PLURISUBHARMONIC NONCOMMUTATIVE RATIONAL FUNCTIONS

HARRY DYM, J. WILLIAM HELTON[1], IGOR KLEP[2], SCOTT MCCULLOUGH[3], AND JURIJ VOLČIČ[4]

Abstract. A noncommutative (nc) function in $x_1, \dots, x_{\tt g}, x_1^*, \dots, x_{\tt g}^*$ is called plurisubharmonic (plush) if its nc complex Hessian takes only positive semidefinite values on an nc neighborhood of 0. The main result of this paper shows that an nc rational function is plush if and only if it is a composite of a convex rational function with an analytic (no x_j^*) rational function. The proof is entirely constructive. Further, a simple computable necessary and sufficient condition for an nc rational function to be plush is given in terms of its minimal realization.

1. Introduction

This article establishes a representation theorem (Theorem 1.3) for free noncommutative (nc) plurisubharmonic rational functions and an effective criterion (Theorem 1.4) for an nc rational function to be plurisubharmonic. Plurisubharmonic functions are multivariate analogs of subharmonic functions and are central objects in several complex variables [DAn93, For17], in part because of their connection to pseudoconvex domains. Our interest in nc plurisubharmonic rational functions stems from their connection to free domains that can be transformed, via a proper nc rational mapping, to a convex free domain. Free domains and free maps are basic objects studied in free analysis [AM15b, BMV18, MT16, MS08, PT-D17, Pop08, Pop10, SSS18], a quantized analog of classical analysis.

1.1. **Basic notation and terminology.** Let $\langle x, x^* \rangle$ denote the free monoid generated by the $2{\tt g}$ freely noncommuting variables $x_1, \dots, x_{\tt g}, x_1^*, \dots, x_{\tt g}^*$. Elements of $\langle x, x^* \rangle$ are **words**. There is a natural involution * on $\langle x, x^* \rangle$ determined by $x_j \mapsto x_j^*$ and, $(uv)^* = v^* u^*$ for words $u, v \in \langle x, x^* \rangle$. Let $\mathbb{C}\langle x, x^* \rangle$ denote the free algebra of finite $\mathbb{C}$-linear combinations of elements of $\langle x, x^* \rangle$. Elements of $\mathbb{C}\langle x, x^* \rangle$ are **(nc) polynomials**. Thus an nc polynomial p

2000 *Mathematics Subject Classification.* 47A56, 46L07, 32A99, 46L89.

Key words and phrases. Plurisubharmonic function, noncommutative rational function, realization, convex function, free analysis.

[1]Research supported by the NSF grant DMS 1500835.

[2]Supported by the Slovenian Research Agency grants J1-8132, N1-0057 and P1-0222. Partially supported by the Marsden Fund Council of the Royal Society of New Zealand.

[3]Research supported by NSF grant DMS-1764231.

[4]Supported by the Deutsche Forschungsgemeinschaft (DFG) Grant No. SCHW 1723/1-1. Supported by the NSF grant DMS 1954709.

has the form

$$p = \sum_{w \in \langle x, x^* \rangle} p_w \, w, \tag{1.1}$$

where the sum is finite and $p_w \in \mathbb{C}$. The involution * extends to an involution on $\mathbb{C}\langle x, x^* \rangle$. For p of the form (1.1),

$$p^* = \sum \overline{p_w} \, w^*.$$

A polynomial $p \in \mathbb{C}\langle x, x^* \rangle$ is symmetric if $p^* = p$ and is **analytic** if it contains only the x variables and none of the x^* variables. In this latter case we write $p(x) \in \mathbb{C}\langle x \rangle$ instead of $p(x, x^*) \in \mathbb{C}\langle x, x^* \rangle$.

