomials.

We use $\mathbb{C}^{\ell\times\nu}\langle x\rangle_k$ to denote free polynomials of degree $\leq k$. Here the degree of a word is its **length**. The free polynomial P is evaluated at an $X \in \mathbb{S}_n^g$ by

$$P(X) = \sum_{w\in\langle x\rangle} B_w \otimes w(X) \in \mathbb{C}^{\ell n\times\mu n},$$

where $\otimes$ denotes the (Kronecker) tensor product.

There is a natural **involution** * on words that reverses the order. This involution extends to $\mathbb{C}^{\ell\times\nu}\langle x\rangle$ by

$$P^* = \sum_w B_w^* w^* \in \mathbb{C}^{\mu\times\ell}\langle x\rangle.$$

If $\mu = \ell$ and $P^* = P$, then P is **symmetric**. Note that if $P \in \mathbb{C}^{\ell\times\ell}\langle x\rangle$ is symmetric, and $X \in \mathbb{S}_n^g$, then $P(X) \in \mathbb{C}^{\ell n\times\ell n}$ is a hermitian matrix.

2.3. **Free Semialgebraic Sets.** The **nonnegativity set** $\mathcal{D}_P \subseteq \mathbb{S}^g$ of a symmetric free polynomial is the sequence of sets

$$\mathcal{D}_P(n) = \{X \in \mathbb{S}_n^g : P(X) \succeq 0\}.$$

It is readily checked that $\mathcal{D}_P$ is a free set. By analogy with (commutative) real algebraic geometry, we call $\mathcal{D}_P$ a basic **free semialgebraic set**. Often it is assumed that $P(0) \succ 0$. The free set $\mathcal{D}_P$ has the additional property that it is **closed with respect to restriction to reducing subspaces**; that is, if $X \in \mathcal{D}_P(n)$ and $\mathcal{H} \subseteq \mathbb{C}^n$ is a reducing (equivalently invariant) subspace for X of dimension m, then that X restricted to $\mathcal{H}$ is in $\mathcal{D}_P(m)$.

2.4. **Free Convexity.** In the case that Γ is matrix convex, it is easy to show that Γ is levelwise convex. More generally, if $A^\ell = (A_1^\ell, \dots, A_g^\ell)$ are in $\Gamma(n_\ell)$ for $1 \leq \ell \leq k$, then $A = \bigoplus_{\ell=1}^k A^\ell \in \Gamma(n)$, where $n = \sum n_\ell$. Hence, if V_ℓ are $n_\ell \times m$ matrices (for some m) such that $V = (V_1^* \ \dots \ V_k^*)^*$ is an isometry (equivalently $\sum_{\ell=1}^k V_\ell^* V_\ell = I_m$), then

$$V^*AV = \sum_{\ell=1}^k V_\ell^* A^\ell V_\ell \in \Gamma(m). \tag{2.1}$$

A sum as in equation (2.1) is a **matrix (free) convex combination** of the g-tuples $\{A^\ell : \ell = 1, \dots, k\}$.

Lemma 2.1 ([HKM16, Lemma 2.3])**.** *Suppose Γ is a free subset of $\mathbb{S}^g$.*

(1) *If Γ is closed with respect to restriction to reducing subspaces, then the following are equivalent:*
  (i) *Γ is matrix convex; and*
  (ii) *Γ is levelwise convex.*
(2) *If Γ is (nonempty and) matrix convex, then $0 \in \Gamma(1)$ if and only if Γ is* **closed with respect to (*simultaneous*) conjugation by contractions**.

Convex subsets of $\mathbb{R}^g$ are defined as intersections of half-spaces and are thus described by linear functionals. Analogously, matrix convex subsets of $\mathbb{S}^g$ are defined by linear pencils; cf. [EW97, HM12]. We next present basic facts about linear pencils and their associated matrix convex sets.

2.4.1. *Linear Pencils.* Recall the definition (see equation (1.2)) of the (affine) linear pencil $L_A(x)$ associated to a tuple $A = (A_0, \dots, A_g) \in \mathbb{S}_k^{g+1}$. In the case that $A_0 = 0$; i.e., $A = (A_1, ..., A_g) \in \mathbb{S}_k^g$, let

$$\Lambda_A(x) = \sum_{j=1}^{g} A_j x_j$$

denote the corresponding **homogeneous (truly) linear pencil** and

$$\mathfrak{L}_A = I - \Lambda_A$$

the associated **monic linear pencil**.

The pencil L_A (also $\mathfrak{L}_A$) is a free polynomial with matrix coefficients, so is naturally evaluated on $X \in \mathbb{S}_n^g$ using (Kronecker's) tensor product yielding equation (1.3). The free semialgebraic set $\mathcal{D}_{L_A}$ is easily seen to be matrix convex. We will refer to $\mathcal{D}_{L_A}$ as a **free spectrahedron** or **free LMI domain** and say that a free set Γ is **freely LMI representable** if there is a linear pencil L such that $\Gamma = \mathcal{D}_L$. In particular, if Γ is freely LMI representable with a monic $\mathfrak{L}_A$, then 0 is in the interior of $\Gamma(1)$.

The following is a special case of a theorem due to Effros and Winkler [EW97]. (See also [HKM13a, Theorem 3.1].) Given a free set Γ, if $0 \in \Gamma(1)$, then $0 \in \Gamma(n)$ for each n. In this case we will write $0 \in \Gamma$.

Theorem 2.2. *If $\mathcal{C} = (\mathcal{C}(n))_{n\in\mathbb{N}} \subseteq \mathbb{S}^g$ is a closed matrix convex set containing 0 and $Y \in \mathbb{S}_m^g$ is not in $\mathcal{C}(m)$, then there is a monic linear pencil $\mathfrak{L}$ of size m such that $\mathfrak{L}(X) \succeq 0$ for all $X \in \mathcal{C}$, but $\mathfrak{L}(Y) \not\succeq 0$.*

By the following result from [HM12], linear matrix inequalities account for matrix convexity of free semialgebraic sets.

Theorem 2.3. *Fix p a symmetric matrix polynomial. If $p(0) \succ 0$ and the strict positivity set $\mathfrak{P}_p = \{X : p(X) \succ 0\}$ of p is bounded, then $\mathfrak{P}_p$ is matrix convex if and only if if is freely LMI representable with a monic pencil.*

2.5. **A Convex Positivstellensatz and LMI domination.** Positivstellensätze are pillars of real algebraic geometry [BCR98]. We next recall the Positivstellensatz for a free polynomial p. It is the algebraic certificate for nonnegativity of p on the free spectrahedron $\mathcal{D}_L$ from [HKM12]. It is "perfect" in the sense that p is only assumed to be nonnegative on $\mathcal{D}_L$, and we obtain degree bounds on the scale of $\deg(p)/2$ for the polynomials involved in the positivity certificate. In Section 5, we will extend this Positivstellensatz to free spectrahedrops (i.e., projections of free spectrahedra). See Theorem 5.1.

Theorem 2.4. *Suppose $\mathfrak{L}$ is a monic linear pencil. A matrix polynomial p is positive semidefinite on $\mathcal{D}_{\mathfrak{L}}$ if and only if it has a weighted sum of squares representation with optimal degree bounds:*

$$p = s^*s + \sum_j^{\text{finite}} f_j^* \mathfrak{L} f_j,$$

where s, f_j are matrix polynomials of degree no greater than $\frac{\deg(p)}{2}$.

In particular, if $\mathfrak{L}_A$, $\mathfrak{L}_B$ are monic linear pencils, then $\mathcal{D}_{\mathfrak{L}_B} \subseteq \mathcal{D}_{\mathfrak{L}_A}$ if and only if there exists a positive integer μ and a contraction V such that

$$A = V^*(I_\mu \otimes B)V. \tag{2.2}$$

In the case $\mathcal{D}_{\mathfrak{L}_B}$ is bounded, V can be chosen to be an isometry.

Proof. The first statement is [HKM12, Theorem 1.1]. Applying this result to the **LMI domination problem** $\mathcal{D}_{\mathfrak{L}_B} \subseteq \mathcal{D}_{\mathfrak{L}_A}$, we see $\mathcal{D}_{\mathfrak{L}_B} \subseteq \mathcal{D}_{\mathfrak{L}_A}$ is equivalent to

$$\mathfrak{L}_A(x) = S^*S + \sum_{j=1}^{\mu} V_j^* \mathfrak{L}_B(x) V_j \tag{2.3}$$

for some matrices S, V_j; i.e.,

$$I = S^*S + \sum_j V_j^* V_j = S^*S + V^*V \tag{2.4}$$

$$A = \sum_{j=1}^{\mu} V_j^* B V_j = V^*(I_\mu \otimes B)V, \tag{2.5}$$

where V is the block column matrix of the V_j. Equation (2.4) simply says that V is a contraction, and (2.5) is (2.2). The last statement is proved in [HKM13a]. Alternately, as is shown in [HKM12, Proposition 4.2], if $\mathcal{D}_{\mathfrak{L}_B}$ is bounded, then there are finitely many matrices W_j such that

$$I = \sum W_j^* \mathfrak{L}_B(x) W_j.$$

Writing $S^*S = \sum (W_j S)^* \mathfrak{L}_B(x)(W_j S)$ and substituting into equation (2.3) completes the proof. ∎

Example 8.1 shows that it is not necessarily possible to choose V an isometry in equation (2.5) in absence of additional hypothesis on the tuple B.

3. Completely Positive Interpolation

Theorem 3.4 provides a solution to three cp interpolation problems in terms of concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver. The unital cp interpolation problem comes from efforts to understand matrix convex sets that arise in convex optimization. Its solution plays an important role in the proof of the main result on the polar dual of a free spectrahedrop, Theorem 4.11, via its appearance in the proof of Proposition 4.14.

The trace preserving and trace non-increasing cp interpolation problems arise in quantum information theory and the study of quantum channels, where one is interested in sending one (finite) set of prescribed quantum states to another.

3.1. **Basics of Completely Positive Maps.** This subsection collects basic facts about completely positive (cp) maps $\phi : \mathcal{S} \to M_d$, where $\mathcal{S}$ is a subspace of M_n closed under conjugate transpose (see for instance [Pau02]) containing a positive definite matrix. Thus $\mathcal{S}$ is a **operator system**.

Suppose $\mathcal{S}$ is a subspace of M_n closed under conjugate transpose, $\phi : \mathcal{S} \to M_d$ is a linear map and ℓ is a positive integer. The (ℓ-th) **ampliation** $\phi_\ell : M_\ell(\mathcal{S}) \to M_\ell(M_d)$ of ϕ is defined by by applying ϕ entrywise,

$$\phi_\ell(S_{j,k}) = \big(\phi(S_{j,k})\big) .$$

The map ϕ is **symmetric** if $\phi(S^*) = \phi(S)^*$ and it is **completely positive** if each ϕ_ℓ is positive in the sense that if $S \in M_\ell(\mathcal{S})$ is positive semidefinite, then so is $\phi_\ell(S) \in M_\ell(M_d)$. In what follows, often $\mathcal{S}$ is a subspace of $\mathbb{S}_n$ (and is thus automatically closed under the conjugate transpose operation).

The **Choi matrix** of a mapping $\phi : M_n \to M_d$ is the $n \times n$ block matrix with $d \times d$ matrix entries given by

$$(C_\phi)_{i,j} = \big(\phi(E_{i,j})\big)_{i,j}.$$

On the other hand, a matrix $C = (C_{i,j}) \in M_n(M_d)$ determines a mapping $\phi_C : M_n \to M_d$ by $\phi_C(E_{i,j}) = C_{i,j} \in M_d$. A matrix C is **a Choi matrix for** $\phi : \mathcal{S} \to M_d$, if the mapping ϕ_C agrees with ϕ on $\mathcal{S}$.

Theorem 3.1. *For $\phi : M_n \to M_d$, the following are equivalent:*

(a) *ϕ is completely positive;*
(b) *the Choi matrix C_ϕ is positive semidefinite.*

Suppose $\mathcal{S} \subseteq M_n$ is an operator system. For a symmetric $\phi : \mathcal{S} \to M_d$, the following are equivalent:

(i) *ϕ is completely positive;*

(ii) *ϕ_d is positive;*
(iii) *there exists a completely positive mapping $\Phi : M_n \to M_d$ extending ϕ;*
(iv) *there is a positive semidefinite Choi matrix for ϕ;*
(v) *there exists $n \times d$ matrices $V_1, \dots, V_{nd}$ such that*

$$\phi(A) = \sum V_j^* A V_j. \tag{3.1}$$

Finally, for a subspace $\mathcal{S}$ of M_n, a mapping $\phi : \mathcal{S} \to M_d$ has a completely positive extension $\Phi : M_n \to M_d$ if and only if ϕ has a positive semidefinite Choi matrix.

Lemma 3.2. *The cp mapping $\phi : M_n \to M_d$ as in (3.1) is*

(a) *unital (that is, $\phi(I_n) = I_d$) if and only if*

$$\sum_j V_j^* V_j = I;$$

(b) *trace preserving if and only if*

$$\sum_j V_j V_j^* = I;$$

(c) *trace non-increasing for positive semidefinite matrices (i.e., $\operatorname{tr}(\phi(P)) \leq \operatorname{tr}(P)$ for all positive semidefinite P) if and only if*

$$\sum_j V_j V_j^* \preceq I.$$

Proof. We prove (c) and leave items (a) and (b) as an easy exercise for the reader. For $A \in M_n$,

$$\operatorname{tr}(\phi(A)) = \sum_j \operatorname{tr}(V_j^* A V_j) = \operatorname{tr}\big(A \sum_j V_j V_j^*\big).$$

Hence the trace non-increasing property for ϕ is equivalent to

$$\operatorname{tr}\big(P(I - \sum_j V_j V_j^*)\big) \geq 0$$

for all positive semidefinite P, i.e., $I - \sum_j V_j V_j^* \succeq 0$. ■

Proposition 3.3. *The linear mapping $\phi : M_n \to M_d$ is*

(a) *unital (that is, $\phi(I_n) = I_d$) if and only if its Choi matrix C satisfies*

$$\sum_{j=1}^n C_{j,j} = I;$$

(b) *trace preserving if and only if its Choi matrix C satisfies*

$$(\operatorname{tr}(C_{i,j}))_{i,j=1}^n = I_n;$$

(c) *trace non-increasing for positive semidefinite matrices (i.e.,* $\operatorname{tr}(\phi(P)) \leq \operatorname{tr}(P)$ *for all positive semidefinite* P*) if and only if*

$$(\operatorname{tr}(C_{i,j}))_{i,j} \preceq I_n,$$

where C *is the Choi matrix for* ϕ*.*

Proof. Statement (a) follows from

$$\phi(I_n) = \phi\big(\sum_{j=1}^n E_{j,j}\big) = \sum_{j=1}^n C_{j,j},$$

where C is the Choi matrix for ϕ. Here $E_{i,j}$ denote the matrix units,

For (b), let $X = \sum_{i,j=1}^n \alpha_{i,j} E_{i,j}$. Then

$$\operatorname{tr}(X) = \sum_{i=1}^n \alpha_{i,i}$$
$$\operatorname{tr}(\phi(X)) = \sum_{i,j=1}^n \alpha_{i,j} \operatorname{tr}(C_{i,j}).$$

Since $\operatorname{tr}(\phi(X)) = \operatorname{tr}(X)$ for all X, this linear system yields $\operatorname{tr}(C_{i,j}) = \delta_{i,j}$ for all i, j.

Finally, for statement (c), if ϕ is trace non-increasing, choosing $X = xx^*$ a rank one matrix, $X = (x_i x_j)$, we find that

$$\sum x_i x_j \operatorname{tr}(C_{i,j}) = \operatorname{tr}(\phi(X)) \leq \operatorname{tr}(X) = \sum x_i^2.$$

Hence $I - (\operatorname{tr}(C_{i,j})) \succeq 0$. Conversely, if $I - (\operatorname{tr}(C_{i,j})) \succeq 0$, then for any positive semidefinite rank one matrix X, the computation above shows that $\operatorname{tr}(\phi(X)) \leq \operatorname{tr}(X)$. Finally, use the fact that any positive semidefinite matrix is a sum of rank one positive semidefinite matrices to complete the proof. ■

The Arveson extension theorem [Arv69] says that any cp (resp. ucp) map on an operator system extends to a cp (resp. ucp) map on the full algebra. Example 8.2 shows that a TPCP map need not extend to a TPCP map on the full algebra.

3.2. **Quantum Interpolation Problems and Semidefinite Programming.** The cp interpolation problem is formulated as follows. Given $A^1 \in \mathbb{S}_n^g$ and given A^2 in $\mathbb{S}_m^g$, does there exist a cp map $\Phi : M_n \to M_m$ such that

$$A_\ell^2 = \Phi(A_\ell^1) \quad \text{for} \quad \ell = 1, \dots, g?$$

One can require further that

(1) Φ be **unital**, or
(2) Φ be **trace preserving**, or

(3) Φ be **trace non-increasing** in the sense that $\mathrm{tr}(\Phi(P)) \le \mathrm{tr}(P)$ for positive semidefinite P.

Our solutions to these interpolation problems are formulated as concrete LMIs that can be solved with a standard semidefinite programming (SDP) solver. They are equivalent to, but stated quite differently than, the earlier results in [AG15].

Theorem 3.4. *Suppose, for $\ell = 1, \dots, g$ the matrices $A^1_\ell \in \mathbb{S}_n$ and $A^2_\ell \in \mathbb{S}_m$ are symmetric. Let $\alpha^\ell_{p,q}$ denote the (p,q) entry of A^1_ℓ.*

There exists a cp map $\Phi : M_n \to M_m$ that solves the interpolation problem

$$\Phi(A^1_\ell) = A^2_\ell, \quad \ell = 1, \cdots, g$$

if and only if the following feasibility semidefinite programming problem has a solution:

$$(C_{p,q})^n_{p,q=1} := C \succeq 0, \qquad \forall \ell = 1, \dots, g : \sum_{p,q}^{n} \alpha^\ell_{p,q} C_{p,q} = A^2_\ell, \tag{3.2}$$

for the unknown $mn \times mn$ symmetric matrix $C = (C_{p,q})^n_{p,q=1}$ consisting of $m \times m$ blocks $C_{p,q}$. Furthermore,

(1) *the map Φ is unital if and only if in addition to (3.2)*

$$\sum_{p=1}^{n} C_{p,p} = I_m; \tag{3.3}$$

(2) *the map Φ is a quantum channel if and only if in addition to (3.2)*

$$(\mathrm{tr}(C_{p,q}))_{p,q} = I_n; \tag{3.4}$$

(3) *the map Φ is a quantum operation if and only if, in addition to (3.2),*

$$(\mathrm{tr}(C_{p,q}))_{p,q} \preceq I_n. \tag{3.5}$$

In each case the constraints on C are LMIs, and the set of solutions C constitute a bounded spectrahedron.