Differentiation of elements of $\mathbb{C}\langle x, x^* \rangle$ is described as follows. Let $h_1, \dots, h_{\mathtt{g}}, h_1^*, \dots, h_{\mathtt{g}}^*$ denote a second $2\mathtt{g}$-tuple of freely noncommuting variables. For $p \in \mathbb{C}\langle x, x^* \rangle$, the **partial of** p **with respect to** x and the **partial of** p **with respect to** x^* are, respectively,

$$p_x(x, x^*)[h, h^*] = \lim_{t \to 0} \frac{p(x + th, x^*) - p(x, x^*)}{t},$$

$$p_{x^*}(x, x^*)[h, h^*] = \lim_{t \to 0} \frac{p(x, (x + th)^*) - p(x, x^*)}{t}.$$

There are four second order partial derivatives. Each lies in $\mathbb{C}\langle x, x^*, h, h^* \rangle$. The mixed partial

$$p_{x,x^*}(x, x^*)[h, h^*] = \lim_{t \to 0} \frac{p_x(x, (x + th)^*)[h, h^*] - p_x(x, x^*)[h, h^*]}{t} \tag{1.2}$$

is the **complex Hessian** of p.

Example 1.1. Consider the polynomial $q(x, x^*) = 1 + 2x_1 x_2^* x_1^* x_2$. Its derivative with respect to x is,

$$q_x(x, x^*)[h, h^*] = 2h_1 x_2^* x_1^* x_2 + 2x_1 x_2^* x_1^* h_2 \in \mathbb{C}\langle x, x^*, h, h^* \rangle.$$

and its complex Hessian is,

$$q_{x,x^*}(x, x^*)[h, h^*] = 2h_1 h_2^* x_1^* x_2 + 2h_1 x_2^* h_1^* x_2 + 2x_1 h_2^* x_1^* h_2 + 2x_1 x_2^* h_1^* h_2. \qquad \square$$

Example 1.2. As a general example, given analytic polynomials $f_j(x)$, the complex Hessian of

$$Q(x, x^*) = \sum_j \zeta_j^*(x) \zeta_j(x)$$

is

$$Q_{x,x^*}(x, x^*)[h, h^*] = \sum_j \zeta_j^*(x)[h] \zeta_j(x)[h]. \qquad \square$$

Let $M_n(\mathbb{C})^{\mathtt{g}}$ denote the set of $\mathtt{g}$-tuples $X = (X_1, \dots, X_{\mathtt{g}})$ of $n \times n$ matrices over $\mathbb{C}$. Let $M(\mathbb{C})^{\mathtt{g}}$ denote the sequence $(M_n(\mathbb{C})^{\mathtt{g}})_n$. An element p of $\mathbb{C}\langle x, x^* \rangle$ is naturally evaluated at a tuple $X \in M(\mathbb{C})^{\mathtt{g}}$ by simply replacing x_j by X_j and x_j^* by X_j^*. The involution on $\mathbb{C}\langle x, x^* \rangle$ and evaluation on $M(\mathbb{C})^g$ is compatible with matrix adjoint; that is,

$$p^*(X, X^*) = p(X, X^*)^*.$$

Moreover, it is well known and easy to see that p is symmetric if and only if $p(X,X^*)^* = p(X,X^*)$ for all $X \in M(\mathbb{C})^{\mathrm{g}}$.

The derivatives of p involve both x and h variables and are thus evaluated at pairs $(X,H) \in M(\mathbb{C})^{2\mathrm{g}}$. Moreover, the derivatives of p are compatible with differentiation after evaluation. For example,

$$p_x(X,X^*)[H,H^*] = \lim_{t\to 0} \frac{p(X+tH,X^*) - p(X,X^*)}{t}.$$

A polynomial $p \in \mathbb{C}\langle x,x^*\rangle$ is **(matrix) positive** if $p(X,X^*) \succeq 0$ for all $X \in M(\mathbb{C})^{\mathrm{g}}$. Here $T \succeq 0$ indicates the selfadjoint matrix T is positive semidefinite. For example, for the polynomial Q of Example 1.2,

$$Q_{x,x^*}(X,X^*)[H,H^*] = \sum_j (\zeta_x(X)[H])^* \zeta_x(X)[H] \succeq 0.$$

Thus Q_{x,x^*} is matrix positive.

A polynomial $p \in \mathbb{C}\langle x,x^*\rangle$ is **plurisubharmonic**, abbreviated **plush**, if its complex Hessian is matrix positive. By the main result of [Gre12] (see also [GHV11]), if $p \in \mathbb{C}\langle x,x^*\rangle$ is plush, then p has the (canonical) form,

$$p(x,x^*) = \ell(x) + \ell(x)^* + \sum_{j=1}^{N} \zeta_j(x)^*\zeta_j(x) + \sum_{k=1}^{M} \eta_k(x)\eta_k(x)^*, \tag{1.3}$$

for some affine linear analytic ℓ and analytic $\zeta_j, \eta_k \in \mathbb{C}\langle x\rangle$.

A symmetric polynomial $f \in \mathbb{C}\langle x,x^*\rangle$ is **convex** if

$$\frac{F(X)+F(Y)}{2} - F\left(\frac{X+Y}{2}\right) \succeq 0$$

for all X, Y, where $F(X) = f(X,X^*)$. The **(full) Hessian** of f is

$$f''(x,x^*)[h,h^*] := f_{x,x}(x,x^*)[h,h^*] + 2f_{x,x^*}(x,x^*)[h,h^*] + f_{x^*,x^*}(x,x^*)[h,h^*], \tag{1.4}$$

Convexity of f is equivalent to matrix positivity of its full Hessian [HM04, Theorem 2.4]. Furthermore, by [HM04, Theorem 3.1], f is convex if and only if there exists an affine linear analytic polynomial $\ell \in \mathbb{C}\langle x\rangle$ and linear polynomials $\varphi_j \in \mathbb{C}\langle x,x^*\rangle$ such that

$$f(x,x^*) = \ell(x) + \ell(x)^* + \sum_j \varphi_j(x,x^*)^*\varphi_j(x,x^*).$$

Hence, writing $\varphi_j(x,x^*) = w_j(x) + y_j(x)^*$, if f is convex, then there exists an analytic (quadratic) polynomial $u(x)$, a positive integer M, and linear analytic polynomials w_j and v_j such that

$$f(x,x^*) = u(x) + u(x)^* + \sum_{j=1}^{M} w_j(x)^*w_j(x) + \sum_{j=1}^{M} y_j(x)y_j(x)^*.$$

There is an intimate connection between convex and plush polynomials. Using variables $z = (u, w_1, \ldots, w_N, y_1, \ldots, y_M)$ and the formal adjoints $z^* = (u^*, w_1^*, \ldots, w_N^*, y_1^*, \ldots, y_N^*)$, the discussion above shows

$$f(z, z^*) = u + u^* + \sum_{j=1}^{N} w_j^* w_j + \sum_{j=1}^{M} y_j y_j^* \tag{1.5}$$

is convex. Further, for the polynomial p of (1.3) and f from (1.5),

$$p(x, x^*) = f(q(x), q(x)^*),$$

where q is the analytic mapping,

$$q(x) = \left[\ell(x),\ \zeta_1(x),\ \ldots,\ \zeta_N(x),\ \eta_1(x),\ \ldots,\ \eta_M(x)\right].$$

Thus, if p is plush, then p is the composition of an analytic polynomial map with a convex polynomial. The converse is evidently true. The main result of this paper establishes the analog of this result for nc rational functions.

1.2. **Noncommutative rational functions.** A **descriptor realization** [BGM05, HMV06, K-VV09] of an nc rational function $r \in \mathbb{C}\langle\!\langle x, x^* \rangle\!\rangle$ [BR11, Coh95] regular at 0 is an expression of the form

$$r(x, x^*) = c^*(J - \Lambda_A(x) - \Lambda_B(x^*))^{-1} b, \tag{1.6}$$

where, for some positive integer d, the $d \times d$ matrix J is invertible, $b, c \in \mathbb{C}^d$, $A, B \in M_d(\mathbb{C})^{\mathrm{g}}$ and

$$\Lambda_A(x) = \sum_{j=1}^{\mathrm{g}} A_j x_j.$$

As an example,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 - x_1 + x_2^* & x_2 \\ x_2^* & -1 + x_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(1 - x_1 + x_2^* - x_2(-1 + x_2)^{-1} x_2^*\right)^{-1}.$$

The $d \times d$ matrix-valued polynomial $\Lambda_A(x) \in M_d(\mathbb{C}\langle x \rangle)$ is evaluated at a g-tuple $X \in M(\mathbb{C})^{\mathrm{g}}$ via the tensor product. Thus if $X \in M_n(\mathbb{C})^{\mathrm{g}}$, then

$$\Lambda_A(X) = \sum_{j=1}^{\mathrm{g}} A_j \otimes X_j \in M_d(\mathbb{C}) \otimes M_n(\mathbb{C}).$$