Remark 3.5. In the unital case the obtained spectrahedron is free. Namely, for fixed $A^1 \in \mathbb{S}^g_n$, the sequence of solution sets to (3.2) and (3.3) parametrized over m is a free spectrahedron. See Proposition 4.14 for details. In the two quantum cases, for each m, the solutions $\mathcal{D}(m)$ at level m form a spectrahedron, but the sequence $\mathcal{D} = (\mathcal{D}(m))_m$ is in general not a free spectrahedron since it fails to respect direct sums. □

Proof. This interpolation result is a consequence of Theorem 3.1. Let $\mathcal{S}$ denote the span of $\{A^1_\ell\}$ and ϕ the mapping from $\mathcal{S}$ to M_m defined by $\phi(A^1_\ell) = A^2_\ell$. This mapping has a completely positive extension $\Phi : M_n \to M_m$ if and only if it has a positive semidefinite Choi matrix. The conditions on C evidently are exactly those needed to say that C is a positive semidefinite Choi matrix for ϕ.

The additional conditions in (3.3) and (3.4) (i.e., $\phi(I_n) = I_m$ and trace preservation) are clearly linear, so produce a spectrahedron in $\mathbb{S}_{mn}$. Both spectrahedra are bounded. Indeed, in each case $C_{p,p} \preceq I_m$, so $C \preceq I_{mn}$. Likewise, the additional condition in (3.5) is an LMI constraint, producing a bounded spectrahedron. ∎

We note that cp maps between subspaces of matrix algebras in the absence of positive definite elements were treated in [HKN14, Section 8]; see also [KS13, KTT13].

4. Free Spectrahedrops and Polar Duals

This section starts by recalling the definition of a free spectrahedrop as the coordinate projection of a spectrahedron. It then continues with a review of free polar duals [EW97] and their basic properties before turning to two main results, stated now without technical hypotheses. Firstly, a free convex set is, in a canonical sense, generated by a finite set (equivalently a single point) if and only if it is the polar dual of a free spectrahedron (Theorem 4.6). Secondly, the polar dual of a free spectrahedrop is again a free spectrahedrop (Theorem 4.11).

4.1. **Projections of Free Spectrahedra: Free Spectrahedrops.** Let L be a linear pencil in the variables $(x_1, \dots, x_g; y_1, \dots, y_h)$. Thus, for some d and $d \times d$ hermitian matrices $D, \Omega_1, \dots, \Omega_g, \Gamma_1, \dots, \Gamma_h$,

$$L(x, y) = D + \sum_{j=1}^{g} \Omega_j x_j + \sum_{\ell=1}^{h} \Gamma_\ell y_\ell.$$

The set

$$\operatorname{proj}_x \mathcal{D}_L(1) = \{x \in \mathbb{R}^g : \exists\, y \in \mathbb{R}^h \text{ such that } L(x, y) \succeq 0\}$$

is known as a **spectrahedral shadow** or a **semidefinite programming (SDP) representable set** [BPR13] and the representation afforded by L is an **SDP representation**. SDP representable sets are evidently convex and lie in a middle ground between LMI representable sets and general convex sets. They play an important role in convex optimization [Nem06]. In the case that $S \subseteq \mathbb{R}^g$ is closed semialgebraic and satisfies some mild additional hypothesis, it is proved in [HN10] based upon the Lasserre–Parrilo construction ([Las09, Par06]) that the convex hull of S is SDP representable.

Given a linear pencil L, let $\operatorname{proj}_x \mathcal{D}_L = (\operatorname{proj}_x \mathcal{D}_L(n))_n$ denote the free set

$$\operatorname{proj}_x \mathcal{D}_L(n) = \{X \in \mathbb{S}_n^g : \exists\, Y \in \mathbb{S}_n^h \text{ such that } L(X, Y) \succeq 0\}.$$

We call a set of the form $\operatorname{proj}_x \mathcal{D}_L$ a **free spectrahedrop** and $\mathcal{D}_L$ an **LMI lift** of $\operatorname{proj}_x \mathcal{D}_L$. Thus a free spectrahedrop is a coordinate projection of a free spectrahedron. Clearly, free spectrahedrops are matrix convex. In particular, they are closed with respect to restrictions to reducing subspaces.

Lemma 4.1 ([HKM16, §4.1])**.** *If $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_L$ is a free spectrahedrop containing $0 \in \mathbb{R}^g$ in the interior of $\mathcal{K}(1)$, then there exists a monic linear pencil $\mathfrak{L}(x, y)$ such that*

$$\mathcal{K} = \operatorname{proj}_x \mathcal{D}_{\mathfrak{L}} = \{X \in \mathbb{S}^g : \exists Y \in \mathbb{S}^h : \mathfrak{L}(X, Y) \succeq 0\}.$$

If, in addition, $\mathcal{D}_L$ is bounded, then we may further ensure $\mathcal{D}_{\mathfrak{L}}$ is bounded.

If the free spectrahedrop $\mathcal{K}$ is closed and bounded, and contains 0 in its interior, then there is a monic linear pencil $\mathfrak{L}$ such that $\mathcal{D}_{\mathfrak{L}}$ is bounded and $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_{\mathfrak{L}}$. See Theorem 4.11.

Let $p = 1 - x_1^2 - x_2^4$. It is well known that $\mathcal{D}_p(1) = \{(x, y) \in \mathbb{R}^2 : 1 - x_1^2 - x_2^4 \geq 0\}$ is a spectrahedral shadow. On the other hand, $\mathcal{D}_p(2)$ is not convex (in the usual sense) and hence $\mathcal{D}_p$ is not a spectrahedrop. Further details can be found in Example 8.3.

4.2. **Basics of Polar Duals.** By precise analogy with the classical $\mathbb{R}^g$ notion, the **free polar dual** $\mathcal{K}^\circ = (\mathcal{K}^\circ(n))_n$ of a free set $\mathcal{K} \subseteq \mathbb{S}^g$ is

$$\mathcal{K}^\circ(n) := \{A \in \mathbb{S}_n^g : \ \mathfrak{L}_A(X) = I \otimes I - \sum_j^g A_j \otimes X_j \succeq 0 \text{ for all } X \in \mathcal{K}\}.$$

Given $\varepsilon > 0$, consider the free ε ball centered at 0,

$$\mathcal{N}_\varepsilon := \{X \in \mathbb{S}^g : \|X\| \leq \varepsilon\} = \Big\{X : \varepsilon^2 I \succeq \sum_j X_j^2\Big\}.$$

It is easy to see that its polar dual is bounded. In fact,

$$\mathcal{N}_{\frac{1}{g\varepsilon}} \subseteq \mathcal{N}_\varepsilon^\circ \subseteq \mathcal{N}_{\frac{\sqrt{g}}{\varepsilon}}.$$

We say that 0 **is in the interior** of the subset $\Gamma \subseteq \mathbb{S}^g$ if Γ contains some free ε ball centered at 0.

Lemma 4.2. *Suppose $\mathcal{K} \subseteq \mathbb{S}^g$ is matrix convex. The following are equivalent.*

(i) $0 \in \mathbb{R}^g$ *is in the interior of* $\mathcal{K}(1)$*;*
(ii) $0 \in \mathbb{S}_n^g$ *is in the interior of* $\mathcal{K}(n)$ *for some* n*;*
(iii) $0 \in \mathbb{S}_n^g$ *is in the interior of* $\mathcal{K}(n)$ *for all* n*;*
(iv) 0 *is in the interior of* $\mathcal{K}$*.*

Proof. It is clear that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). Assume (ii) holds. There is an $\varepsilon > 0$ with $\mathcal{N}_\varepsilon(n) \subseteq \mathcal{K}(n)$. Since $\mathcal{K}$ is closed with respect to restriction to reducing subspaces, and

$$\mathcal{N}_\varepsilon(1) \oplus \cdots \oplus \mathcal{N}_\varepsilon(1) \subseteq \mathcal{N}_\varepsilon(n),$$

we see $\mathcal{N}_\varepsilon(1) \subseteq \mathcal{K}(1)$, i.e., (i) holds.

Now suppose (i) holds, i.e., $\mathcal{N}_\varepsilon(1) \subseteq \mathcal{K}(1)$ for some $\varepsilon > 0$. We claim that $\mathcal{N}_{\varepsilon/g^2} \subseteq \mathcal{K}$. Let $X \in \mathcal{N}_{\varepsilon/g^2}$ be arbitrary. It is clear that

$$\left[-\frac{\varepsilon}{g}, \frac{\varepsilon}{g}\right]^g \subseteq \mathcal{K}(1),$$

hence $[-\varepsilon/g, \varepsilon/g]^g \otimes I_n \subseteq \mathcal{K}(n)$. Since each X_j has norm $\leq \varepsilon/g^2$, matrix convexity of $\mathcal{K}$ implies that

$$(0, \dots, 0, gX_j, 0, \dots, 0) \in \mathcal{K}$$

and thus

$$X = \frac{1}{g}\big((gX_1, 0, \dots, 0) + \cdots + (0, \dots, 0, gX_g)\big) \in \mathcal{K}. \qquad \blacksquare$$

For the readers' convenience, the following proposition lists some properties of $\mathcal{K}^\circ$. The bipolar result of item (6) is due to [EW97]. Given Γ_α, a collection of matrix convex sets, it is readily verified that $\Gamma = (\Gamma(n))_n$ defined by $\Gamma(n) = \bigcap_\alpha \Gamma_\alpha(n)$ is again matrix convex. Likewise, if Γ is matrix convex, then so is its closure $\overline{\Gamma} = (\overline{\Gamma(n)})_n$. Given a subset $\mathcal{K}$ of $\mathbb{S}^g$, let $\mathrm{co}^{\mathrm{mat}}\mathcal{K}$ denote the intersection of all matrix convex sets containing $\mathcal{K}$. Thus, $\mathrm{co}^{\mathrm{mat}}\mathcal{K}$ is the smallest matrix convex set containing $\mathcal{K}$. Likewise, $\overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K} = \overline{\mathrm{co}^{\mathrm{mat}}\mathcal{K}}$ is the smallest closed matrix convex set containing $\mathcal{K}$. Details, and an alternate characterization of the matrix convex hull of a free set $\mathcal{K}$, can be found in [HKM16].

Proposition 4.3. *Suppose $\mathcal{K} \subseteq \mathbb{S}^g$.*

(1) *$\mathcal{K}^\circ$ is a closed matrix convex set containing* 0;
(2) *if 0 is in the interior of $\mathcal{K}$, then $\mathcal{K}^\circ$ is bounded;*
(3) *$\mathcal{K}(n) \subseteq \mathcal{K}^{\circ\circ}(n)$ for all n; that is, $\mathcal{K} \subseteq \mathcal{K}^{\circ\circ}$;*
(4) *$\mathcal{K}$ is bounded if and only if 0 is in the interior of $\mathcal{K}^\circ$;*
(5) *if there is an m such that $0 \in \mathcal{K}(m)$, then $\mathcal{K}^{\circ\circ} = \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K}$;*
(6) *if $\mathcal{K}$ is a closed matrix convex set containing 0, then $\mathcal{K} = \mathcal{K}^{\circ\circ}$; and*
(7) *if $\mathcal{K}$ is matrix convex, then $\mathcal{K}(1)^\circ = \mathcal{K}^\circ(1)$.*

Proof. Matrix convexity in (1) is straightforward.

If $\mathcal{K}$ has 0 in its interior, then there is a small free neighborhood $\mathcal{N}_\varepsilon$ of 0 inside $\mathcal{K}$. Hence $\mathcal{K}^\circ \subseteq \mathcal{N}_\varepsilon^\circ = \mathcal{N}_{1/\varepsilon}$ is bounded.

Item (3) is a tautology. Indeed, if $X \in \mathcal{K}(n)$, then we want to show $\mathfrak{L}_X(A) \succeq 0$ whenever $\mathfrak{L}_A(Y) \succeq 0$ for all Y in $\mathcal{K}$. But this follows simply from the fact that $\mathfrak{L}_X(A)$ and $\mathfrak{L}_A(X)$ are unitarily equivalent.

If $\mathcal{K}$ is bounded, then it is evident that 0 is in the interior of $\mathcal{K}^\circ$. If 0 is in the interior of $\mathcal{K}^\circ$, then, by item (2), $\mathcal{K}^{\circ\circ}$ is bounded. By item (3), $\mathcal{K} \subseteq \mathcal{K}^{\circ\circ}$ and thus $\mathcal{K}$ is bounded.

To prove (5), first note that $0 \in \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K}(m)$ and since $\overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K}(m)$ is matrix convex, $0 \in \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K}(1)$. Now suppose $W \notin \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K}$. The Effros-Winkler matricial Hahn-Banach Theorem 2.2 produces a monic linear pencil $\mathfrak{L}_A$ (with the size of A no larger than the size of W) separating W from $\overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K}$; that is, $\mathfrak{L}_A(W) \not\succeq 0$ and $\mathfrak{L}_A(X) \succeq 0$ for $X \in \mathrm{co}^{\mathrm{mat}}\mathcal{K}$. Hence $A \in \mathcal{K}^\circ$. Using the unitary equivalence of $\mathfrak{L}_W(A)$ and $\mathfrak{L}_A(W)$ it follows that $\mathfrak{L}_W(A) \not\succeq 0$, and thus $W \notin \mathcal{K}^{\circ\circ}$. Thus, $\mathcal{K}^{\circ\circ} \subseteq \overline{\mathrm{co}}^{\mathrm{mat}}\mathcal{K}$. The reverse inclusion follows from item (3).

Finally, suppose $\mathcal{K}$ is matrix convex and $y \in \mathcal{K}(1)^\circ$. Thus, $\sum y_j x_j = \langle y, x\rangle \le 1$ for all $x \in \mathcal{K}(1)$. Given $X \in \mathcal{K}(m)$ and a unit vector $v \in \mathbb{C}^m$, since $v^*Xv \in \mathcal{K}(1)$,

$$1 \ge \sum y_j v^* X_j v.$$

Hence,

$$v^*\big(I - \sum y_j X_j\big)v \ge 0$$

for all unit vectors v. So $y \in \mathcal{K}^\circ(1)$. The reverse inclusion is immediate. ∎

Corollary 4.4. *If $\mathcal{K} \subseteq \mathbb{S}^g$, then $\mathcal{K}^{\circ\circ} = \overline{\mathrm{co}}^{\mathrm{mat}}\big(\mathcal{K} \cup \{0\}\big)$. Here $0 \in \mathbb{R}^g$.*

Proof. Note that $\mathcal{K}^\circ = (\mathcal{K} \cup \{0\})^\circ$ and hence,

$$\mathcal{K}^{\circ\circ} = (\mathcal{K} \cup \{0\})^{\circ\circ}.$$

By item (5) of Proposition 4.3,

$$\overline{\mathrm{co}}^{\mathrm{mat}}\big(\mathcal{K} \cup \{0\}\big) = (\mathcal{K} \cup \{0\})^{\circ\circ}.$$ ∎

Lemma 4.5. *Suppose $\mathcal{K} \subseteq \mathbb{S}^{g+h}$, and consider its image $\mathrm{proj}\,\mathcal{K} \subseteq \mathbb{S}^g$ under the projection $\mathrm{proj} : \mathbb{S}^{g+h} \to \mathbb{S}^g$. A tuple $A \in \mathbb{S}^g$ is in $(\mathrm{proj}\,\mathcal{K})^\circ$ if and only if $(A, 0) \in \mathcal{K}^\circ$.*

Proof. Note that $A \in (\mathrm{proj}\,\mathcal{K})^\circ$ if and only if for all $X \in \mathrm{proj}\,\mathcal{K}$ we have $\mathfrak{L}_A(X) \succeq 0$ if and only if $\mathfrak{L}_{(A,0)}(X, Y) \succeq 0$ for all $X \in \mathrm{proj}\,\mathcal{K}$ and all $Y \in \mathbb{S}^h$ if and only if $\mathfrak{L}_{(A,0)}(X, Y) \succeq 0$ for all $(X, Y) \in \mathcal{K}$ if and only if $(A, 0) \in \mathcal{K}^\circ$. ∎

The polar dual of the set $\{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1^2 - x_2^4 \ge 0\}$ is computed and seen not to be a spectrahedron in Example 8.4.

4.3. **Polar Duals of Free Spectrahedra.** The next theorem completely characterizes finitely generated matrix convex sets $\mathcal{K}$ containing 0 in their interior. Namely, such sets are exactly polar duals of bounded free spectrahedra.

Theorem 4.6. *Suppose $\mathcal{K}$ is a closed matrix convex set with 0 in its interior. If there is an $\Omega \in \mathcal{K}$ such that for each $X \in \mathcal{K}$ there is a $\mu \in \mathbb{N}$ and an isometry V such that*

$$X_j = V^*(I_\mu \otimes \Omega_j)V, \tag{4.1}$$

then

$$\mathcal{K}^\circ = \mathcal{D}_{\mathfrak{L}_\Omega}, \tag{4.2}$$

where $\mathfrak{L}_\Omega$ is the monic linear pencil $\mathfrak{L}_\Omega(x) = I - \sum \Omega_j x_j$.

Conversely, if there is an Ω such that (4.2) holds, then $\Omega \in \mathcal{K}$ and, for each $X \in \mathcal{K}$, there is an isometry V such that (4.1) holds.

A variant of Theorem 4.6 in which the condition that 0 is in the interior of $\mathcal{K}$ is replaced by the weaker hypothesis that 0 is merely in $\mathcal{K}$ and of course with a slightly weaker conclusion, is stated as a separate result, Proposition 4.9 below.

Lemma 4.7. *Suppose $\Omega \in \mathbb{S}_d^g$ and consider the monic linear pencil $\mathfrak{L}_\Omega = I - \sum \Omega_j x_j$.*

(1) *Let $\Omega' = \Omega \oplus 0$ where $0 \in \mathbb{S}_d^g$. A tuple $X \in \mathbb{S}^g$ is in $\mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ if and only if there is an isometry V such that*
$$X_j = V^*(I \otimes \Omega_j')V.$$
(2) *If $\mathcal{D}_{\mathfrak{L}_\Omega}$ is bounded, then $X \in \mathbb{S}^g$ is in $\mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ if and only if there is an isometry V such that equation (4.1) holds.*

Remark 4.8. As an alternate of (2), $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ if and only if there exists a contraction V such that equation (4.1) holds. □

Proof. Note that $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$ if and only if $\mathcal{D}_{\mathfrak{L}_\Omega} \subseteq \mathcal{D}_{\mathfrak{L}_X}$. Thus if $\mathcal{D}_{\mathfrak{L}_\Omega}$ is bounded, then the result follows directly from the last part of Theorem 2.4. On the other hand, if X has the representation of equation (4.1), then evidently $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$.