The descriptor realization of (1.6) is naturally evaluated at any tuple $X \in M_n(\mathbb{C})^{\mathrm{g}}$ for which $J - \Lambda_A(X) - \Lambda_B(X^*)$ is invertible as

$$r(X, X^*) = (c^* \otimes I_n)\ (J \otimes I_n - \Lambda_A(X) - \Lambda_B(X^*))^{-1}\ (b \otimes I_n).$$

In particular $0 \in \mathbb{C}^{\mathrm{g}}$ is in the domain of r, a property we glorify by saying r is **regular at** 0.

If r from (1.6) is symmetric in that $r = r^*$, then it admits a **symmetric descriptor realization**

$$r(x, x^*) = c^*(K - \Lambda_B(x) - \Lambda_{B^*}(x^*))^{-1}c, \tag{1.7}$$

where the $d \times d$ matrix K is a **signature matrix** ($K^2 = I_d,\ K^* = K$). If r from (1.6) is **analytic**, i.e., has no x^* variables, then we may take $B = 0$ in which case

$$r(x) = c^*(J - \Lambda_A(x))^{-1}b.$$

For the purposes of this article, nc rational functions that are regular at 0 can be identified with any one of their descriptor realizations.

The definitions of derivatives for polynomials naturally extend to symmetric and analytic rational functions. Formulas for the derivative, Hessian and complex Hessian of a symmetric descriptor realization are given in Subsection 2.1. In particular, a (symmetric) rational function r is defined to be plush in a neighborhood of 0 if its complex Hessian is matrix positive in a neighborhood of 0. Likewise the notion of convexity for nc polynomials extends to nc rational functions.

1.3. **Main results.** We now state the main results of this article.

Theorem 1.3. *A symmetric nc rational function r in $\mathtt{g}$ variables that is regular at 0 is plush in a neighborhood of 0 if and only if there exists a positive integer $\mathtt{h}$, a convex nc rational function f in $\mathtt{h}$ variables and an analytic nc rational mapping $q : M(\mathbb{C})^{\mathtt{g}} \dashrightarrow M(\mathbb{C})^{\mathtt{h}}$ such that $r = f \circ q$.*

The realization of (1.7) is **minimal** if

$$\operatorname{span}\{w(B_1K, \dots, B_{\mathtt{g}}K, B_1^*K, \dots, B_{\mathtt{g}}^*K)c : w \in <x, \tilde{x}>\} = \mathbb{C}^d,$$

where $\langle x, \tilde{x}\rangle$ is the free monoid on the $2\mathtt{g}$ freely noncommuting variables $(x_1, \dots, x_{\mathtt{g}}, \tilde{x}_1, \dots, \tilde{x}_{\mathtt{g}})$. An nc rational function regular at 0 admits a minimal realization, which is readily computable and unique up to *similarity* and in the symmetric case unique up to *unitary* similarity; see [HMV06, Section 4] or [Vol18, Section 6], Remark 1.7.

Given a tuple $E \in M_d(\mathbb{C})^{\mathtt{g}}$, let $\operatorname{rng} E$ denote the span of the ranges of the E_j. We can now state our second main result.

Theorem 1.4. *Assuming the realization of (1.7) is minimal, r is plush in a neighborhood of 0 if and only if PKP and P_*KP_* are both positive semidefinite, where P and P_* are the orthogonal projections onto $\operatorname{rng} B$ and $\operatorname{rng} B^*$ respectively.*

Remark 1.5. Since minimal realizations for nc rational functions are efficiently computable, Theorem 1.4 implies that so is determining whether an nc rational function is plush. ☐

There is one further result that merits inclusion in this introduction. In [HMV06] and [PT-D] (see also [PT-D17]) nc rational functions that are convex in a neighborhood of 0 are characterized in terms of *butterfly representations*. Below is an alternate characterization in the spirit of Theorem 1.4.

Theorem 1.6. *Assuming the realization of* (1.7) *is minimal, r is convex in a neighborhood of* 0 *if and only if QKQ is positive semidefinite, where Q is the orthogonal projection onto* $\operatorname{rng} B + \operatorname{rng} B^*$.

1.4. **Background and motivation.** Given φ, a perhaps matrix-valued symmetric nc rational function, let $\mathfrak{P}_\varphi(n) = \{X \in M_n(\mathbb{C})^{\mathtt{g}} : \varphi(X) \succ 0\}$. Let $\mathfrak{P}_\varphi$ denote the sequence $(\mathfrak{P}_\varphi(n))_n$. In the case φ is a polynomial, $\mathfrak{P}_\varphi$ is the free analog of a basic semialgebraic set. In several complex variables, Levi pseudoconvex sets are described in terms of plurisubharmonic functions. Pushing this analogy, if φ is plush, then we say $\mathfrak{P}_\varphi$ is a **free pseudoconvex set**. Free pseudoconvex sets are natural for the free analog of several complex variables, particularly as domains for uniform polynomial approximation [AM15a, AM15b] (see also [BMV18, AHKM18]). However, our primary motivation for studying nc plush functions and free pseudoconvex sets arises in another way.

Given a tuple $B \in M_r(\mathbb{C})^{\mathtt{g}}$ and $X \in M_n(\mathbb{C})^{\mathtt{g}}$, let

$$L_B(X) = I_r \otimes I_n - \sum B_j \otimes X_j - \sum B_j^* \otimes X_j^*$$

and let

$$\mathfrak{P}_B(n) = \{X \in M_n(\mathbb{C})^{\mathtt{g}} : L_B(X) \succ 0\}.$$

It is evident that each $\mathfrak{P}_B(n)$ is a convex subset of $M_n(\mathbb{C})^{\mathtt{g}}$. The set $\mathfrak{P}_B(1) \subseteq \mathbb{C}^{\mathtt{g}}$ is a **spectrahedron**. Thus spectrahedra form a class of convex subsets more general than polytopes, but yet with a type of finitary representation. Spectrahedra appear in several branches of mathematics, such as convex optimization and real algebraic geometry [BPR13]. They also play a key role in the solution of the Kadison-Singer paving conjecture [MSS15], and the solution of the Lax conjecture [HV07]. It is natural to call the sequence $\mathfrak{P}_B = (\mathfrak{P}_B(n))_n$ a **free spectrahedron**. Free spectrahedra arise naturally in applications such as systems engineering [dOHMP09] and control theory [HKMS19]. They are also intimately connected to the theories of matrix convex sets, operator algebras and operator systems and completely positive maps [EW97, HKM17, Pau02, PSS18].

By the main result of [HM14] and also [HM12], each $\mathfrak{P}_\varphi(n)$ is convex if and only if $\mathfrak{P}_\varphi$ is a free spectrahedron; that is, there exists a d and tuple $B \in M_d(\mathbb{C})^{\mathtt{g}}$ such that $\mathfrak{P}_\varphi = \mathfrak{P}_B$. In particular, a basic free semialgebraic set is convex if and only if it is a free spectrahedron.

Motivated by systems engineering considerations [SIG96], a problem is to determine, given a free semialgebraic set $\mathfrak{P}_\varphi$ that is not necessarily convex, if there is a free spectrahedron $\mathfrak{P}_B$ and an analytic nc rational mapping $q : \mathfrak{P}_\varphi \to \mathfrak{P}_B$ that is proper, or better still bianalytic. Informally, the problem is to achieve convexity via change of variables. Note that, in any case, the matrix-valued rational function $\psi = L_B \circ q$ is plush and if q is bianalytic, then $\mathfrak{P}_\varphi = \mathfrak{P}_\psi$. On the other hand, if φ is plush, then by Theorem 1.3 there exists a convex function f in $\mathtt{h}$ variables and an analytic rational mapping $q : M(\mathbb{C})^{\mathtt{g}} \dashrightarrow M(\mathbb{C})^{\mathtt{h}}$ such that $\varphi = f \circ q$. Now the set $\mathfrak{P}_{-f} \subseteq M(\mathbb{C})^h$ is convex and hence, by [HM14], there exists $A \in M_d(\mathbb{C})^{\mathtt{g}}$ such that $\mathfrak{P}_{-f} = \mathfrak{P}_A$. Further, $q : \mathfrak{P}_\varphi \to \mathfrak{P}_A$ is proper. Summarizing, there is a proper analytic rational change of variables from $\mathfrak{P}_\varphi$ to a convex set if and only if there is a plush rational function ψ such that $\mathfrak{P}_\varphi = \mathfrak{P}_{-\psi}$.