If $\mathcal{D}_{\mathfrak{L}_\Omega}$ is not necessarily bounded and $X \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ(m)$, then, by Theorem 2.4,

$$X = \sum_{j=1}^{\mu} V_j^* \Omega V_j,$$

for some μ and operators $V_j : \mathbb{C}^m \to \mathbb{C}^n$ such that

$$I - \sum V_j^* V_j \succeq 0.$$

There is a $\nu > \mu$ and $m \times n$ matrices $V_{\mu+1}, \dots, V_\nu$ such that

$$\sum_{j=1}^{\nu} V_j^* V_j = I.$$

For $1 \le j \le \mu$, let

$$W_j = \begin{pmatrix} V_j \\ 0 \end{pmatrix}$$

and similarly for $\mu < j \le \nu$, let $W_j = \begin{pmatrix} 0 & V_j^* \end{pmatrix}^*$. With this choice of W, note that $\sum W_j^* W_j = I_m$ and

$$\sum W_j^* \Omega_j' W_j = \sum W_j^* (\Omega_j \oplus 0) W_j = \sum_{j=1}^{\nu} V_j^* \Omega_j V_j = X_j. \tag{4.3}$$

If X has the representation as in equation (4.3) and $\mathfrak{L}_\Omega(Y) \succeq 0$, then

$$\mathfrak{L}_X(Y) = \sum_j (W_j \otimes I)^* \mathfrak{L}_{\Omega'}(Y)(W_j \otimes I).$$

On the other hand,

$$\mathfrak{L}_{\Omega'}(Y) = \mathfrak{L}_\Omega(Y) \oplus I \succeq 0.$$

Hence $X \in \mathcal{D}^\circ_{\mathfrak{L}_\Omega}$. ■

Proof of Theorem 4.6. Suppose first (4.2) holds for some $\Omega \in \mathbb{S}^g_n$. Since $\mathcal{D}^\circ_{\mathfrak{L}_\Omega} = \mathcal{K}$ and evidently $\Omega \in \mathcal{D}^\circ_{\mathfrak{L}_\Omega}$, it follows that $\Omega \in \mathcal{K}$. Since 0 is assumed to be in the interior of $\mathcal{K}$, its polar dual $\mathcal{K}^\circ = \mathcal{D}_{\mathfrak{L}_\Omega}$ is bounded by Proposition 4.3. Thus, if $X \in \mathcal{K} = \mathcal{D}^\circ_{\mathfrak{L}_\Omega}$, then by Lemma 4.7, X has a representation as in equation (4.1).

Conversely, assume that $\Omega \in \mathcal{K}$ has the property that any $X \in \mathcal{K}$ can be represented as in (4.1). Consider the matrix convex set

$$\Gamma = \{V^*(I_\mu \otimes \Omega)V : \mu \in \mathbb{N},\ V^*V = I\}.$$

Since $\Omega \in \mathcal{K}$, it follows that $\Gamma \subseteq \mathcal{K}$. On the other hand, the hypothesis is that $\mathcal{K} \subseteq \Gamma$. Hence $\mathcal{K} = \Gamma$. Now, for $\mathfrak{L}_X$ a monic linear pencil, $\mathfrak{L}_X(\Omega) \succeq 0$ if and only if

$$\mathfrak{L}_X\big(V^*(I_\mu \otimes \Omega)V\big) = (V \otimes I)^*\, \mathfrak{L}_X(I_\mu \otimes \Omega)\, (V \otimes I) \succeq 0$$

over all choices of μ and isometries V. Thus, $X \in \mathcal{K}^\circ$ if and only if $\mathfrak{L}_X(\Omega) \succeq 0$. On the other hand, $\mathfrak{L}_X(\Omega)$ is unitarily equivalent to $\mathfrak{L}_\Omega(X)$. Thus $X \in \mathcal{K}^\circ$ if and only if $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$. ■

Proposition 4.9. *Suppose $\mathcal{K}$ is a closed matrix convex set containing* 0*. If there is a $\Omega \in \mathcal{K}$ such that for each $X \in \mathcal{K}$ there is a $\mu \in \mathbb{N}$ and an isometry V such that*

$$X_j = V^*(I_\mu \otimes \Omega_j')V, \tag{4.4}$$

then

$$\mathcal{K}^\circ = \mathcal{D}_{\mathfrak{L}_\Omega}, \tag{4.5}$$

where $\mathfrak{L}_\Omega$ *is the monic linear pencil* $\mathfrak{L}_\Omega(x) = I - \sum \Omega_j x_j$*. Here* $\Omega' = \Omega \oplus 0$ *as in Lemma 4.7.*

Conversely, if there is an Ω *such that equation (4.5) holds, then* $\Omega \in \mathcal{K}$ *and for each* $X \in \mathcal{K}$ *there is an isometry* V *such that equation (4.4) holds.*

Proof. Suppose first (4.5) holds for some $\Omega \in \mathbb{S}_n^g$. Since $\mathcal{D}_{\mathfrak{L}_\Omega}^\circ = \mathcal{K}$ and evidently $\Omega \in \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$, it follows that $\Omega \in \mathcal{K}$. By Lemma 4.7, if $X \in \mathcal{K} = \mathcal{D}_{\mathfrak{L}_\Omega}^\circ$, then X has a representation as in equation (4.4).

Conversely, assume that Ω has the property that any $X \in \mathcal{K}$ can be represented as in (4.4). Consider the matrix convex set

$$\Gamma = \big\{V^*(I_\mu \otimes \Omega')V : \mu \in \mathbb{N},\, V^*V = I\big\}.$$

Since $0, \Omega \in \mathcal{K}$, it follows that $\Omega' = \Omega \oplus 0 \in \mathcal{K}$ and thus $\Gamma \subseteq \mathcal{K}$. On the other hand, the hypothesis is that $\mathcal{K} \subseteq \Gamma$. Hence $\mathcal{K} = \Gamma$. Now, for $\mathfrak{L}_X$ a monic linear pencil, $\mathfrak{L}_X(\Omega) \succeq 0$ if and only if $\mathfrak{L}_X(\Omega') \succeq 0$ if and only if

$$\mathfrak{L}_X\big(V^*(I_\mu \otimes \Omega')V\big) = (V \otimes I)^*\, \mathfrak{L}_X(I_\mu \otimes \Omega')\,(V \otimes I) \succeq 0$$

over all choices of μ and isometries V. Thus, $X \in \mathcal{K}^\circ$ if and only if $\mathfrak{L}_X(\Omega) \succeq 0$. On the other hand, $\mathfrak{L}_X(\Omega)$ is unitarily equivalent to $\mathfrak{L}_\Omega(X)$. Thus $X \in \mathcal{K}^\circ$ if and only if $X \in \mathcal{D}_{\mathfrak{L}_\Omega}$. ∎

Remark 4.10.

(1) For perspective, in the classical (not free) situation when $g = 2$, it is known that $K \subseteq \mathbb{R}^2$ has an LMI representation if and only if K° is a numerical range [Hen10, HS12]. It is well known that the polar dual of a spectrahedron is not necessarily a spectrahedron. This is the case even in $\mathbb{R}^g$, cf. [BPR13, Section 5] or Example 8.4.

(2) In the commutative case the polar dual of a spectrahedron (more generally, of a spectrahedral shadow) is a spectrahedral shadow, see [GN11] or [BPR13, Chapter 5].

(3) It turns out that the Ω in Theorem 4.6 can be taken to be an extreme point of $\mathcal{K}$ in a very strong free sense. We refer to [Far00, Kls14, WW99] for more on matrix extreme points. □

4.4. **The Polar Dual of a Free Spectrahedrop is a Free Spectrahedrop.** This subsection contains a duality result for free spectrahedrops (Theorem 4.11) and several of its corollaries.

It can happen that $\mathcal{D}_\mathfrak{L}$ is not bounded, but the projection $\mathcal{K} = \mathrm{proj}_x \mathcal{D}_\mathfrak{L}$ is. Corollary 4.13 says that a free spectrahedrop is closed and bounded if and only if it is the projection of some bounded free spectrahedron. For expositional purposes, it is convenient to introduce the following terminology. A free spectrahedrop $\mathcal{K}$ is called **stratospherically bounded** if there is a linear pencil $\mathfrak{L}$ such that $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_{\mathfrak{L}}$, and $\mathcal{D}_{\mathfrak{L}}$ is bounded.

Theorem 4.11. *Suppose $\mathcal{K}$ is a closed matrix convex set containing* 0.

(1) *If $\mathcal{K}$ is a free spectrahedrop and* 0 *is in the interior of $\mathcal{K}$, then $\mathcal{K}^\circ$ is a stratospherically bounded free spectrahedrop.*
(2) *If $\mathcal{K}^\circ$ is a free spectrahedrop containing* 0 *in its interior, then $\mathcal{K}$ is a stratospherically bounded free spectrahedrop.*

In particular, if $\mathcal{K}$ is a bounded free spectrahedrop with 0 *in its interior, then both $\mathcal{K}$ and $\mathcal{K}^\circ$ are stratospherically bounded free spectrahedrops* (*with* 0 *in their interiors*)*.*

Before presenting the proof of the theorem we state a few corollaries and Proposition 4.14 needed in the proof.

Corollary 4.12. *Given $\Omega \in \mathbb{S}_d^g$, let $\mathfrak{L}_\Omega$ denote the corresponding monic linear pencil. The free set $\mathcal{D}^\circ_{\mathfrak{L}_\Omega}$ is a stratospherically bounded free spectrahedrop.*

Proof. The set $\mathcal{D}_{\mathfrak{L}_\Omega}$ is (trivially) a free spectrahedrop with 0 in its interior. Thus, by Theorem 4.11, $\mathcal{D}^\circ_{\mathfrak{L}_\Omega}$ is a stratospherically bounded free spectrahedrop. ∎

Corollary 4.13. *A free spectrahedrop $\mathcal{K} \subseteq \mathbb{S}^g$ is closed and bounded if and only if it is stratospherically bounded.*

Proof. Implication ($\Leftarrow$) is obvious. ($\Rightarrow$) Let us first reduce to the case where $\mathcal{K}(1)$ has nonempty interior. If $\mathcal{K}(1)$ has empty interior, then it is contained in a proper affine hyperplane $\{\ell = 0\}$ of $\mathbb{R}^g$. Here ℓ is an affine linear functional. In this case we can solve for one of the variables thereby reducing the codimension of $\mathcal{K}(1)$. (Note that $\ell = 0$ on $\mathcal{K}(1)$ implies $\ell = 0$ on $\mathcal{K}$, cf. [HKM16, Lemma 3.3].)

Now let $\hat{x} \in \mathbb{R}^g$ be an interior point of $\mathcal{K}(1)$. Consider the translation

$$\tilde{\mathcal{K}} = \mathcal{K} - \hat{x} = \bigcup_{n\in\mathbb{N}} \big\{X - \hat{x} I_n : X \in \mathcal{K}(n)\big\}. \tag{4.6}$$

Clearly, $\tilde{\mathcal{K}}$ is a bounded free spectrahedrop with 0 in its interior. Hence by Theorem 4.11, it is stratospherically bounded. Translating back, we see $\mathcal{K}$ is a stratospherically bounded free spectrahedrop. ∎

Each stratospherically bounded free spectrahedrop is closed, since it is the projection of a (levelwise) compact spectrahedron. Hence a bounded free spectrahedrop $\mathcal{K}$ will not be stratospherically bounded if it is not closed. For a concrete example, consider the linear pencil

$$L(x,y) = \begin{pmatrix} 2-x & 1 \\ 1 & 2-y \end{pmatrix} \oplus \big(2+x\big),$$

and let $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_L$. Thus

$$\mathcal{K} = \big\{X \in \mathbb{S} : -2 \preceq X \prec 2\big\}$$

is bounded but not closed.

Proposition 4.14. *Given $\Omega \in \mathbb{S}_d^g$ and $\Gamma \in \mathbb{S}_d^h$, the sequence $\mathcal{K} = (\mathcal{K}(n))_n$,*

$$\mathcal{K}(n) = \big\{A \in \mathbb{S}_n^g : A = V^*(I_\mu \otimes \Omega)V, \quad 0 = V^*(I_\mu \otimes \Gamma)V \text{ for some isometry } V \text{ and } \mu \le nd\big\},$$

is a stratospherically bounded free spectrahedrop

Let $\mathfrak{L}_{(\Omega,\Gamma)}$ denote the monic linear pencil corresponding to (Ω,Γ). The free set

$$\mathcal{C} = \big\{A : (A,0) \in \mathcal{D}^{\circ}_{\mathfrak{L}_{(\Omega,\Gamma)}}\big\}$$

is a stratospherically bounded free spectrahedrop.

Proof. Let $\mathcal{S}$ denote the span of $\{I, \Omega_1, \dots, \Omega_g, \ \Gamma_1, \dots, \Gamma_h\}$. Thus $\mathcal{S}$ is an operator system in M_d (the fact that $I \in \mathcal{S}$ implies $\mathcal{S}$ contains a positive definite element). Let

$$\phi : \mathcal{S} \to M_n$$

denote the linear mapping determined by

$$I \mapsto I, \quad \Omega_j \mapsto A_j, \quad \text{and} \quad \Gamma_\ell \mapsto 0.$$

Observe that, by Theorem 3.1, $A \in \mathcal{K}(n)$ if and only if ϕ has a completely positive extension $\Phi : M_d \to M_n$. Theorem 3.4 expresses existence of such a Φ as a (unital) cp interpolation problem in terms of a free spectrahedron. For the reader's convenience we write out this critical LMI explicitly. Let ω_{pq}^j denote the (p,q)-entry of Ω_j and γ_{pq}^ℓ the (p,q) entry of Γ_ℓ. For a complex matrix (or scalar) Q we use $\hat{Q}$ to denote its real part and $i\check{Q}$ for its imaginary part. Thus $\hat{Q} = \frac{1}{2}(Q+Q^*)$ and $\check{Q} = -\frac{i}{2}(Q-Q^*)$.

Now A is in $\mathcal{K}(n)$ if and only if there exists $n \times n$ matrices $C_{p,q}$ satisfying

(i) $\sum_{p,q=1}^d E_{p,q} \otimes C_{p,q} \succeq 0$;
(ii) $\sum_{p=1}^d C_{p,p} = I_n$;
(iii) $\sum_{p,q=1}^d \omega_{pq}^\ell C_{p,q} = A_\ell$ for $\ell = 1, \dots, g$; and
(iv) $\sum_{p,q=1}^d \gamma_{pq}^\ell C_{p,q} = 0$ for $\ell = 1, \dots, h$.

Since the $C_{p,q}$ for $p \neq q$ are not hermitian matrices, we rewrite the system (i) – (iv) into one with hermitian unknowns $\hat{C}_{p,q}$ and $\check{C}_{p,q}$. Property (i) transforms into

$$\sum_{p,q} (\hat{E}_{p,q} \otimes \hat{C}_{p,q} - \check{E}_{p,q} \otimes \check{C}_{p,q}) + i(\hat{E}_{p,q} \otimes \check{C}_{p,q} + \check{E}_{p,q} \otimes \hat{C}_{p,q}) \succeq 0,$$

i.e.,

$$\begin{aligned}&\sum_{p,q}\hat{E}_{p,q}\otimes\hat{C}_{p,q}-\check{E}_{p,q}\otimes\check{C}_{p,q}\succeq 0\\&\sum_{p,q}\hat{E}_{p,q}\otimes\check{C}_{p,q}+\check{E}_{p,q}\otimes\hat{C}_{p,q}=0.\end{aligned}\tag{4.7}$$

In item (ii) we simply replace $C_{p,p}$ with $\hat{C}_{p,p}$,

$$\sum_{p=1}^{d}\hat{C}_{p,p}=I_n.\tag{4.8}$$

Properties (iii) and (iv) are handled similarly to (i). Thus

$$\begin{aligned}&\sum_{p,q}\hat{\omega}^{\ell}_{pq}\hat{C}_{p,q}-\check{\omega}^{\ell}_{pq}\check{C}_{p,q}=A_\ell\\&\sum_{p,q}\hat{\omega}^{\ell}_{pq}\check{C}_{p,q}+\check{\omega}^{\ell}_{pq}\hat{C}_{p,q}=0,\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}&\sum_{p,q}\hat{\gamma}^{\ell}_{pq}\hat{C}_{p,q}-\check{\gamma}^{\ell}_{pq}\check{C}_{p,q}=A_\ell\\&\sum_{p,q}\hat{\gamma}^{\ell}_{pq}\check{C}_{p,q}+\check{\gamma}^{\ell}_{pq}\hat{C}_{p,q}=0.\end{aligned}\tag{4.10}$$

Thus, $\mathcal{K}$ is the linear image of the explicitly constructed free spectrahedron (in the variables $\hat{C}_{p,q}$ and $\check{C}_{p,q}$) given by (4.7) – (4.10). Moreover, items (i) and (ii) together imply $0\preceq C_{p,p}\preceq I$. It now follows that $\|\hat{C}_{p,q}\|,\|\check{C}_{p,q}\|\leq 1$ for all p,q. Thus, this free spectrahedron is bounded. It is now routine to verify that $\mathcal{K}$ is a projection of a bounded free spectrahedron and is thus a stratospherically bounded free spectrahedrop.

Let $\Omega'=\Omega\oplus 0$ and $\Gamma'=\Gamma\oplus 0$ where $0\in\mathbb{S}^g_d$. Note that

$$\mathcal{D}_{\mathfrak{L}_{(\Omega,\Gamma)}}=\mathcal{D}_{\mathfrak{L}_{(\Omega',\Gamma')}}.$$

By Lemma 4.7, $(A,0)\in\mathcal{D}^{\circ}_{\mathfrak{L}_{(\Omega,\Gamma)}}$ if and only if

$$A\in\big\{B:\exists\,\mu\in\mathbb{N}\text{ and an isometry }V\text{ such that }B=V^*(I_\mu\otimes\Omega')V,\ 0=V^*(I_\mu\otimes\Gamma')V\big\}.$$

By the first part of the proposition (applied to the tuple (Ω',Γ')), it follows that $\mathcal{C}$ is a stratospherically bounded free spectrahedrop. ■

We are now ready to give the proof of Theorem 4.11.

Proof of Theorem 4.11. Suppose $\mathcal{K}$ is a free spectrahedrop with 0 in its interior. By Lemma 4.1, there exists (Ω, Γ), a pair of tuples of matrices, such that

$$\mathcal{K} = \{X : \exists Y \text{ such that } (X,Y) \in \mathcal{D}_{\mathfrak{L}_{(\Omega,\Gamma)}}\} = \operatorname{proj}_x \mathcal{D}_{\mathfrak{L}_{(\Omega,\Gamma)}},$$

where $\mathfrak{L}_{(\Omega,\Gamma)}(x,y)$ is the monic linear pencil associated to (Ω,Γ).

Observe, $A \in \mathcal{K}^\circ$ if and only if for each $X \in \mathcal{K}$,

$$\mathfrak{L}_A(X) \succeq 0.$$

Thus, $A \in \mathcal{K}^\circ$ if and only if

$$\mathfrak{L}_{(A,0)}(X,Y) \succeq 0$$

for all $(X,Y) \in \mathcal{D}_{\mathfrak{L}_{(\Omega,\Gamma)}}$ if and only if

$$(A,0) \in \mathcal{D}^\circ_{\mathfrak{L}_{(\Omega,\Gamma)}}.$$

Summarizing, $A \in \mathcal{K}^\circ$ if and only if $(A,0) \in \mathcal{D}^\circ_{\mathfrak{L}_{(\Omega,\Gamma)}}$. Thus, by the second part of Proposition 4.14, $\mathcal{K}^\circ$ is a stratospherically bounded free spectrahedrop.

Because $\mathcal{K}$ contains 0 and is a closed matrix convex set, $\mathcal{K}^{\circ\circ} = \mathcal{K}$ by Proposition 4.3. Thus, if $\mathcal{K}^\circ$ is a free spectrahedrop with 0 in its interior, then, by what has already been proved, $\mathcal{K}^{\circ\circ} = \mathcal{K}$ is a stratospherically bounded free spectrahedrop.