Of course, in the case there exist distinct bianalytic rational mappings $q : \mathfrak{P}_\varphi \to \mathfrak{P}_B$, and $s : \mathfrak{P}_\varphi \to \mathfrak{P}_E$, then there is a non-trivial bianalytic rational mapping $t : \mathfrak{P}_B \to \mathfrak{P}_E$. The articles [AHKM18, HKMV20] classify, up to some mild hypotheses, the triples $(\mathfrak{P}_B, \mathfrak{P}_E, t)$ where $t : \mathfrak{P}_B \to \mathfrak{P}_E$ is an nc rational bianalytic mapping. Automorphisms of free domains such as balls have been considered by a number of authors including [MT16, MS08, Pop10, SSS18].

1.5. **Readers' guide.** Beyond this introduction, the paper is organized as follows. Formulas for various derivatives of a symmetric descriptor realization, a canonical decomposition of the complex Hessian and a preliminary version of Theorem 1.4 are collected in the next section, Section 2. Theorem 1.4 is proved in Section 3. Theorem 1.6 is proved in Section 4 and the half of Theorem 1.3 that says the composition of a convex rational function and an analytic rational function is plush is obtained as a corollary. The proof of Theorem 1.3 is completed in Section 5. We conclude this introduction with the following remark.

Remark 1.7. Throughout the text we will refer to several existing realization theoretic structural theorems, for example on convex polynomials, rational functions, etc., that are scattered across the literature. However, in this paper we consider functions in variables x and x^*, while in the existing literature most statements involve symmetric or hermitian variables, or variables x and x^T evaluated on real matrices. The reason these results can be applied in the present setting has two justifications. Firstly, for each of the required statements, the version for symmetric variables (and symmetric matrix functions) and the version for hermitian variables (and hermitian matrix functions) have essentially the same proofs; in some cases, e.g. [Vol18], this was outlined explicitly. Secondly, to each function f in $\mathtt{g}$ variables $x_1, \dots, x_{\mathtt{g}}$ and their adjoints $x_1^*, \dots, x_{\mathtt{g}}^*$ one can associate a function s in $2\mathtt{g}$ hermitian variables $y_1, \dots, y_{2\mathtt{g}}$ via

$$s(y_1, \dots, y_{2\mathtt{g}}) = f(y_1 + iy_{\mathtt{g}+1}, \dots, y_{\mathtt{g}} + iy_{2\mathtt{g}}, y_1 - iy_{\mathtt{g}+1}, \dots, y_{\mathtt{g}} - iy_{2\mathtt{g}}),$$
$$f(x_1, \dots, x_{\mathtt{g}}, x_1^*, \dots, x_{\mathtt{g}}^*) = s\left(\frac{x_1 + x_1^*}{2}, \dots, \frac{x_{\mathtt{g}} + x_{\mathtt{g}}^*}{2}, \frac{x_1 - x_1^*}{2i}, \dots, \frac{x_{\mathtt{g}} - x_{\mathtt{g}}^*}{2i}\right).$$

These transforms then enable us to freely move between the (x, x^*)-setting and the hermitian setting from the preceding papers. □

2. Plush preliminaries

Let r denote a symmetric descriptor realization as in (1.7). As preliminary results and background, this section contains formulas for the derivative, complex Hessian and (full)

Hessian of r; a precisely stated preliminary version of Theorem 1.3; and a discussion of minimal descriptor realizations.

2.1. **Derivatives and the Hessians.** Given r as in (1.7), let

$$\Delta(x) = (K - \Lambda_B(x) - \Lambda_{B^*}(x^*))^{-1}, \tag{2.1}$$

and given $X \in M_n(\mathbb{C})^{\mathrm{g}}$ and assuming the inverse exists,

$$\Delta(X) = (K \otimes I_n - \Lambda_B(X) - \Lambda_{B^*}(X^*))^{-1}. \tag{2.2}$$

Thus $r(x) = c^*\Delta(x)c$ and $r(X,X^*) = (c \otimes I_n)^*\Delta(X)(c \otimes I_n)$. Straightforward direct calculation shows that the derivative r_x with respect to x, the complex Hessian r_{x,x^*} and the full Hessian r'' of r are given by

$$\begin{aligned} r_x(x,x^*)[h,h^*] &= c^*\ \Delta(x)\ \Lambda_B(h)\ \Delta(x)\ c \\ r_{x,x^*}(x,x^*)[h,h^*] &= c^*\ \Delta(x)\ \Lambda_B(h)^*\ \Delta(x)\ \Lambda_B(h)\ \Delta(x)\ c \\ &\quad + c^*\ \Delta(x)\ \Lambda_B(h)\ \Delta(x)\ \Lambda_B(h)^*\ \Delta(x)\ c, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} r''(x,x^*)[h,h^*] &= r_{x,x}[h,h^*] + 2r_{x,x^*}[h,h^*] + r_{x^*,x^*}[h,h^*] \\ &= 2\Big[c^*\Delta(x)\Lambda_B(h)\Delta(x)\Lambda_B(h)\Delta(x)c + c^*\Delta(x)\Lambda_B(h)^*\Delta(x)\Lambda_B(h)\Delta(x)c \\ &\quad + c^*\Delta(x)\Lambda_B(h)\Delta(x)\Lambda_B(h)^*\Delta(x)c + c^*\Delta(x)\Lambda_B(h)^*\Delta(x)\Lambda_B(h)^*\Delta(x)c\Big] \\ &= 2c^*\Delta(x)\left(\Lambda_B(h) + \Lambda_B(h)^*\right)\Delta(x)\left(\Lambda_B(h) + \Lambda_B(h)^*\right)\Delta(x)c, \end{aligned} \tag{2.4}$$

respectively.

2.2. **Decomposing the complex Hessian.** A subset $\Omega \subseteq M(\mathbb{C})^{\mathrm{g}}$ is a sequence $\Omega = (\Omega(n))_n$, where $\Omega(n) \subseteq M_n(\mathbb{C})^{\mathrm{g}}$. The set Ω is **closed with respect to direct sums** if $X \in \Omega(n)$ and $Y \in \Omega(m)$ implies

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = \left(\begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}, \dots, \begin{bmatrix} X_{\mathrm{g}} & 0 \\ 0 & Y_{\mathrm{g}} \end{bmatrix} \right) \in \Omega(n+m).$$

Recall, by definition, the descriptor realization r as in (1.7) is **plush on** Ω if $r_{x,x^*}(X,X^*)[H,H^*] \succeq 0$ for each n, each $X \in \Omega(n)$ and each $H \in M_n(\mathbb{C})^{\mathrm{g}}$. (See equation (1.2).) That is, r is plush on Ω if its complex Hessian takes positive semidefinite values on Ω. Given $X, \widetilde{X}, H \in M_n(\mathbb{C})^{\mathrm{g}}$, let

$$\begin{aligned} r_{\downarrow}(X,\widetilde{X})[H] &= C^*\Delta(X)\Lambda_B(H)^*\Delta(\widetilde{X})\Lambda_B(H)\Delta(X)C, \\ r_{\uparrow}(X,\widetilde{X})[H] &= C^*\Delta(\widetilde{X})\Lambda_B(H)\Delta(X)\Lambda_B(H)^*\Delta(\widetilde{X})C, \end{aligned}$$

where $C = c \otimes I_n$.

Proposition 2.1. *Suppose $\Omega \subseteq M(\mathbb{C})^{\mathtt{g}}$ is closed with respect to direct sums. Then the nc rational function r as in (1.7) is plush on Ω if and only if*

$$r_{\downarrow}(X,\widetilde{X})[H] \succeq 0 \quad \text{and} \quad r_{\uparrow}(X,\widetilde{X})[H] \succeq 0 \tag{2.5}$$

for all $X, \widetilde{X} \in \Omega$ and $H \in M(\mathbb{C})^{\mathtt{g}}$.