Finally, if $\mathcal{K}$ is a bounded free spectrahedrop with 0 in its interior, then $\mathcal{K}^\circ$ contains 0 in its interior and is a stratospherically bounded free spectrahedrop. Hence, $\mathcal{K} = \mathcal{K}^{\circ\circ}$ is also a stratospherically bounded free spectrahedrop. ■

Note that the polar dual of a free spectrahedron is a matrix convex set generated by a singleton (Theorem 4.6) and is a free spectrahedrop by the above corollary.

Corollary 4.15. *Let $\mathfrak{L}$ denote the monic linear pencil associated with (Ω,Γ). If $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_{\mathfrak{L}}$ is bounded, then its polar dual is the free set given by*

$$\begin{aligned}\mathcal{K}^\circ(n) &= \{A \in \mathbb{S}^g_n : (A,0) \in \mathcal{D}^\circ_{\mathfrak{L}}\}\\ &= \{A \in \mathbb{S}^g_n : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ s.t. } A = V^*(I_\mu \otimes \Omega)V, \quad 0 = V^*(I_\mu \otimes \Gamma)V\}.\end{aligned}$$

Whether or not $\mathcal{K}$ is bounded, its polar dual is the free set

$$\begin{aligned}\mathcal{K}^\circ(n) &= \{A \in \mathbb{S}^g_n : (A,0) \in \mathcal{D}^\circ_{\mathfrak{L}}\}\\ &= \{A \in \mathbb{S}^g_n : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ s.t. } A = V^*(I_\mu \otimes \Omega')V, \quad 0 = V^*(I_\mu \otimes \Gamma')V\},\end{aligned}$$

where $\Omega' = \Omega \oplus 0$ and $\Gamma' = \Gamma \oplus 0$, as in Lemma 4.7.

Proof. From the proof of Theorem 4.11, $\mathcal{K}^\circ = \{A : (A,0) \in \mathcal{D}^\circ_{\mathfrak{L}}\}$. Writing $\mathfrak{L} = \mathfrak{L}_\Delta$, by Lemma 4.7 (whether or not $\mathcal{D}_{\mathfrak{L}}$ is bounded),

$$\mathcal{D}^\circ_{\mathfrak{L}} = \{X : \exists \mu \in \mathbb{N} \text{ and an isometry } V \text{ such that } X = V^*(I_\mu \otimes \Delta')V\}.$$

To obtain the stronger conclusion under the assumption that $\mathcal{K}$ is bounded, an additional argument along the lines of [HKM13a, §3.1] is needed; see also [Za+, Theorem 2.12]. Let $(A, 0) \in \mathcal{D}_{\mathcal{L}}^{\circ}$. We need to show that the unital linear map

$$\tau : \operatorname{span}\{I, \Omega_1, \dots, \Omega_g, \Gamma_1, \dots, \Gamma_h\} \to \operatorname{span}\{I, A_1, \dots, A_g\}$$
$$\Gamma_j \mapsto A_j, \quad \Gamma_k \mapsto 0$$

is completely positive. Assume

$$I \otimes X_0 + \sum_j \Omega_j \otimes X_j + \sum_k \Gamma_k \otimes Y_k \succeq 0 \tag{4.11}$$

for some hermitian $X_0, \dots, X_g, Y_1, \dots, Y_h$. In particular, $X_0 = X_0^*$. We claim that $X_0 \succeq 0$. Suppose $X_0 \not\succeq 0$. By compressing we may reduce to $X_0 \prec 0$. From (4.11) it now follows that

$$I \otimes tX_0 + \sum_j \Omega_j \otimes tX_j + \sum_k \Gamma_k \otimes tY_k \succeq 0$$

for every $t > 0$. Since $tX_0 \prec 0$, this implies

$$I \otimes I + \sum_j \Omega_j \otimes tX_j + \sum_k \Gamma_k \otimes tY_k \succeq 0,$$

whence

$$(tX_1, \dots, tX_g) \in \mathcal{K}$$

for every $t > 0$. If $(X_1, \dots, X_g) \neq 0$ this contradicts the boundedness of $\mathcal{K}$. Otherwise $(X_1, \dots, X_g) = 0$, and

$$\sum_k \Gamma_k \otimes Y_k \succ -I \otimes X_0 \succ 0.$$

Hence for any tuple $(X_1, \dots, X_g)$ of hermitian matrices of the same size as the Y_k,

$$I \otimes I + \sum_j \Omega_j \otimes X_j + \sum_k \Gamma_k \otimes tY_k \succeq 0$$

for some $t > 0$. This again contradicts the boundedness of $\mathcal{K}$. Thus $X_0 \succeq 0$.

By adding a small multiple of the identity to X_0 there is no harm in assuming $X_0 \succ 0$. Hence multiplying (4.11) by $X_0^{-\frac{1}{2}}$ from the left and right yields the tuple $X_0^{-\frac{1}{2}}(X_1, \dots, X_g, Y_1, \dots, Y_h)X_0^{-\frac{1}{2}} \in \mathcal{D}_{\mathcal{L}}$. Since $(A, 0) \in \mathcal{D}_{\mathcal{L}}^{\circ}$,

$$\mathcal{L}_{(A,0)}\big(X_0^{-\frac{1}{2}}(X_1, \dots, X_g, Y_1, \dots, Y_h)X_0^{-\frac{1}{2}}\big) = I \otimes I + \sum_k A_k \otimes X_0^{-\frac{1}{2}} X_k X_0^{-\frac{1}{2}} \succeq 0.$$

Multiplying with $X_0^{\frac{1}{2}}$ on the left and right gives

$$I \otimes X_0 + \sum_k A_k \otimes X_k \succeq 0,$$

as required. ■

To each subset $\Gamma \subseteq \mathbb{S}^g$ we associate its interior $\operatorname{int}\Gamma = (\operatorname{int}\Gamma(n))_{n\in\mathbb{N}}$, where $\operatorname{int}\Gamma(n)$ denotes the interior of $\Gamma(n)$ in the Euclidean space $\mathbb{S}^g_n$. We say Γ has nonempty interior if there is n with $\operatorname{int}\Gamma(n) \neq \varnothing$.

Corollary 4.16. *If $\mathcal{K} \subseteq \mathbb{S}^g$ is a bounded free spectrahedrop, then $\overline{\mathcal{K}}$ is a free spectrahedrop.*

Proof. As in the proof of Corollary 4.13, we may assume the interior of $\mathcal{K}$ is nonempty. This implies there is a $\hat{x} \in \mathbb{R}^g$ in the interior of $\mathcal{K}(1)$. Consider the translation $\tilde{\mathcal{K}} = \mathcal{K} - \hat{x}$ as in (4.6). This is a free spectrahedrop containing 0 in its interior. Hence its closure $\overline{\tilde{\mathcal{K}}} = \tilde{\mathcal{K}}^{\circ\circ}$ is a free spectrahedrop by Theorem 4.11. Thus so is $\overline{\mathcal{K}} = \overline{\tilde{\mathcal{K}}} + \hat{x}$. ∎

Corollary 4.17. *If $\mathcal{K} \subseteq \mathbb{S}^g$ is a free spectrahedrop with nonempty interior, then $\mathcal{K}^\circ$ is a free spectrahedrop.*

Proof. We are assuming $\mathcal{K} = \operatorname{proj} \mathcal{D}_{L_A}$. Applying Lemma 4.5 gives

$$\mathcal{K}^\circ = \{B \in \mathbb{S}^g : (B, 0) \in \mathcal{D}^\circ_{L_A}\}.$$

So if we prove $\mathcal{D}^\circ_{L_A}$ is a free spectrahedrop, then

$$\mathcal{K}^\circ = \operatorname{proj}\left(\mathcal{D}^\circ_{L_A} \bigcap \left(\mathbb{S}^g \otimes \{0\}^h\right)\right)$$

is the intersection of two free spectrahedrops, so is a free spectrahedrop. Thus without loss of generality we may take $\mathcal{K} = \mathcal{D}_{L_A}$ and proceed. We will demonstrate the corollary holds in this case as a consequence of the Convex Positivstellensatz, Theorem 2.4.

Suppose $\hat{x} \in \mathbb{R}^g$ is in the interior of $\mathcal{D}_{L_A}$. Without loss of generality we may assume

$$L_0 = L_A(\hat{x}) \succ 0,$$

cf. [HV07]. Define the monic linear pencil

$$\begin{aligned} \mathfrak{L}(y) &= L_0^{-\frac{1}{2}} L(y + \hat{x}) L_0^{-\frac{1}{2}} \\ &= I + \sum_{j=1}^{g} L_0^{-\frac{1}{2}} A_j L_0^{-\frac{1}{2}} y_j. \end{aligned}$$

By definition, a tuple $\Omega \in \mathbb{S}^g$ is in $\mathcal{D}^\circ_{L_A}$ if and only if $\mathcal{D}_{L_A} \subseteq \mathcal{D}_{\mathfrak{L}_\Omega}$. Equivalently, with

$$L(y) = \big(I + \sum_{j=1}^{g} \Omega_j \hat{x}_j\big) + \sum_{j=1}^{g} \Omega_j y_j,$$

$\mathcal{D}_{\mathfrak{L}} \subseteq \mathcal{D}_L$. By Theorem 2.4, there is a $S \succeq 0$ and matrices V_k with

$$L(y) = S + \sum_k V_k^* \mathfrak{L}(y) V_k.$$

That is,

$$\big(I+\sum_{j=1}^{g}\Omega_j\hat{x}_j\big)\succeq\sum_k V_k^*V_k,\quad\text{and}\quad\Omega_j=\sum_k V_k^*L_0^{-\frac12}A_jL_0^{-\frac12}V_k,\ j=1,\dots,g.$$

Equivalently, there is a completely positive mapping Φ satisfying

$$\Phi\big(L_0^{-\frac12}A_jL_0^{-\frac12}\big)=\Omega_j,\ j=1,\dots,g \tag{4.12}$$

$$\Phi(I)\preceq I+\sum_j\Omega_j\hat{x}_j. \tag{4.13}$$

As in Theorem 3.4 we now employ the Choi matrix C. Conditions (4.12) translate into linear constraints on the block entries C_{ij} of C. Similarly, (4.13) transforms into an LMI constraint on the entries of C. Thus C provides a free spectrahedral lift of $\mathcal{D}_{L_A}^\circ$. ■

4.5. **The Free Convex Hull of a Union.** In this subsection we prove that the convex hull of a union of free spectrahedrops is again a free spectrahedrop.

Proposition 4.18. *If $\mathcal{S}_1,\dots,\mathcal{S}_t\subseteq\mathbb{S}^g$ are stratospherically bounded free spectrahedrops and each contains 0 in its interior, then $\overline{\text{co}}^{\text{mat}}(\mathcal{S}_1\cup\cdots\cup\mathcal{S}_t)$ is a stratospherically bounded free spectrahedrop with 0 in its interior.*

Proof. Let $\mathcal{K}=\mathcal{S}_1\cup\cdots\cup\mathcal{S}_t$. Then

$$\mathcal{K}^\circ=\mathcal{S}_1^\circ\cap\cdots\cap\mathcal{S}_t^\circ.$$

Since each $\mathcal{S}_j$ is a stratospherically bounded free spectrahedrop with 0 in interior, the same holds true for $\mathcal{S}_j^\circ$ by Theorem 4.11. It is clear that these properties are preserved under a finite intersection, so $\mathcal{K}^\circ$ is again a stratospherically bounded free spectrahedrop with 0 in its interior. Hence, by Proposition 4.3),

$$(\mathcal{K}^\circ)^\circ=\overline{\text{co}}^{\text{mat}}\mathcal{K}$$

is a stratospherically bounded free spectrahedrop with 0 in its interior by Theorem 4.11. ■

5. Positivstellensatz for Free Spectrahedrops

This section focuses on polynomials positive on a free spectrahedrop, extending our Positivstellensatz for free polynomials positive on free spectrahedra, Theorem 2.4, to a Convex Positivstellensatz for free spectrahedrops, Theorem 5.1.

Let $\mathfrak{L}$ denote a monic linear pencil of size d,

$$\mathfrak{L}(x,y)=I+\sum_{j=1}^{g}\Omega_jx_j+\sum_{k=1}^{h}\Gamma_ky_k, \tag{5.1}$$

and let $\mathcal{K} = \operatorname{proj}_x \mathcal{D}_{\mathfrak{L}}$. If μ is a positive integer and $q_\ell \in \mathbb{C}^{d\times\mu}\langle x\rangle$ (and so are polynomials in the x variables only), and if $\sum_\ell q_\ell(x)^*\Gamma_k q_\ell(x) = 0$ for each k, then

$$\sum_\ell q_\ell^*(x)\mathfrak{L}(x,y)q_\ell(x)$$

is a polynomial in the x variables and is thus in $\mathbb{C}^{\mu\times\mu}\langle x\rangle$. For positive integers μ and r we define the **truncated quadratic module** in $\mathbb{C}^{\mu\times\mu}\langle x\rangle$ associated to $\mathfrak{L}$ and $\mathcal{K}$ by

$$M_x^\mu(\mathfrak{L})_r = \Big\{ \sum_\ell q_\ell^*\mathfrak{L}q_\ell + \sigma \ : \ q_\ell \in \mathbb{C}^{d\times\mu}\langle x\rangle_r,\ \sigma \in \Sigma_r^\mu\langle x\rangle,\ \sum_\ell q_\ell^*\Gamma_k q_\ell = 0 \text{ for all } k\Big\}.$$

Here $\Sigma_r^\mu = \Sigma_r^\mu\langle x\rangle$ denotes the set of all sums of hermitian squares h^*h for $h \in \mathbb{C}^{\mu\times\mu}\langle x\rangle_r$. It is easy to see $M_x^\mu(\mathfrak{L}) = \bigcup_{r\in\mathbb{N}} M_x^\mu(\mathfrak{L})_r$ is a quadratic module in $\mathbb{C}^{\mu\times\mu}\langle x\rangle$.

The main result of this section is the following Positivstellensatz:

Theorem 5.1. *A symmetric polynomial $p \in \mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+1}$ is positive semidefinite on $\mathcal{K}$ if and only if $p \in M_x^\mu(\mathfrak{L})_r$.*

Remark 5.2. Several remarks are in order.

(1) In case there are no y-variables in $\mathfrak{L}$, Theorem 5.1 reduces to the Convex Positivstellensatz of [HKM12].
(2) If $r = 0$, i.e., p is linear, then Theorem 5.1 reduces to Corollary 4.15.
(3) A Positivstellensatz for commutative polynomials *strictly positive* on spectrahedrops was established by Gouveia and Netzer in [GN11]. A major distinction is that the degrees of the q_i and σ in the commutative theorem behave very badly.
(4) Observe that $\mathcal{K}$ is in general not closed. Thus Theorem 5.1 yields a "perfect" Positivstellensatz for certain non-closed sets. □

5.1. **Proof of Theorem 5.1.** We begin with some auxiliary results.

Proposition 5.3. *With $\mathfrak{L}$ a monic linear pencil as in (5.1), $M_x^\mu(\mathfrak{L})_r$ is a closed convex cone in the set of all symmetric polynomials in $\mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+1}$.*

The convex cone property is obvious. For the proof that this cone is closed, it is convenient to introduce a norm compatible with $\mathfrak{L}$.

Given $\varepsilon > 0$, let

$$\mathcal{B}_\varepsilon^g(n) := \big\{X \in \mathbb{S}_n^g : \|X\| \le \varepsilon\big\}.$$

There is an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, if $(X,Y) \in \mathbb{S}_n^{g+h}$ and $\|(X,Y)\| \le \varepsilon$, then $\mathfrak{L}(X,Y) \succeq \frac{1}{2}$. In particular, $\mathcal{B}_\varepsilon^{g+h} \subseteq \mathcal{D}_{\mathfrak{L}}$. Using this ε we norm matrix polynomials in $g+h$ variables by

$$\|\!|p(x,y)|\!\| := \max\big\{\|p(X,Y)\| : (X,Y) \in \mathcal{B}_\varepsilon^{g+h}\big\}. \tag{5.2}$$

(Note that by the nonexistence of polynomial identities for matrices of all sizes, $|||p(x,y)||| = 0$ iff $p(x,y) = 0$; cf. [Row80, §2.5, §1.4]. Furthermore, on the right-hand side of (5.2) the maximum is attained because the bounded free semialgebraic set $\mathcal{B}_\varepsilon^{g+h}$ is levelwise compact and matrix convex; see [HM04, Section 2.3] for details). Note that if $f \in \mathbb{C}^{d\times\mu}\langle x\rangle_\beta$ and if $|||f(x)^*\mathfrak{L}(x,y)f(x)||| \le N^2$, then $|||f^*f||| \le 2N^2$.

Proof of Proposition 5.3. Suppose (p_n) is a sequence from $M_x^\mu(\mathfrak{L})_r$ that converges to some symmetric $p \in \mathbb{C}^{\mu\times\mu}\langle x\rangle$ of degree at most $2r+1$. By Caratheodory's convex hull theorem (see e.g. [Bar02, Theorem I.2.3]), there is an M such that for each n there exist matrix-valued polynomials $r_{n,i} \in \mathbb{C}^{\mu\times\mu}\langle x\rangle_r$ and $t_{n,i} \in \mathbb{C}^{d\times\mu}\langle x\rangle_r$ such that

$$p_n = \sum_{i=1}^{M} r_{n,i}^* r_{n,i} + \sum_{i=1}^{M} t_{n,i}^*\, \mathfrak{L}(x,y)\, t_{n,i}.$$

Since $|||p_n||| \le N^2$, it follows that $|||r_{n,i}||| \le N$ and likewise $\left|\left|\left| t_{n,i}^*\mathfrak{L}(x,y)t_{n,i} \right|\right|\right| \le N^2$. In view of the remarks preceding the proof, we obtain $|||t_{n,i}||| \le \sqrt{2}N$ for all i, n. Hence for each i, the sequences $(r_{n,i})$ and $(t_{n,i})$ are bounded in n. They thus have convergent subsequences. Passing to one of these subsequential limits finishes the proof. ■

Next is a variant of the Gelfand-Naimark-Segal (GNS) construction.

Proposition 5.4. *If $\lambda : \mathbb{C}^{\nu\times\nu}\langle x\rangle_{2k+2} \to \mathbb{C}$ is a linear functional that is nonnegative on Σ_{k+1}^ν and positive on $\Sigma_k^\nu \setminus \{0\}$, then there exists a tuple $X = (X_1, \dots, X_g)$ of hermitian operators on a Hilbert space $\mathcal{X}$ of dimension at most $\nu\sigma_\#(k) = \nu \dim \mathbb{C}\langle x\rangle_k$ and a vector $\gamma \in \mathcal{X}^{\oplus\nu}$ such that*

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle \tag{5.3}$$

for all $f \in \mathbb{C}^{\nu\times\nu}\langle x\rangle_{2k+1}$, where $\langle \_, \_ \rangle$ is the inner product on $\mathcal{X}$. Further, if λ is nonnegative on $M_x^\nu(\mathfrak{L})_k$, then X is in the closure $\overline{\mathcal{K}}$ of the free spectrahedrop $\mathcal{K}$ coming from $\mathfrak{L}$.

Conversely, if $X = (X_1, \dots, X_g)$ is a tuple of symmetric operators on a Hilbert space $\mathcal{X}$ of dimension N, the vector $\gamma \in \mathcal{X}^{\oplus\nu}$, and k is a positive integer, then the linear functional $\lambda : \mathbb{C}^{\nu\times\nu}\langle x\rangle_{2k+2} \to \mathbb{C}$ defined by

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle$$

is nonnegative on Σ_{k+1}^ν. Further, if $X \in \overline{\mathcal{K}}$, then λ is nonnegative also on $M_x^\nu(\mathfrak{L})_k$.

Proof. The first part of the forward direction is standard, see e.g. [HKM12, Proposition 2.5]. In the course of the proof one constructs X_j as the operators of multiplication by x_j on a Hilbert space $\mathcal{X}$, that, as a set, is $\mathbb{C}\langle x\rangle_k^{1\times\nu}$ (the set of row vectors of length ν whose entries are polynomials of degree at most k). The vector space $\mathcal{X}^{\oplus\nu}$ in which γ lies is $\mathbb{C}\langle x\rangle_k^{\nu\times\nu}$ and γ can be thought of as the identity matrix in $\mathbb{C}\langle x\rangle_k^{\nu\times\nu}$. Indeed, the (column) vector γ has j-th entry the row vector with j-th entry the empty set (which plays the role of multiplicative identity) and zeros elsewhere.

In particular, for $p \in \mathcal{X} = \mathbb{C}\langle x\rangle_k^{1\times\nu}$, we have $p = p(X)\gamma$. Let σ denote the dimension of $\mathcal{X}$ (which turns out to be ν times the dimension of $\mathbb{C}\langle x\rangle_k$).

We next assume that λ is nonnegative on $M_x^\nu(\mathfrak{L})_k$ and claim that then $X \in \overline{\mathcal{K}}$. Assume otherwise. Then, as $\overline{\mathcal{K}}$ is closed matrix convex (and $\mathcal{K}$ contains 0 since $\mathfrak{L}$ is monic), the matricial Hahn-Banach Theorem 2.2 applies: there is a monic linear pencil $\mathfrak{L}_\Lambda$ of size σ such that $\mathfrak{L}_\Lambda|_{\mathcal{K}} \succeq 0$ and $\mathfrak{L}_\Lambda(X) \not\succeq 0$. In particular, $\mathcal{D}_{\mathfrak{L}_\Lambda} \supseteq \mathcal{K}$, whence

$$\mathcal{D}_{\mathfrak{L}_\Lambda}^\circ \subseteq \mathcal{K}^\circ.$$

By Corollary 4.15,

$$\mathcal{K}^\circ(n) = \Big\{A \in \mathbb{S}_n^g : \exists \mu \in \mathbb{N}\ \exists \text{isometry } V : \sum_{j=1}^{\mu} V_j^*\Gamma V_j = 0, \sum_{j=1}^{\mu} V_j^*\Omega V_j = A\Big\}. \tag{5.4}$$

Since $\Lambda \in \mathcal{K}^\circ$, there is an isometry W with

$$\sum_{j=1}^{\eta} W_j^*\Gamma W_j = 0, \quad \sum_{j=1}^{\eta} W_j^*\Omega W_j = \Lambda.$$

Here, $W = \operatorname{col}(W_1,\dots,W_\eta)$ for some η, and $W_j \in \mathbb{C}^{d\times\sigma}$.

Since $\mathfrak{L}_\Lambda(X) \not\succeq 0$, there is $u \in \mathbb{C}^\sigma \otimes \mathcal{X}$ with

$$u^* L_\Lambda(X) u < 0. \tag{5.5}$$

Let

$$u = \sum_i e_i \otimes v_i,$$

where $e_i \in \mathbb{C}^\sigma$ are the standard basis vectors, and $v_i \in \mathcal{X}$. By the construction of X and γ, there is a polynomial $p_i \in \mathbb{C}\langle x\rangle_k^{1\times\nu}$ with $v_i = p_i(X)\gamma$. Now (5.5) can be written as follows:

$$\begin{aligned} 0 > u^*\mathfrak{L}_\Lambda(X)u &= \big(\sum_i e_i \otimes v_i\big)^* \mathfrak{L}_\Lambda(X)\big(\sum_j e_j \otimes v_j\big) \\ &= \sum_{i,j,\ell} \big(e_i \otimes v_i\big)^*(W_\ell \otimes I)^*\mathfrak{L}(X,Y)\big(W_\ell \otimes I\big)\big(e_j \otimes v_j\big) \\ &= \sum_{i,j,\ell} \big(W_\ell e_i \otimes p_i(X)\gamma\big)^*\mathfrak{L}(X,Y)\big(W_\ell e_j \otimes p_j(X)\gamma\big). \end{aligned} \tag{5.6}$$

Letting $\vec{p}_\ell(x) = \sum_j W_\ell e_j \otimes p_j(x) \in \mathbb{C}^{d\times\nu}\langle x\rangle_k$, (5.6) is further equivalent to

$$0 > \sum_\ell \big(\vec{p}_\ell(X)\gamma\big)^*\mathfrak{L}(X,Y)\big(\vec{p}_\ell(X)\gamma\big) = \lambda(q), \tag{5.7}$$

where $q = \sum_\ell \vec{p}_\ell(x)^* \mathfrak{L}(x,y)\vec{p}_\ell(x)$ is a matrix polynomial only in x by (5.4), and thus $q \in M_x^\mu(\mathfrak{L})_k$. But now (5.7) contradicts the nonnegativity of λ on $M_x^\mu(\mathfrak{L})_k$.

The converse is obvious. ∎

Proof of Theorem 5.1. Let $\operatorname{Sym}\mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+1}$ denote the symmetric elements of $\mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+1}$. Arguing by contradiction, suppose $p \in \operatorname{Sym}\mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+1}$ and $p|_{\mathcal{K}} \succeq 0$, but $p \notin M_x^\mu(\mathfrak{L})_r$. By the scalar Hahn-Banach theorem and Proposition 5.3, there is a strictly separating positive (real) linear functional $\lambda : \operatorname{Sym}\mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+1} \to \mathbb{R}$ nonnegative on $M_x^\mu(\mathfrak{L})_r$. We first extend λ to a (complex) linear functional on the whole $\mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+1}$ by sending $q + is \mapsto \lambda(q) + i\lambda(s)$ for symmetric q, s. We then extend λ to a linear functional (still called λ) on $\mathbb{C}^{\mu\times\mu}\langle x\rangle_{2r+2}$ by mapping

$$E_{ij} \otimes u^* v \mapsto \begin{cases} 0 & \text{if } i \neq j \text{ or } u \neq v \\ C & \text{otherwise,} \end{cases}$$

where $i, j = 1, \ldots, \mu$, and $u, v \in \langle x\rangle$ are of length $r+1$. For $C > 0$ large enough, this λ will be nonnegative on Σ_{r+1}^μ. Perturbing λ if necessary, we may further assume λ is strictly positive on $\Sigma_r^\mu \setminus \{0\}$. Now applying Proposition 5.4 yields a matrix tuple $X \in \overline{\mathcal{K}}$ and a vector γ satisfying (5.3) (with $k = r$). But then

$$0 > \lambda(p) = \langle p(X)\gamma, \gamma\rangle \geq 0,$$

a contradiction. ∎

6. Tracial Sets

While this papers original motivation arose from considerations of free optimization as it appears in linear systems theory, determining the matrix convex hull of a free set has an analog in quantum information theory, see [LP11]. In free optimization, the relevant maps are completely positive and *unital* (ucp). In quantum information theory, the relevant maps are completely positive and *trace preserving* (CPTP) or *trace non-increasing*. This section begins by recalling the two *quantum interpolation problems* from Subsection 3.2 before reformulating these problem in terms of *tracial hulls*. Corresponding duality results are the topic of the next section.

Recall a quantum channel is a cp map Φ from M_n to M_k that is trace preserving,

$$\operatorname{tr}(\Phi(X)) = \operatorname{tr}(X).$$

The dual Φ' of Φ is the mapping from M_k to M_n defined by

$$\operatorname{tr}(\Phi(X)Y^*) = \operatorname{tr}(X\Phi'(Y)^*).$$

Lemma 6.1 ([LP11, Proposition 1.2])**.** *Φ' is a quantum channel cp if and only if Φ is unital cp.*

Recall the cp interpolation problem from Subsection 3.2. It asks, given $A \in \mathbb{S}_n^g$ and given B in $\mathbb{S}_m^g$: does there exist a unital cp map $\Phi : M_n \to M_m$ such that $B_j = \Phi(A_j)$ for $j = 1, \dots, g$? The set of solutions B for a given A is the matrix convex hull of A. The versions of the interpolation problem arising in quantum information theory [Ha11, Kle07, NCSB98] replace unital with trace preserving or trace non-increasing. Namely, does $B_j = \Phi(A_j)$ for $j = 1, \dots, g$ for some trace preserving (resp. trace non-increasing) cp map $\Phi : M_n \to M_m$? The set of all solutions B for a given A is the **tracial hull** of A. Thus,

$$\text{thull}(A) = \{B : \Phi(A) = B \ \text{ for some trace preserving cp map } \Phi\}. \tag{6.1}$$

We define the **contractive tracial hull** of a tuple A by

$$\text{cthull}(A) = \{B : \Phi(A) = B \ \text{ for some cp trace non-increasing map } \Phi\}.$$

The article [LP11] determines when $B \in \text{thull}(A)$ for $g = 1$ (see Section 3.2). For any $g \geq 0$ the paper [AG15, Section 3] converts this problem to an LMI suitable for semidefinite programming; see Theorem 3.4 here for a similar result.

While the unital and trace preserving (or trace non-increasing) interpolation problems have very similar formulations, tracial hulls possess far less structure than matrix convex hulls. Indeed, as is easily seen, tracial hulls need not be convex (levelwise) and contractive tracial hulls need not be closed with respect to direct sums. Tracial hulls are studied in Subsection 6.1, and contractive tracial hulls in Subsection 6.2. Section 7 contains "tracial" notions of half-space and corresponding Hahn-Banach type separation theorems.

6.1. **Tracial Sets and Hulls.** A set $\mathcal{Y} \subseteq \mathbb{S}^g$ is **tracial** if $Y \in \mathcal{Y}(n)$ and if C_ℓ are $m \times n$ matrices such that

$$\sum C_\ell^* C_\ell = I_n,$$

then $\sum C_j Y C_j^* \in \mathcal{Y}(m)$. The **tracial hull** of a subset $\mathcal{S} \subseteq \mathbb{S}^g$ is the smallest tracial set containing $\mathcal{S}$, denoted $\text{thull}(\mathcal{S})$. Note that, in the case that $\mathcal{S}$ is a singleton, this definition is consistent with the definition afforded by equation (6.1).

The following lemma is an easy consequence of a theorem of Choi, stated in [Pau02, Proposition 4.7]. It caps the number of terms needed in a convex combination to represent a given matrix tuple Z in the tracial hull of T. Hence it is an analog of Caratheodory's convex hull theorem (see e.g. [Bar02, Theorem I.2.3]).

Lemma 6.2. *Suppose $T \in \mathbb{S}_n^g$ and $C_1, \dots, C_N$ are $m \times n$ matrices making $\sum C_\ell^* C_\ell = I_n$. If $Z = \sum_{\ell=1}^N C_\ell T C_\ell^*$, then there exists $m \times n$ matrices $V_1, \dots, V_{mn}$ such that $\sum V_\ell^* V_\ell = I_n$ and*

$$Z = \sum_{\ell=1}^{mn} V_\ell T V_\ell^*.$$

Proof. The mapping $\Phi : M_n \to M_m$ defined by

$$\Phi(X) = \sum C_\ell X C_\ell^*$$

is completely positive. Hence, by [Pau02, Proposition 4.7], there exist (at most) nm matrices $V_j : \mathbb{C}^m \to \mathbb{C}^n$ such that

$$\Phi(X) = \sum_{\ell=1}^{mn} V_\ell X V_\ell^*.$$

In particular,

$$Z = \Phi(T) = \sum V_\ell T V_\ell^*.$$

Further, for all $m \times m$ matrices X,

$$\begin{aligned}\operatorname{tr}(X) &= \operatorname{tr}\big(X \sum C_\ell^* C_\ell\big) = \operatorname{tr}\big(\sum C_\ell X C_\ell^*\big) = \operatorname{tr}\big(\Phi(X)\big) \\ &= \operatorname{tr}\big(\sum V_\ell X V_\ell^*\big) = \operatorname{tr}\big(X \sum V_\ell^* V_\ell\big).\end{aligned}$$

It follows that $\sum V_\ell^* V_\ell = I$. ■

Lemma 6.3. *For $\mathcal{S} = \{T\}$ a singleton,*

$$\operatorname{thull}(\{T\}) = \Big\{\sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell = I\Big\}.$$

Moreover, this set is closed (levelwise).

The tracial hull of a subset $\mathcal{S} \subseteq \mathbb{S}^g$ is

$$\operatorname{thull}(\mathcal{S}) = \Big\{\sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell = I,\, T \in \mathcal{S}\Big\} = \bigcup_{T \in \mathcal{S}} \operatorname{thull}(\{T\}).$$

If $\mathcal{S}$ is a finite set, then the tracial hull of $\mathcal{S}$ is closed.

Proof. The first statement follows from the observation that $\{\sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell = I\}$ is tracial.

To prove the moreover, suppose T has size n and suppose Z^k is a sequence from $\mathcal{Y}(m)$. By Lemma 6.2 for each k there exists nm matrices $V_{k,\ell}$ of size $n \times m$ such that

$$Z^k = \sum_\ell V_{k,\ell} T V_{k,\ell}^*$$

and each $V_{k,\ell}$ is a contraction. Hence, by passing to a subsequence, we can assume, that for each fixed ℓ, the sequence $(V_{k,\ell})_k$ converges to some W_ℓ. Hence Z^k converges to $Z = \sum_\ell W_\ell T W_\ell^*$. Also, since $\sum_\ell V_{k,\ell}^* V_{k,\ell} = I$ for each k, we have $\sum_\ell W_\ell^* W_\ell = I$, whence $Z \in \mathcal{Y}(m)$.

To prove the second statement, let $\mathcal{S} \subseteq \mathbb{S}^g$ be given. Evidently,

$$\mathcal{S} \subseteq \bigcup_{T \in \mathcal{S}} \operatorname{thull}(\{T\}) \subseteq \operatorname{thull}(\mathcal{S}).$$

Hence it suffices to show that $\bigcup_{T\in\mathcal{S}} \operatorname{thull}(\{T\})$ is itself tracially convex. To this end, suppose $X \in \bigcup_{T\in\mathcal{S}} \operatorname{thull}(\{T\})$ and $C_1, \dots, C_N$ with $\sum C_\ell^* C_\ell = I$ are given (and of the appropriate sizes). There is a $S \in \mathcal{S}$ such that $X \in \operatorname{thull}(\{S\})$. Hence, by the first part of the lemma, $\sum C_\ell X C_\ell^* \in \operatorname{thull}(\{S\}) \subseteq \bigcup_{T\in\mathcal{S}} \operatorname{thull}(\{T\})$ and the desired conclusion follows.

The final statement of the lemma follows by combining its first two assertions and using the fact that the closure of a finite union is the finite union of the closures. ∎

6.2. **Contractively Tracial Sets and Hulls.** A set $\mathcal{Y} \subseteq \mathbb{S}^g$ is **contractively tracial** if $Y \in \mathcal{Y}(m)$ and if C_ℓ are $n \times m$ matrices such that

$$\sum C_\ell^* C_\ell \preceq I_m, \tag{6.2}$$

then $\sum C_j Y C_j^* \in \mathcal{Y}(n)$. Note that, in this case, $\mathcal{Y}$ is closed under unitary conjugation and compression to subspaces, but not necessarily direct sums. It is clear that intersections of contractively tracial sets are again contractively tracial.

In the case $\mathcal{S}$ is a singleton, the **contractive tracial hull** of a set $\mathcal{S}$, defined as the smallest contractively tracial set containing $\mathcal{S}$, is consistent with our earlier definition in terms of cp maps.

Lemma 6.4. *The contractive tracial hull of a subset $\mathcal{S} \subseteq \mathbb{S}^g$ is*

$$\operatorname{cthull}(\mathcal{S}) = \Big\{ \sum C_\ell T C_\ell^* : \sum C_\ell^* C_\ell \preceq I,\, T \in \mathcal{S} \Big\} = \bigcup_{T\in\mathcal{S}} \operatorname{cthull}(\{T\}).$$

If $\mathcal{S}$ is a finite set, then the contractive tracial hull of $\mathcal{S}$ is closed.

Proof. Proof is the same as for Lemma 6.3, so is omitted. ∎

Tracial and contractively tracial sets are not necessarily convex, as Example 8.6 illustrates, and they are not necessarily free sets because they may not respect direct sums. Lemma 6.5 below explains the relation between these two failings. Recall, a subset $\mathcal{Y}$ of $\mathbb{S}^g$ is levelwise convex if each $\mathcal{Y}(n)$ is convex (in the usual sense as a subset of $\mathbb{S}^g_n$). Say that $\mathcal{Y}$ is **closed with respect to convex direct sums** if given ℓ and $Y^1, \dots, Y^\ell \in \mathcal{Y}$ and given $\lambda_1, \dots, \lambda_\ell \geq 0$ with $\sum \lambda_j \leq 1$,

$$\oplus_j \lambda_j Y^j \in \mathcal{Y}.$$

Lemma 6.5. *If $\mathcal{Y}$ is contractively tracial, then $\mathcal{Y}$ is levelwise convex if and only if $\mathcal{Y}$ is closed with respect to convex direct sums.*

Proof. Suppose each $\mathcal{Y}(m)$ is convex. Given $Y^j \in \mathcal{Y}(m_j)$ for $1 \leq j \leq \ell$, let $m = \sum m_j$. Consider, the block operator column W_j embedding $\mathbb{C}^{m_j}$ into $\mathbb{C}^m = \bigoplus_j \mathbb{C}^{m_j}$. Note that

$W_j^* W_j = I_{m_j}$ and thus contractively tracial implies $W_j Y^j W_j^* \in \mathcal{Y}(m)$. Hence, given $\lambda_j \geq 0$ with $\sum \lambda_j = 1$, convexity of $\mathcal{Y}(m)$ (in the ordinary sense), implies

$$\bigoplus_j \lambda_j Y^j = \sum \lambda_j W_j Y^j W_j^* \in \mathcal{Y}(m).$$

To prove the converse, suppose $Y^j \in \mathcal{Y}(n)$ and $m = \ell n$. In this case, $\sum W_j W_j^* = I_n$ and hence tracial implies,

$$\sum W_j^* \big(\bigoplus \lambda_j Y^j \big) W_j = \sum \lambda_j Y^j \in \mathcal{Y}(n). \qquad \blacksquare$$

6.3. **Classical Duals of Free Convex Hulls and of Tracial Hulls.** This subsection gives properties of the classical polar dual of matrix convex hulls and tracial hulls. Real linear functionals $\lambda : \mathbb{S}_n^g \to \mathbb{R}$ are in one-one correspondence with elements $B \in \mathbb{S}_n^g$ via the pairing,

$$\lambda(X) = \operatorname{tr}\big(\sum B_j X_j \big), \quad X = (X_1, \dots, X_g).$$

Write λ_B for this λ. To avoid confusion with the free polar duals appearing earlier in this article, let $\mathcal{U}^{\circ c}$ denote the **conventional polar dual** of a subset $\mathcal{U} \subseteq \mathbb{S}_n^g$. Thus,

$$\mathcal{U}^{\circ c} = \{ B \in \mathbb{S}_n^g : \lambda_B(X) \leq 1 \text{ for all } X \in \mathcal{U} \}.$$

Lemma 6.6. *Suppose $A \in \mathbb{S}_n^g$.*

(i) $\operatorname{co}^{\mathrm{mat}}(A)^{\circ c} = \big\{ Y : \operatorname{thull}(Y) \subseteq \{A\}^{\circ c} \big\}$;
(ii) $\operatorname{thull}(A)^{\circ c} = \big\{ Y : \{A\}^{\circ c} \supseteq \operatorname{co}^{\mathrm{mat}}(Y) \big\}$; *and*
(iii) $\operatorname{thull}(B) \subseteq \operatorname{thull}(A)$ *if and only if* $\{A\}^{\circ c} \supseteq \operatorname{co}^{\mathrm{mat}}(Y)$ *implies* $\{B\}^{\circ c} \supseteq \operatorname{co}^{\mathrm{mat}}(Y)$.

Proof. The first formula:

$$\begin{aligned} \operatorname{co}^{\mathrm{mat}}(A)^{\circ c} &= \big\{ Y : 1 - \operatorname{tr}(\sum_j V_j^* A V_j \ Y) \geq 0, \ \sum_j V_j^* V_j = I \big\} \\ &= \big\{ Y : 1 - \operatorname{tr}(A \ \sum_j V_j Y V_j^*) \geq 0, \ \sum_j V_j^* V_j = I \big\} \\ &= \{ Y : 1 - \operatorname{tr}(AG) \geq 0, \ G \in \operatorname{thull}(Y) \} \\ &= \big\{ Y : \{A\}^{\circ c} \supseteq \operatorname{thull}(Y) \big\}. \end{aligned}$$

The second formula:

$$\begin{aligned} \operatorname{thull}(A)^{\circ c} &= \big\{ Y : 1 - \operatorname{tr}(\sum_j V_j^* A V_j \ Y) \geq 0, \ \sum_j V_j V_j^* = I \big\} \\ &= \big\{ Y : 1 - \operatorname{tr}(A \ \sum_j V_j \ Y V_j^*) \geq 0, \ \sum_j V_j V_j^* = I \big\} \\ &= \big\{ Y : \{A\}^{\circ c} \supseteq \operatorname{co}^{\mathrm{mat}}(Y) \big\}. \end{aligned}$$

The third formula: $\operatorname{thull}(B) \subseteq \operatorname{thull}(A)$ if and only if $\operatorname{thull}(B)^{\circ c} \supseteq \operatorname{thull}(A)^{\circ c}$ if and only if

$$\big\{Y : \{B\}^{\circ c} \supseteq \operatorname{co}^{\operatorname{mat}}(Y)\big\} \supseteq \big\{Y : \{A\}^{\circ c} \supseteq \operatorname{co}^{\operatorname{mat}}(Y)\big\},$$

if and only if $\{A\}^{\circ c} \supseteq \operatorname{co}^{\operatorname{mat}}(Y)$ and $\{B\}^{\circ c} \supseteq \operatorname{co}^{\operatorname{mat}}(Y)$. ■

7. Tracial Spectrahedra and an Effros-Winkler Separation Theorem

Classically, convex sets are delineated by half-spaces. In this section a notion of half-space suitable in the tracial context – we call these *tracial spectrahedra* – are introduced. Subsection 7.3 contains a free Hahn-Banach separation theorem for tracial spectrahedra. The section concludes with applications of this Hahn-Banach theorem. Subsection 7.4 suggests several notions of duality based on the tracial separation theorem from Subsection 7.3. Subsection 7.5 studies free (convex) cones.

7.1. Tracial Spectrahedra.

Polar duality considerations in the trace non-increasing context lead naturally to inequalities of the type,

$$I \otimes T - \sum_{j=1}^{g} B_j \otimes Y_j \succeq 0,$$

for tuples $B, Y \in \mathbb{S}^g$ and a positive semidefinite matrix T with trace at most one. Two notions, in a sense dual to one another, of half-space are obtained by fixing either B or Y.

Given $B \in \mathbb{S}^g_k$, let

$$\begin{aligned}\mathfrak{H}_B &= \bigcup_{m\in\mathbb{N}} \big\{Y \in \mathbb{S}^g_m : \exists T \succeq 0,\ \operatorname{tr}(T) \le 1,\quad I \otimes T - \sum B_j \otimes Y_j \succeq 0\big\}\\ &= \bigcup_{m\in\mathbb{N}} \big\{Y \in \mathbb{S}^g_m : \exists T \succeq 0,\ \operatorname{tr}(T) = 1,\quad I \otimes T - \sum B_j \otimes Y_j \succeq 0\big\}.\end{aligned}$$

We call sets of the form $\mathfrak{H}_B$ **tracial spectrahedra**. Tracial spectrahedra obtained by fixing Y, and parameterizing over B, appear in Subsubsection 7.4.2.

Proposition 7.1. *Let $B \in \mathbb{S}^g_k$ be given.*

(a) *The set $\mathfrak{H}_B$ is contractively tracial;*
(b) *For each m, the set $\mathfrak{H}_B(m)$ is convex; and*
(c) *For each m, the set $\mathfrak{H}_B(m)$ is closed.*

In summary, $\mathfrak{H}_B$ is levelwise compact and closed, and is contractively tracial.

Remark 7.2. Of course $\mathfrak{H}_B$ is not a free set since, in particular, it is not closed with respect to direct sums. □

Proof. Suppose $Y \in \mathfrak{H}_B(m)$ and C_ℓ satisfying equation (6.2) are given. There is an $m \times m$ positive semidefinite matrix T with trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$

It follows that

$$0 \preceq I \otimes \sum_\ell C_\ell T C_\ell^* - \sum_j B_j \otimes \sum_\ell C_\ell Y_j C_\ell^*.$$

Note that $T' = \sum_\ell C_\ell T C_\ell^* \succeq 0$ and

$$\operatorname{tr}(T') = \operatorname{tr}\left(T \sum C_\ell^* C_\ell\right) = \operatorname{tr}\left(T^{\frac{1}{2}} C_\ell^* C_\ell T^{\frac{1}{2}}\right) \le \operatorname{tr}(T) \le 1.$$

Hence $\sum C_\ell Y C_\ell^* \in \mathcal{Y}(n)$ and item (a) of the proposition is proved.

To prove item (b), suppose both Y^1 and Y^2 are in $\mathcal{Y}_B$. To each there is an associated positive semidefinite matrix of trace at most one, say T_1 and T_2. If $0 \le s_1, s_2 \le 1$ and $s_1 + s_2 = 1$, then $T = \sum s_\ell T_\ell$ is positive semidefinite and has trace at most one. Moreover, with $Y = \sum s_j Y^j$,

$$I \otimes T - \sum_j B_j \otimes \left(\sum s_\ell Y_j^\ell\right) = \sum_\ell s_\ell \left(I \otimes T_\ell - \sum_j B_j \otimes Y_j^\ell\right) \succeq 0.$$

To prove (c), suppose the sequence $(Y^k)_k$ from $\mathfrak{H}_B(m)$ converges to $Y \in \mathbb{S}_m^g$. For each k there is a positive semidefinite matrix T_k of trace at most one such that

$$I \otimes T_k - \Lambda_B(Y^k) \succeq 0.$$

Choose a convergent subsequence of the T_k with limit T. This T witnesses $Y \in \mathfrak{H}_B(m)$. ∎

To proceed toward the separation theorem we start with some preliminaries.

7.2. **An Auxiliary Result.** Given a positive integer n, let $\mathcal{T}_n$ denote the positive semidefinite $n \times n$ matrices of trace one. Each $T \in \mathcal{T}_n$ corresponds to a state on M_n via the trace,

$$M_n \ni A \mapsto \operatorname{tr}(AT). \tag{7.1}$$

Conversely, to each state φ on M_n we can assign a matrix T such that φ is the map (7.1). Note that $\mathcal{T}_n$ is a convex, compact subset of $\mathbb{S}_n$, the symmetric $n \times n$ matrices.

The following lemma is a version of [EW97, Lemma 5.2]. An affine (real) linear mapping $f : \mathbb{S}_n \to \mathbb{R}$ is a function of the form $f(x) = a_f + \lambda_f(x)$, where λ_f is (real) linear and $a_f \in \mathbb{R}$.

Lemma 7.3. *Suppose $\mathcal{F}$ is a convex set of affine linear mappings $f : \mathbb{S}_n \to \mathbb{R}$. If for each $f \in \mathcal{F}$ there is a $T \in \mathcal{T}_n$ such that $f(T) \ge 0$, then there is a $\mathfrak{T} \in \mathcal{T}_n$ such that $f(\mathfrak{T}) \ge 0$ for every $f \in \mathcal{F}$.*

Proof. For $f \in \mathcal{F}$, let

$$B_f = \{T \in \mathcal{T}_n : f(T) \geq 0\} \subseteq \mathcal{T}_n.$$

By hypothesis each B_f is non-empty and it suffices to prove that

$$\bigcap_{f \in \mathcal{F}} B_f \neq \varnothing.$$

Since each B_f is compact, it suffices to prove that the collection $\{B_f : f \in \mathcal{F}\}$ has the finite intersection property. Accordingly, let $f_1, \dots, f_m \in \mathcal{F}$ be given. Arguing by contradiction, suppose

$$\bigcap_{j=1}^{m} B_{f_j} = \varnothing.$$

Define $F : \mathbb{S}_n \to \mathbb{R}^m$ by

$$F(T) = (f_1(T), \dots, f_m(T)).$$

Then $F(\mathcal{T}_n)$ is both convex and compact because $\mathcal{T}_n$ is both convex and compact and each f_j, and hence F, is affine linear. Moreover, $F(\mathcal{T}_n)$ does not intersect

$$\mathbb{R}^m_{\geq 0} = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : x_j \geq 0 \text{ for each } j\}.$$

Hence there is a linear functional $\lambda : \mathbb{R}^m \to \mathbb{R}$ such that

$$\lambda\big(F(\mathcal{T}_n)\big) < 0 \quad \text{ and } \quad \lambda\big(\mathbb{R}^m_{\geq 0}\big) \geq 0.$$

There exists $\lambda_j \in \mathbb{R}$ such that $\lambda(x) = \sum \lambda_j x_j$. Since $\lambda\big(\mathbb{R}^m_{\geq 0}\big) \geq 0$ it follows that each $\lambda_j \geq 0$ and since $\lambda \neq 0$, for at least one k, $\lambda_k > 0$. Without loss of generality, it may be assumed that $\sum \lambda_j = 1$. Let

$$f = \sum \lambda_j f_j.$$

Since $\mathcal{F}$ is convex, it follows that $f \in \mathcal{F}$. On the other hand, $f(T) = \lambda(F(T))$. Hence if $T \in \mathcal{T}_n$, then $f(T) < 0$. Thus, for this f there does not exist a $T \in \mathcal{T}_n$ such that $f(T) \geq 0$, a contradiction which completes the proof. ■

7.3. **A Tracial Spectrahedron Separating Theorem.** The following lemma is proved by a variant of the Effros-Winkler construction of separating LMIs (i.e., the matricial Hahn-Banach Theorem) in the theory of matrix convex sets.

Lemma 7.4. *Fix positive integers m, n, and suppose that $\mathcal{S}$ is a nonempty subset of $\mathbb{S}^g_m$. Let $\mathcal{U}$ denote the subset of $\mathbb{S}^g_n$ consisting of all tuples of the form*

$$\sum_{\ell=1}^{\mu} C_\ell Y^\ell C_\ell^*,$$

where each C_ℓ is $n \times m$, each $Y^\ell \in \mathcal{S}$ and $\sum C_\ell^* C_\ell \preceq I$. If $B \in \mathbb{S}_n^g$ is in the conventional polar dual of $\mathcal{U}$, then there exists a positive semidefinite $m \times m$ matrix T with trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0$$

for every $Y \in \mathcal{S}$.

Proof. Recall the definition of λ_B from Subsection 6.3. Given C_ℓ and Y^ℓ as in the statement of the lemma, define $f_{C,Y} : \mathbb{S}_m^g \to \mathbb{R}$ by

$$f_{C,Y}(X) = \operatorname{tr}\big(\sum C_\ell X C_\ell^*\big) - \lambda_B\big(\sum C_\ell Y C_\ell^*\big).$$

Let $\mathcal{F} = \{f_{C,Y} : C, Y\}$. Thus $\mathfrak{F}$ is a set of affine (real) linear mappings from $\mathbb{S}_m^g$ to $\mathbb{R}$. To show that $\mathcal{F}$ is convex, suppose, for $1 \le s \le N$, $C^s = (C_1^s, \dots, C_{\mu_s}^s)$ is a tuple of $n \times m$ matrices, for $1 \le s \le N$ and $1 \le j \le \mu_s$ the matrices $Y^{s,j}$ are in $\mathcal{S}$ and and $\lambda_1, \dots, \lambda_N$ are positive numbers with $\sum \lambda_s = 1$. In this case,

$$\sum \lambda_s f_{C^s, Y^{s,\cdot}} = f_{C,Y}$$

for

$$C = \Big(\frac{1}{\sqrt{\lambda_s}} C_\ell^s\Big)_{s,\ell}, \quad Y = \Big(Y^{s,\ell}\Big)_{s,\ell}.$$

Hence $\mathcal{F}$ is convex.

Given $n \times m$ matrices $C_1, \dots, C_\mu$ and $Y^1, \dots, Y^\mu \in \mathcal{S}$, let $D = \sum C_\ell^* C_\ell$. Assuming D has norm one, there is a unit vector γ such that $\|D\gamma\| = \|D\| = 1$. Choose $T = \gamma\gamma^*$. Thus $T \in \mathcal{T}_m$. Moreover,

$$\operatorname{tr}\big(\sum C_\ell T C_\ell^*\big) = \operatorname{tr}(TD) = \langle D\gamma, \gamma\rangle = 1.$$

Thus, using the assumption that B is in $\mathcal{U}^{\circ c}$,

$$f_{C,Y}(T) = 1 - \lambda_B\big(\sum C_\ell Y^\ell C_\ell^*\big) \ge 0.$$

If D is not of norm one, a simple scaling argument gives the same conclusion; that is,

$$f_{C,Y}(T) \ge 0.$$

Thus, for each $f \in \mathcal{F}$ there exists a $T \in \mathcal{T}_m$ such that $f(T) \ge 0$. By Lemma 7.3, it follows that there is a $\mathfrak{T} \in \mathcal{T}_m$ such that $f_C(\mathfrak{T}) \ge 0$ for all C and Y; i.e.,

$$\operatorname{tr}\big(\sum C_\ell \mathfrak{T} C_\ell^*\big) - \lambda_B\big(\sum C_\ell Y^\ell C_\ell^*\big) \ge 0, \tag{7.2}$$

regardless of the norm of $\sum C_\ell^* C_\ell$.

Now the aim is to show that

$$\Delta := I \otimes \mathfrak{T} - \sum_j B_j \otimes Y_j \succeq 0$$

for every $Y \in \mathcal{S}$. Accordingly, let $Y \in \mathcal{S}$ and $\gamma = \sum e_s \otimes \gamma_s \in \mathbb{R}^n \otimes \mathbb{R}^m$ be given. Compute,

$$\langle \Delta\gamma, \gamma\rangle = \sum_s \langle \mathfrak{T}\gamma_s, \gamma_s\rangle - \sum_j \sum_{s,t} (B_j)_{s,t} \langle Y_j \gamma_s, \gamma_t\rangle.$$

Now let Γ^* denote the matrix with s-th column γ_s. Hence Γ is $n \times m$ and

$$\begin{aligned}\lambda_B(\Gamma Y \Gamma^*) &= \operatorname{tr}\Big(\sum B_j (\Gamma Y_j \Gamma^*)\Big) \\ &= \sum_j \sum_{s,t} (B_j)_{s,t} \langle Y_j \gamma_s, \gamma_t\rangle.\end{aligned}$$

Similarly,

$$\operatorname{tr}(\Gamma \mathfrak{T} \Gamma^*) = \sum_s \langle \mathfrak{T}\gamma_s, \gamma_s\rangle.$$

Thus, using the inequality (7.2),

$$\langle \Delta\gamma, \gamma\rangle = \operatorname{tr}(\Gamma \mathfrak{T} \Gamma^*) - \lambda_B(\Gamma Y \Gamma^*) \geq 0.$$

It is in this last step that the contractively tracial, not just tracial is needed, so that it is not necessary for $\Gamma^*\Gamma$ to be a multiple of the identity. ■

Proposition 7.5. *If $\mathcal{Y} \subseteq \mathbb{S}^g$ is contractively tracial and if $B \in \mathbb{S}^g_n$ is in the conventional polar dual $\mathcal{Y}(n)^{\circ c}$ of $\mathcal{Y}(n)$, then $\mathcal{Y} \subseteq \mathfrak{H}_B$.*

Proof. Suppose $\mathcal{Y}$ is contractively tracial and $Y \in \mathcal{Y}(m)$. Letting $\mathcal{S} = \{Y\}$ in Lemma 7.4, it follows that there is a T such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$

Thus, $Y \in \mathfrak{H}_B$ and the proof is complete. ■

We are now ready to state the separation result for closed levelwise convex tracial sets.

Theorem 7.6.

(i) *If $\mathcal{Y} \subseteq \mathbb{S}^g$ is contractively tracial, levelwise convex, and if $Z \in \mathbb{S}^g_m$ is not in the closure of $\mathcal{Y}(m)$, then there exists a $B \in \mathbb{S}^g_m$ such that $\mathcal{Y} \subseteq \mathfrak{H}_B$, but $Z \notin \mathfrak{H}_B$. Hence,*

$$\overline{\mathcal{Y}} = \bigcap \{\mathfrak{H}_B : \mathfrak{H}_B \supseteq \mathcal{Y}\} = \bigcap_{n\in\mathbb{N}} \bigcap_{B\in\mathcal{Y}(n)^{\circ c}} \mathfrak{H}_B.$$

(ii) *The levelwise closed convex contractively tracial hull of a subset $\mathcal{Y}$ of $\mathbb{S}^g$ is*

$$\bigcap \{\mathfrak{H}_B : \mathfrak{H}_B \supseteq \mathcal{Y}\}.$$

Proof. To prove item (i), suppose $Z \in \mathbb{S}^g_m$ but $Z \notin \overline{\mathcal{Y}(m)}$. Since $\mathcal{Y}$ is levelwise convex, there is λ_B such that $\lambda_B(Y) \le 1$ for all $Y \in \mathcal{Y}(m)$, but $\lambda_B(Z) > 1$ by the usual Hahn-Banach separation theorem for closed convex sets. Thus B is in the conventional polar dual of $\mathcal{Y}(m)^{\circ c}$. From Proposition 7.5, $\mathcal{Y} \subseteq \mathfrak{H}_B$.

On the other hand, if $T \in \mathcal{T}_m$ and $\{e_1, \dots, e_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then, with $e = \sum e_s \otimes e_s \in \mathbb{R}^m \otimes \mathbb{R}^m$,

$$\big\langle (I \otimes T - \sum B_j \otimes Z_j)e, e\big\rangle = \operatorname{tr}(T) - \operatorname{tr}\big(\sum B_j Z_j\big) \;=\; 1 - \lambda_B(Z) < 0.$$

Hence $Z \notin \mathfrak{H}_B$ and the conclusion follows.

To prove item (ii), first note, letting $\mathcal{I}$ denote the intersection of the $\mathfrak{H}_B$ that contain $\mathcal{Y}$, that $\mathcal{Y} \subseteq \mathcal{I}$. Since the intersection of tracial spectrahedra is levelwise closed and convex, and contractively tracial, the levelwise closed convex tracial hull $\mathcal{H}$ of $\mathcal{Y}$ is also contained in $\mathcal{I}$. On the other hand, from (i),

$$\mathcal{H} = \bigcap \{\mathfrak{H}_B : \mathfrak{H}_B \supseteq \mathcal{H}\} \supseteq \mathcal{I} \supseteq \mathcal{H}. \qquad \blacksquare$$

Remark 7.7. The contractive tracial hull of a point. Fix a $Y \in \mathbb{S}^g_n$ and let $\mathcal{Y}$ denote its contractive tracial hull,

$$\mathcal{Y} = \big\{ \sum V_j Y V_j^* : \sum V_j^* V_j \preceq I \big\}.$$

Evidently each $\mathcal{Y}(m)$ (taking $V_j : \mathbb{R}^n \to \mathbb{R}^m$) is a convex set. From Lemma 6.4, $\mathcal{Y}$ is closed. Hence Theorem 7.6 applies and gives a duality description of $\mathcal{Y}$. Namely, $\tilde{Y}$ is in the contractive tracial hull $\mathcal{Y}$ if and only if for each B for which there exists a positive semidefinite T of trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0,$$

there exists a positive semidefinite $\tilde{T}$ of trace at most one such that

$$I \otimes \tilde{T} - \sum B_j \otimes \tilde{Y}_j \succeq 0. \qquad \square$$

7.4. **Tracial Polar Duals.** We now introduce two natural notions of polar duals based on the tracial spectrahedra. Rather than exhaustively studying these duals, we list a few properties to illustrate the possibilities.

7.4.1. *Ex Situ Tracial Dual.* Suppose $\mathcal{K} \subseteq \mathbb{S}^g$. Let $\hat{\mathcal{K}}$ denote its **ex situ tracial dual** defined by

$$\hat{\mathcal{K}} = \bigcap_{B \in \mathcal{K}} \mathfrak{H}_B.$$

Thus,

$$\hat{\mathcal{K}}(n) = \big\{ Y \in \mathbb{S}^g_n : \forall B \in \mathcal{K}\ \exists T \succeq 0 \text{ such that } \operatorname{tr}(T) \le 1 \text{ and } I \otimes T - \sum B_j \otimes Y_j \succeq 0 \big\}.$$

Proposition 7.8. *If $\mathcal{K}$ is matrix convex and each $\mathcal{K}^\circ(n)$ is bounded (equivalently, $\mathcal{K}(1)$ contains 0 in its interior), then*

(i) $\hat{\mathcal{K}}(n) = \{Y \in \mathbb{S}_n^g : \exists T \succeq 0,$ *such that* $\operatorname{tr}(T) \le 1$ *and* $\forall B \in \mathcal{K},\ I \otimes T - \sum B_j \otimes Y_j \succeq 0\}$;
(ii) $\hat{\mathcal{K}}(n) = \{SMS : M \in \mathcal{K}^\circ(n),\ S \succeq 0,\ \operatorname{tr}(S^2) \le 1\}$.

Proof. Suppose K is matrix convex. To prove item (i), let $Y \in \hat{\mathcal{K}}(n)$ be given. For each B, let $\mathcal{T}_B = \{T \in \mathcal{T}_n : I \otimes T - \sum B_j \otimes Y_j \succeq 0\}$. Thus, the hypothesis that $Y \in \hat{\mathcal{K}}(n)$ is equivalent to assuming that for every B in $\mathcal{K}$, the set $\mathcal{T}_B$ is nonempty.

That $\mathcal{T}_B$ is compact will be verified by showing it satisfies the finite intersection property. Now given $B^1, \dots, B^\ell \in \mathcal{K}$, let $B = \bigoplus_k B^k \in \mathcal{K}$. Since $B \in \mathcal{K}$, there is a T such that

$$\bigoplus_k \left(I \otimes T - \sum B_j^k \otimes Y_j\right) = I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$

Hence $T \in \bigcap_{k=1}^\ell \mathcal{T}_{B^k}$. It follows that the collection $\{\mathcal{T}_B : B \in \mathcal{K}\}$ has the finite intersection property and hence there is a $T \in \bigcap_{B \in \mathcal{K}} \mathcal{T}_B$ and the forward inclusion in item (i) follows. The reverse inclusion holds whether or not $\mathcal{K}$ is matrix convex.

To prove item (ii), suppose $Y \in \hat{\mathcal{K}}(n)$. Thus, by what has already been proved, there is a positive semidefinite matrix S such that $\operatorname{tr}(S^2) \le 1$ and

$$I \otimes S^2 - \sum B_j \otimes Y_j \succeq 0, \tag{7.3}$$

for all $B \in \mathcal{K}$. For positive integers k, let S_k^+ denote the inverse of $S + \frac{1}{k}$. Multiplying (7.3) on the left and on the right by $I \otimes S_k^+$ yields

$$I \otimes P - \sum B_j \otimes S_k^+ Y_j S_k^+ \succeq 0,$$

where P is the projection onto the range of S. It follows that $M_k = S_k^+ Y S_k^+ \in \mathcal{K}^\circ(n)$. Since $\mathcal{K}^\circ(n)$ is bounded (by assumption) and closed, it is compact and consequently a subsequence of $(M_k)_k$ converges to some $M \in \mathcal{K}^\circ(n)$. Hence, $Y = SMS$.

Reversing the argument above shows, if $M \in \mathcal{K}^\circ(n)$ and S is positive semidefinite with $\operatorname{tr}(S^2) \le 1$, then $Y = SMS \in \hat{\mathcal{K}}(n)$ and the proof is complete. ∎

Proposition 7.9. *The ex situ tracial dual $\hat{\mathcal{K}}$ of a free spectrahedron $\mathcal{K} = \mathcal{D}_{\mathfrak{L}_\Omega}$ is exactly the set*

$$\left\{\sum_\ell C_\ell^* \Omega C_\ell : \operatorname{tr}\left(\sum C_\ell^* C_\ell\right) \le 1\right\}.$$

Proof. Suppose Y is in the ex situ tracial dual. By Proposition 7.8, there is a positive semidefinite matrix S with $\operatorname{tr}(S^2) \le 1$ and an $M \in \mathcal{K}^\circ$ such that $Y = SMS$. Since $M \in \mathcal{K}^\circ$, by Remark 4.8 there is a positive integer μ and a contraction V such that

$$M = V^*(I_\mu \otimes \Omega)V = \sum_k^\mu V_k^* \Omega V_k.$$

Hence,

$$Y = \sum_k S V_k^* \Omega V_k S.$$

Finally,

$$\operatorname{tr}\left(\sum S V_k^* V_k S\right) \le \operatorname{tr}(S^2) \le 1.$$

Conversely suppose $\operatorname{tr}(\sum C_\ell^* C_\ell) \le 1$ and $Y = \sum C_\ell^* \Omega C_\ell$. Let $T = \sum C_\ell^* C_\ell$ and note that for $B \in \mathcal{K}$,

$$I \otimes T - \sum B_j \otimes Y_j = \sum_\ell C_\ell^* \big(I \otimes I - \sum_j B_j \otimes \Omega_j\big) C_\ell \succeq 0. \qquad \blacksquare$$

7.4.2. *In Situ Tracial Dual.* Given a free set $\mathcal{K} \subseteq \mathbb{S}^g$, we can define another dual set we call the **in situ** $\mathcal{K}^\triangleright = (\mathcal{K}^\triangleright(m))_m$ by

$$\mathcal{K}^\triangleright(m) = \{B \in \mathbb{S}_m^g : \mathcal{K} \subseteq \mathfrak{H}_B\}$$

Equivalently,

$$\mathcal{K}^\triangleright(m) = \Big\{B \in \mathbb{S}_m^g : \forall Y \in \mathcal{K}\, \exists T \succeq 0, \text{ such that } \operatorname{tr}(T) \le 1 \text{ and } I \otimes T - \sum B_j \otimes Y_j \succeq 0\Big\}.$$

Each $\mathcal{K}^\triangleright(m)$ is levelwise convex. Moreover, if $B \in \mathcal{K}^\triangleright$ and $V^*V \preceq I$, then $V^*BV \in \mathcal{K}^\triangleright$. On the other hand, there is no reason to expect that $\mathcal{K}^\triangleright$ is closed with respect to direct sums. Hence it need not be matrix convex.

A subset $\mathcal{Y}$ of $\mathbb{S}^g$ is **contractively stable** if $\sum C_j^* Y C_j \in \mathcal{Y}$ for all $Y \in \mathcal{Y}$ such that $\sum C_j^* C_j \preceq I$. In general, contractively stable sets need not be levelwise convex as Example 8.7 shows.

Proposition 7.10. *The set $\mathcal{K}^\triangleright$ is contractively stable.*

Proof. Suppose $B \in \mathcal{K}^\triangleright(m)$. Let $n \times m$ matrices $C_1, \dots, C_\ell$ such that $\sum C_k^* C_k \preceq I$ be given and consider the $n \times n$ matrix $D = \sum C_k B C_k^*$.

Given $Y \in \mathcal{K}(p)$, there exists a positive semidefinite $p \times p$ matrix T of trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0.$$

Thus,

$$I\otimes T-\sum_{j=1}^{g} D_j\otimes Y_j = I\otimes T-\sum_{j=1}^{g}\sum_k C_k^* B_j C_k\otimes Y_j$$
$$= (I-\sum C_k^*C_k)\otimes T+\sum_k (C_k\otimes I)^*\big(I\otimes T-\sum_j B_j\otimes Y_j\big)(C_k\otimes I)\succeq 0.$$

Hence $D\in\mathcal{K}^{\triangleright}$ and the proof is complete. ■

The **contractive convex hull** of $\mathcal{Y}$ is the smallest levelwise closed set containing $\mathcal{Y}$ that is contractively stable. The following proposition finds the two hulls defined by applying the two notions of tracial polar duals introduced above.

Proposition 7.11. *For $\mathcal{K}\subseteq\mathbb{S}^g$, the set $(\widehat{\mathcal{K}^{\triangleright}})$ is the levelwise closed convex contractively tracial hull of $\mathcal{K}$. Similarly, $\left(\widehat{\mathcal{K}}\right)^{\triangleright}$ is the levelwise closed contractively stable hull of $\mathcal{K}$.*

The proof of the second statement rests on the following companion to Lemma 7.4. Recall, from equation (1.5) the opp-tracial spectrahedron,

$$\mathfrak{H}_Y^{\mathrm{opp}} = \{B : \exists T\succeq 0 \text{ such that } \operatorname{tr}(T)\le 1,\ \ I\otimes T-\sum B_j\otimes Y_j\succeq 0\}.$$

Lemma 7.12. *Fix positive integers m,n, and suppose that $\mathcal{S}$ is a nonempty subset of $\mathbb{S}^g_n$. Let $\mathcal{U}$ denote the subset of $\mathbb{S}^g_m$ consisting of all tuples of the form*

$$\sum_{\ell=1}^{\mu} C_\ell^* B^\ell C_\ell,$$

where each C_ℓ is $n\times m$, each $B^\ell\in\mathcal{S}$ and $\sum C_\ell^*C_\ell\preceq I$.

(1) *If $Y\in\mathbb{S}^g_m$ is in the conventional polar dual of $\mathcal{U}$, then there exists a positive semidefinite $m\times m$ matrix T with trace at most one such that*

$$I\otimes T-\sum B_j\otimes Y_j\succeq 0$$

for every $B\in\mathcal{S}$.

(2) *The tracial spectrahedra $\mathfrak{H}_Y^{\mathrm{opp}}$ are closed and contractively stable.*

(3) *If $\mathcal{K}\subseteq\mathbb{S}^g$ is contractively stable and if $Y\in\mathbb{S}^g_m$ is in the conventional polar dual $\mathcal{K}(m)^{\circ c}$ of $\mathcal{K}(m)$, then $\mathcal{K}\subseteq\mathfrak{H}_Y^{\mathrm{opp}}$.*

(4) *If $\mathcal{K}\subseteq\mathbb{S}^g$ is levelwise closed and convex, and contractively stable, then*

$$\mathcal{K}=\bigcap\{\mathfrak{H}_Y^{\mathrm{opp}} : \mathfrak{H}_Y^{\mathrm{opp}}\supseteq\mathcal{K}\}=\bigcap_n\bigcap\{\mathfrak{H}_Y^{\mathrm{opp}} : Y\in\mathcal{K}(n)^{\circ c}\}.$$

(5) *The levelwise closed and convex contractively stable hull of* $\mathcal{K} \subseteq \mathbb{S}^g$ *is*

$$\bigcap \{\mathfrak{H}_Y^{\text{opp}} : \mathfrak{H}_Y^{\text{opp}} \supseteq \mathcal{K}\}.$$

(6) *For* $\mathcal{K} \subseteq \mathbb{S}^g$*, we have* $Y \in \hat{\mathcal{K}}(n)$ *if and only if* $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$*.*

Proof. The proof of item (1) is similar to the proof of Lemma 7.4 and is omitted. Likewise, the proof of item (2) follows an argument given in the proof of Proposition 7.10.

To prove (3), suppose that $Y \in \mathcal{K}(m)^{\circ c}$. Given $B \in \mathcal{K}(n)$, an application of the first part of the lemma with $\mathcal{S} = \{B\}$ produces an $m \times m$ positive semidefinite matrix T with $\operatorname{tr}(T) \leq 1$ such that $I \otimes T - \sum B_j \otimes Y_j \succeq 0$. Hence, $B \in \mathfrak{H}_Y^{\text{opp}}$.

Moving on to item (4). From (3), if $Y \in \mathcal{K}(m)^{\circ c}$, then $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$. On the other hand, if $Y \in \mathbb{S}_m^g$ and $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$, then, for $B \in \mathcal{K}(m)$,

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0$$

for some positive semidefinite T with trace at most one. In particular, with $e = \sum_{s=1}^m e_s \otimes e_s \in \mathbb{R}^m \otimes \mathbb{R}^m$,

$$0 \leq \langle I \otimes T - \sum B_j \otimes Y_j e, e\rangle = \operatorname{tr}(T) - \lambda_Y(B).$$

Hence $Y \in \mathcal{K}(m)^{\circ c}$. Continuing with the proof of (4), from item (3),

$$\mathcal{K} \subseteq \bigcap_n \bigcap \{\mathfrak{H}_Y^{\text{opp}} : Y \in \mathcal{K}(n)^{\circ c}\}.$$

To establish the reverse inclusion, suppose that C is not in $\mathcal{K}(m)$. Since $\mathcal{K}(m)$ is assumed closed and convex, there exists a $Y \in \mathcal{K}(m)^{\circ c}$, the conventional polar dual of $\mathcal{K}(m)$ (so that $\lambda_Y(\mathcal{K}(m)) \leq 1$) with $\lambda_Y(C) > 1$. In particular, $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$. On the other hand, if T is $m \times m$ and positive semidefinite with trace at most one, then with $e = \sum e_s \otimes e_s$,

$$\langle (I \otimes T - \sum C_j \otimes Y_j)e, e\rangle = \operatorname{tr}(T) - \sum_j \operatorname{tr}(C_j Y_j) = \operatorname{tr}(T) - \lambda_Y(C) < 0.$$

Hence, $C \notin \mathfrak{H}_Y^{\text{opp}}$.

To prove item (5), let $\mathcal{H}$ denote the contractively stable hull of $\mathcal{K}$. Let also $\mathcal{I}$ denote the intersection of tracial spectrahedra $\mathfrak{H}_Y^{\text{opp}}$ such that $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$. Evidently $\mathcal{H} \subseteq \mathcal{I}$. On the other hand, using item (4),

$$\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{I} \subseteq \bigcap \{\mathfrak{H}_Y^{\text{opp}} : \mathfrak{H}_Y^{\text{opp}} \supseteq \mathcal{H}\} = \mathcal{H}.$$

Finally, for item (6), first suppose $Y \in \hat{\mathcal{K}}(n)$. By definition, for each $B \in \mathcal{K}$ there is a positive semidefinite T of trace at most one such that $I \otimes T - \sum B_j \otimes Y_j \succeq 0$. Hence $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$. Conversely, if $B \in \mathfrak{H}_Y^{\text{opp}}$, then $I \otimes T - \sum B_j \otimes Y_j \succeq 0$ for some positive semidefinite T of trace at most one depending on B. Thus, if $\mathcal{K} \subseteq \mathfrak{H}_Y^{\text{opp}}$, then $Y \in \hat{\mathcal{K}}(n)$. ∎

Proof of Proposition 7.11. Since

$$\widehat{(\mathcal{K}^{\triangleright})} = \bigcap_{B\in\mathcal{K}^{\triangleright}} \mathfrak{H}_B = \bigcap_{\mathcal{K}\subseteq\mathfrak{H}_B} \mathfrak{H}_B,$$

item (ii) of Theorem 7.6 gives the conclusion of the first part of the proposition.

Likewise,

$$(\hat{\mathcal{K}})^{\triangleright} = \{B : \hat{\mathcal{K}} \subseteq \mathfrak{H}_B\} = \bigcap_{Y\in\hat{\mathcal{K}}} \mathfrak{H}_Y^{\mathrm{opp}} = \bigcap \{\mathfrak{H}_Y^{\mathrm{opp}} : \mathfrak{H}_Y^{\mathrm{opp}} \supseteq \mathcal{K}\}$$

and the term on the right hand side is, by Lemma 7.12, the closed contractive convex hull of $\mathcal{K}$. ■

7.5. **Matrix Convex Tracial Sets and Free Cones.** In this subsection we introduce and study properties of free (convex) cones.

A subset $\mathcal{S}$ of $\mathbb{S}^g$ is a **free cone** if for all positive integers m, n, ℓ, tuples $T \in \mathcal{S}(n)$ and $n\times m$ matrices $C_1, \dots, C_\ell$, the tuple $\sum C_i^* T C_i$ is in $\mathcal{S}(m)$. The set $\mathcal{S}$ is a **free convex cone** if for all positive integers m, n, ℓ, tuples $T^1, \dots, T^\ell \in \mathcal{S}(n)$ and $n\times m$ matrices $C_1, \dots, C_\ell$, the tuple $\sum_i C_i^* T^i C_i$ lies in $\mathcal{S}(m)$. Finally, a subset $\mathcal{Y}$ of $\mathbb{S}^g$ is a **contractively tracial convex set** if $\mathcal{Y}$ is contractively tracial and given positive integers m, n, μ and $Y^1, \dots, Y^\mu \in \mathcal{Y}(m)$ and $n \times m$ matrices $C_1, \dots, C_\mu$ with $\sum C_j^* C_j \preceq I$, the tuple

$$\sum C_j Y^j C_j^*$$

lies in $\mathcal{Y}(n)$. This condition is an analog to matrix convexity of a set containing 0 which we studied earlier in this paper. Surprisingly:

Proposition 7.13. *Every contractively tracial convex set is a free convex cone.*

For the proof of this proposition we introduce an auxiliary notion and then give a lemma. A subset $\mathcal{Y}$ of $\mathbb{S}^g$ is **closed with respect to identical direct sums** if for each $Y \in \mathcal{Y}$ and positive integer ℓ, the tuple $I_\ell \otimes Y$ is in $\mathcal{Y}$.

Lemma 7.14. *Suppose $\mathcal{Y} \subseteq \mathbb{S}^g$.*

(1) *If $\mathcal{Y}$ is contractively tracial and closed with respect to identical direct sums, then $\mathcal{Y}$ is a free cone.*
(2) *If $\mathcal{Y}$ is contractively tracial and closed with respect to direct sums, then $\mathcal{Y}$ is a free convex cone.*
(3) *If $\mathcal{Y}$ is a tracial set containing 0 which is levelwise convex and closed with respect to identical direct sums, then each $\mathcal{Y}(m)$ is a cone in the ordinary sense.*

Proof. To prove the first statement, let $Y \in \mathcal{Y}(n)$ and a positive integer ℓ be given. Let V_k denote the block $1 \times \ell$ row matrices with $m \times n$ matrix entries with I_n in the k-th position and 0 elsewhere, for $k = 1, \dots, \ell$. It follows that $\sum V_k^* V_k = I$. Since also $I_\ell \otimes Y$ is in $\mathcal{Y}$ and $\mathcal{Y}$ is tracial,

$$\sum V_k \big(Y \otimes I_\ell\big) V_k^* = kY \in \mathcal{Y}(n).$$

Now let positive integers m and ℓ and $m \times n$ matrices $C_1, \dots, C_\ell$ and $Y^1, \dots, Y^\ell \in \mathcal{Y}(n)$ be given. Choose a positive integer k such that each $D_j = \frac{C_j}{\sqrt{k}}$ has norm at most one. Consider M_j equal the block $1 \times \ell$ row matrix with $m \times n$ entries with D_j in the j-th position and 0 elsewhere, for $j = 1, \dots, \ell$. It follows that

$$\sum_j M_j^* M_j = \operatorname{diag}(D_1^* D_1, \dots, D_\ell^* D_\ell) \preceq I.$$

Since $\mathcal{Y}$ is tracial, and assuming either $Y^j = Y^k$ for all j, k and $\mathcal{Y}$ is closed under identical direct sums or assuming that $\mathcal{Y}$ is closed under direct sums, $\oplus_{j=1}^{\ell} Y^j$ is in $\mathcal{Y}$ and hence,

$$\sum M_j (\oplus^\ell Y^j) M_j^* = k \sum_j^{\ell} D_j Y^j D_j^* = \sum C_j Y^y C_j^* \in \mathcal{Y}(n).$$

Thus, in the first case $\mathcal{Y}$ is a free cone and in the second a free convex cone.

To prove the third statement, note that the argument used to prove the first part of the lemma shows, if $\mathcal{Y}$ is a tracial set that is closed with respect to identical direct sums and if each $C_j = I$, then $\ell Y = \sum C_j (Y \otimes I_\ell) C_j^*$ is in $\mathcal{Y}(n)$. If $\mathcal{Y}$ is levelwise convex, since also $0 \in \mathcal{Y}(n)$, it follows that $\mathcal{Y}(n)$ is a convex cone. ∎

Proof of Proposition 7.13. Fix positive integers n and ν. Let $Y^1, \dots, Y^\nu \in \mathcal{Y}(n)$ be given. Let C_ℓ denote the inclusion of $\mathbb{R}^n$ as the ℓ-th coordinate in $\mathbb{R}^{n\nu} = \bigoplus_{i=1}^{\nu} \mathbb{R}^n$. In particular, $C_\ell^* C_\ell = I_n$ and hence, $Z^\ell = C_\ell Y^\ell C_\ell^* \in \mathcal{Y}(n\nu)$ (based only on $\mathcal{Y}$ being a tracial set). Now let V_ℓ denote the block $\nu \times \nu$ matrix with $n \times n$ entries with I_n in the (ℓ, ℓ) position and zeros ($n \times n$ matrices) elsewhere. Note that $\sum V_\ell^* V_\ell = I_{n\nu}$. Hence,

$$\sum V_\ell Z^\ell V_\ell^* = \operatorname{diag}\begin{pmatrix} Y^1 & Y^2 & \dots & Y^\nu \end{pmatrix} \in \mathcal{Y}(n\nu).$$

Thus $\mathcal{Y}$ is closed with to identical direct sums. By the second part of Lemma 7.14, $\mathcal{Y}$ is a free convex cone. ∎

Remark 7.15. If $\mathcal{Y} \subseteq \mathbb{S}^g$ is a cone and if $B \in \mathbb{S}_n^g$ is in the polar dual of the set $\mathcal{U}$ consisting of tuples $\sum C_j Y^j C_j^*$ for $Y^j \in \mathcal{Y}$ and C_j such that $\sum C_j^* C_j \preceq I$, then

$$\sum B_j \otimes Y_j \preceq 0$$

for all $Y \in \mathcal{Y}(m)$. In particular, the polar dual $\mathcal{B} = \mathcal{Y}^\circ$ of a cone $\mathcal{Y}$ is a free convex cone.

To prove this assertion, pick $B \in \mathbb{S}_n^g$ in the polar dual of $\mathcal{U}$. Fix a positive integer m. By Lemma 7.4, there exists a positive semidefinite T with trace at most one such that

$$I \otimes T - \sum B_j \otimes Y_j \succeq 0$$

for all $Y \in \mathcal{Y}(m)$. Since $\mathcal{Y}(m)$ is a cone, $I \otimes T - \sum B_j \otimes t^2 Y_j \succeq 0$ for all real t and hence

$$-\sum B_j \otimes Y_j \succeq 0.$$

It follows that

$$-\sum C^* B_j C \otimes Y_j \succeq 0 \tag{7.4}$$

for any C. The conventional polar dual of a set is convex, which implies convex combinations with various C_j in (7.4) are in $\mathcal{B}$. Hence $\mathcal{B}$ is a free convex cone. $\square$

8. Examples

The examples referenced in the body of the paper are gathered together in this section. Some of the examples consider the scalar level $\Gamma(1) \subseteq \mathbb{R}^g$ of a free set $\Gamma \subseteq \mathbb{S}^g$.

Example 8.1. This example shows that it is not necessarily possible to choose V an isometry in equation (2.5) of Theorem 2.4 if the boundedness assumption on $\mathcal{D}_{\mathfrak{L}_B}$ is omitted. Let $g = 1$, and consider $\mathfrak{L}_A(x) = 1 + x$, $\mathfrak{L}_B(x) = 1 + 2x$. In this case,

$$\mathcal{D}_{\mathfrak{L}_B} = \Big\{X : X \succeq -\frac{1}{2}\Big\} \subseteq \mathcal{D}_{\mathfrak{L}_A} = \{X : X \succeq -1\}.$$

It is clear that there does not exists a μ and isometry V such that $A = V^*(I_\mu \otimes B)V$. This example is in fact representative in the sense that if $\mathfrak{L}_B$ is a monic linear pencil and $\mathcal{D}_{\mathfrak{L}_B}$ is unbounded, then there is a monic linear pencil $\mathfrak{L}_A$ with $\mathcal{D}_{\mathfrak{L}_B} \subseteq \mathcal{D}_{\mathfrak{L}_A}$ for which there does not exist a μ and isometry V such that $A = V^*(I_\mu \otimes B)V$. $\square$

Example 8.2. Here is an example of a trace preserving cp map $\phi : \mathcal{S} \to M_2$ with domain an operator system $\mathcal{S}$ that does not admit an extension to a trace non-increasing cp map $\phi : M_2 \to M_2$. This phenomenon contrasts with the classical Arveson extension theorem [Arv69] which says that any ucp map extends to the full algebra.

Let $\mathcal{S} = \operatorname{span}\{I_2, E_{1,2}, E_{2,1}\}$,

$$V = \begin{pmatrix} \sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix},$$

and consider the cp map $\phi : \mathcal{S} \to M_2$,

$$\phi(A) = V^* A V \quad \text{for } A \in \mathcal{S}.$$

We have

$$\phi(I_2) = V^*V = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}, \quad \phi(E_{1,2}) = \frac{\sqrt{3}}{2}E_{1,2}, \quad \phi(E_{2,1}) = \frac{\sqrt{3}}{2}E_{2,1},$$

so ϕ is trace preserving on $\mathcal{S}$.

Now let us consider a cp extension (still denoted by ϕ) of ϕ to M_2. Letting

$$\phi(E_{1,1}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

the Choi matrix for ϕ is

$$C = \begin{pmatrix} a & b & 0 & \frac{\sqrt{3}}{2} \\ b & c & 0 & 0 \\ 0 & 0 & \frac{1}{2} - a & -b \\ \frac{\sqrt{3}}{2} & 0 & -b & \frac{3}{2} - c \end{pmatrix} \succeq 0.$$

Supposing $\phi : M_2 \to M_2$ is trace non-increasing,

$$\begin{aligned} 1 &= \operatorname{tr}(E_{1,1}) \geq \operatorname{tr}(\phi(E_{1,1})) = a + c \\ 1 &= \operatorname{tr}(E_{2,2}) \geq \operatorname{tr}(\phi(E_{2,2})) = 2 - a - c, \end{aligned}$$

whence $a + c = 1$. Since C is positive semidefinite, the nonnegativity of the diagonal of C now gives us

$$0 \leq a \leq \frac{1}{2}.$$

But then the 2×2 minor

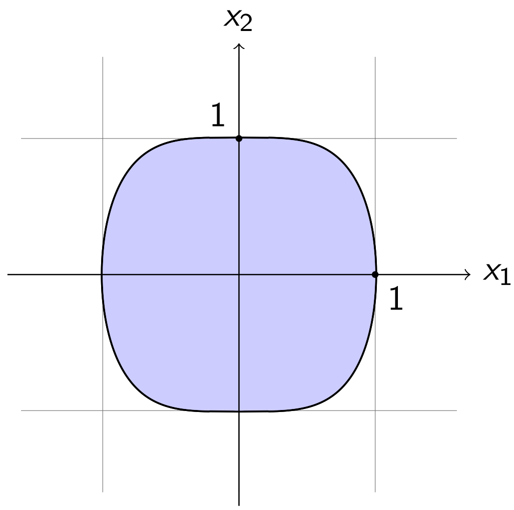
$$\begin{pmatrix} a & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} + a \end{pmatrix}$$

is not positive semidefinite, a contradiction. □

Example 8.3. Consider

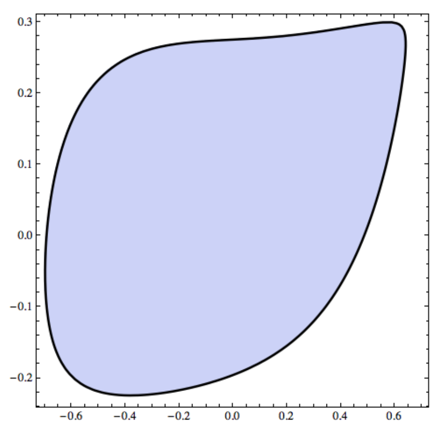
$$p = 1 - x_1^2 - x_2^4.$$

In this case p is symmetric with $p(0) = 1 > 0$.



Bent TV screen $\mathcal{D}_p(1) = \{(x_1, x_2) \in \mathbb{R}^2 : 1 - x_1^2 - x_2^4 \geq 0\}$

The free semialgebraic set $\mathcal{D}_p$ is called the real **bent free TV screen**, or (bent) TV screen for short. While $\mathcal{D}_p(1)$ is convex, it is known that $\mathcal{D}_p$ is not matrix convex, see [DHM07] or [BPR13, Chapter 8]. Indeed, already $\mathcal{D}_p(2)$ is not a convex set.



A non-convex 2-dimensional slice of $\mathcal{D}_p(2)$.

That the set $\mathcal{D}_p(1)$ is a spectrahedral shadow is well known. Indeed, letting

$$L(x_1, x_2, y) = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y \\ x_1 & y & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & x_2 \\ x_2 & y \end{pmatrix},$$

it is readily checked that $\operatorname{proj}_x \mathcal{D}_L(1) = \mathcal{D}_p(1)$. Further, Lemma 4.1 implies that L can be replaced by a monic linear pencil $\mathfrak{L}$. An explicit construction of such an $\mathfrak{L}$ can be found in [HKM16, §7.1]. We remark that $\operatorname{proj}_x \mathcal{D}_L$ strictly contains the matrix convex hull of $\mathcal{D}_p$, cf. [HKM16, §7.1]. □

The next example is one in a classical commutative situation. We refer the reader to [BPR13] for background on classical convex algebraic geometry.

Example 8.4. The polar dual of the bent TV screen $\mathcal{D}_p = \{(X,Y) : 1 - X^2 - Y^4 \succeq 0\}$. We note that $\mathcal{D}_p^\circ(1)$ coincides with the classical polar dual of $\mathcal{D}_p(1)$ by Proposition 4.3, cf. [HKM16, Example 4.7].

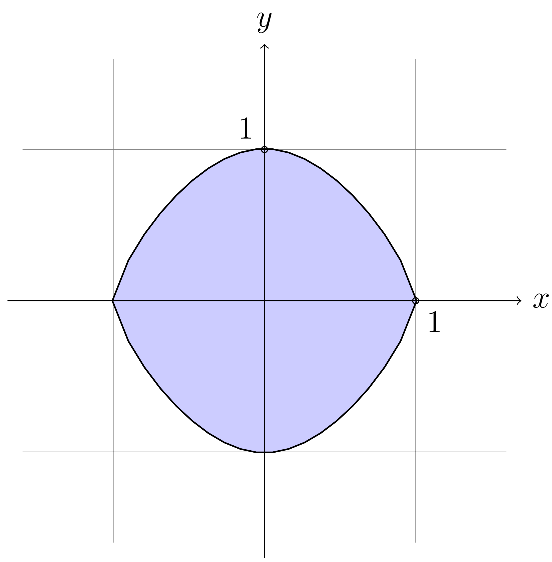
We first find the boundary $\partial\mathcal{D}_p^\circ(1)$ using Lagrange multipliers. Consider a linear function $1-(c_1x+c_2y)$ that is nonnegative but not strictly positive on $\mathcal{D}_p^\circ(1)$ and its values on (the boundary of) $\mathcal{D}_p(1)$. The KarushKuhnTucker (KKT) conditions for first order optimality give

$$1-x^2-y^4=0, \quad c_1=2\lambda x, \quad c_2=4\lambda y^3, \quad 1=c_1x+c_2y.$$

Eliminating x, y, λ leads to the following formula relating c_1, c_2:

$$q(c_1,c_2) := -16c_1^8+48c_1^6-48c_1^4-8c_1^4c_2^4+16c_1^2-20c_1^2c_2^4-c_2^8+c_2^4=0.$$

Thus the boundary of $\mathcal{D}_p^\circ(1)$ is contained in the zero set of q. Since q is irreducible, $\partial\mathcal{D}_p^\circ(1)$ in fact equals the zero set of q. In particular, $\mathcal{D}_p^\circ(1) = \{(x,y)\in\mathbb{R}^2 : q(x,y)\geq 0\}$ is not a spectrahedron, since it fails the line test in [HV07].



Polar dual $\mathcal{D}_p^\circ(1)$ of the bent TV screen. □

Example 8.5. Recall the free bent TV screen is the nonnegativity set $\mathcal{D}_p$ for the polynomial $p = 1 - x^2 - y^4$ (see Example 8.3). Let $\mathcal{K}$ denote the closed matrix convex hull of $\mathcal{D}_p$. Then $\mathcal{K}(1) = \mathcal{D}_p(1)$ and hence, by Proposition 4.3 and Example 8.4, $\mathcal{D}_p^\circ(1) = \mathcal{D}_p(1)^\circ$ is not a spectrahedron. Hence, $\mathcal{D}_p^\circ$ is not a free spectrahedron. In particular, $\mathcal{K}$ cannot be represented by a single Ω as in Theorem 4.6. □

Example 8.6. Tracial and contractively tracial hulls need not be convex (levelwise) as this example shows. Consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

To show that $D = \frac{1}{2}(A+B)$ is not in $\operatorname{thull}(\{A,B\})$, suppose there exists 2×2 matrices $V_1,\dots,V_m$ such that $\sum V_j^* V_j = I$ and

$$\sum V_j A V_j^* = D.$$

On the one hand, the trace of D is zero, on the other hand, $\sum V_j A V_j^*$ has trace 1. Hence D is not in the tracial hull of A. A similar argument shows that D is not in the tracial hull of B. Hence by Lemma 6.3, $D \notin \operatorname{thull}(\{A,B\})$.

Now consider the tuples $A = (A_1, A_2)$ and $B = (B_1, B_2)$ defined by,

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -B_2, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -B_1.$$

In this case $D = \frac{1}{2}(A+B)$ is,

$$D = (D_1, D_2) = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Suppose $\sum C_j^* C_j \preceq I$. Let

$$F_k = \sum C_j A_k C_j^*$$

and note $\operatorname{tr}(F_k) \geq 0$. On the other hand, $\operatorname{tr}(D_k) = 0$. Hence, if $F_k = D_k$, then

$$0 = \operatorname{tr}(F_k) = \sum_j \operatorname{tr}(C_j A_k C_j^*) \geq 0.$$

But then, for each j,

$$0 = \operatorname{tr}\big((C_j(A_1 + A_2)C_j^*\big) = \operatorname{tr}(C_j C_j^*).$$

It follows that $C_j = 0$ for each j and thus $F_k = 0$, a contradiction. Thus, D is not in the contractive tracial hull of A and by symmetry it is not in the contractive tracial hull of B. By Lemma 6.4, D is not in the contractive tracial hull generated by $\{A, B\}$. □

The following example shows a contractively stable set need not be convex.

Example 8.7. Consider the 2×2 matrices A, B from Example 8.6. The smallest contractively stable set containing A, B is the levelwise closed set

$$\mathcal{Y} = \{\sum C_j^* A C_j : \sum C_j^* C_j \preceq I\} \cup \{\sum D_j^* B D_j : \sum D_j^* D_j \preceq I\}.$$

Each matrix in $\mathcal{Y}$ is either positive semidefinite or negative semidefinite, so $\frac{1}{2}(A+B) \notin \mathcal{Y}$. □

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Index

Contents

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