Proof. Given $X, \widetilde{X} \in \Omega(n)$ and $H \in M_n(\mathbb{C})^{\mathtt{g}}$, define

$$\widehat{X} = \begin{bmatrix} X & 0 \\ 0 & \widetilde{X} \end{bmatrix} \qquad \widehat{H} = \begin{bmatrix} 0 & 0 \\ H & 0 \end{bmatrix}.$$

Since Ω is closed with respect to direct sums, $\widehat{X} \in \Omega(2n)$. For notational convenience, let $\Delta = \Delta(X)$, $\widetilde{\Delta} = \Delta(\widetilde{X})$ and $C = c \otimes I_n$ and observe

$$\begin{aligned} r_{x,x^*}(\widehat{X},\widehat{X}^*)[\widehat{H},\widehat{H}^*] = \\ & \begin{bmatrix} C^* & 0 \\ 0 & C^* \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \begin{bmatrix} 0 & \Lambda_B(H)^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \Lambda_B(H) & 0 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \\ & + \begin{bmatrix} C^* & 0 \\ 0 & C^* \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \Lambda_B(H) & 0 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \begin{bmatrix} 0 & \Lambda_B(H)^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \widetilde{\Delta} \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \end{aligned}$$

and thus

$$0 \preceq r_{x,x^*}(\widehat{X},\widehat{X}^*)[\widehat{H},\widehat{H}^*] = \begin{bmatrix} r_{\downarrow}(X,\widetilde{X})[H] & 0 \\ 0 & r_{\uparrow}(X,\widetilde{X})[H] \end{bmatrix},$$

an identity from which the result immediately follows. □

Let $P, P_* : \mathbb{C}^d \to \mathbb{C}^d$ denote the orthogonal projections onto $\operatorname{rng} B$ and $\operatorname{rng} B^*$ respectively.

Corollary 2.2. *If $\Omega \subseteq M(\mathbb{C})^{\mathtt{g}}$ is closed with respect to direct sums and both $P\Delta(X)P$ and $P_*\Delta(X)P_*$ are positive semidefinite for each tuple $X \in \Omega$, then r is plush on Ω.*

Proof. For $X \in \Omega(n)$ and $H \in M_n(\mathbb{C})^{\mathtt{g}}$, since the range of $\Lambda_B(H)\Delta(X)$ lies in $\operatorname{rng} B \otimes \mathbb{C}^n$, the result follows from Proposition 2.1 by choosing $\widetilde{X} = X$ and using either of the inequalities of (2.5). □

For $\mathtt{k}$ a positive integer and $\varepsilon > 0$, the **(column) free ball** $\mathbb{B}_\varepsilon \subseteq M(\mathbb{C})^{\mathtt{k}}$ of radius ε is the sequence $\mathbb{B}_\varepsilon = (\mathbb{B}_\varepsilon(n))_n$ given by

$$\mathbb{B}_\varepsilon(n) = \left\{ X \in M_n(\mathbb{C})^{\mathtt{k}} \colon \sum_{j=1}^{\mathtt{k}} X_j^* X_j \prec \varepsilon^2 I_n \right\} \subseteq M_n(\mathbb{C})^{\mathtt{k}}.$$

Evidently free balls are closed with respect to direct sums. An nc rational mapping $q : M(\mathbb{C})^{\mathtt{k}} \dashrightarrow M(\mathbb{C})^{\mathtt{h}}$ regular at 0 takes the form $q = \begin{bmatrix} q_1 & q_2 & \dots & q_{\mathtt{h}} \end{bmatrix}$, where each $q_j \in \mathbb{C}\langle\!\langle x \rangle\!\rangle$

is regular at 0. Let $q_x(x)[h] = \begin{bmatrix}(q_1)_x(x)[h] & \dots & (q_{\mathtt{h}})_x(x)[h]\end{bmatrix}$. Thus $q_x(X)[H] \in M_n(\mathbb{C})^{\mathtt{h}}$ for $X, H \in M_n(\mathbb{C})^{\mathtt{g}}$.

Corollary 2.3. *If r is a symmetric nc rational function in $\mathtt{h}$ variables that is plush on some free ball and if $q : M(\mathbb{C})^{\mathtt{g}} \dashrightarrow M(\mathbb{C})^{\mathtt{h}}$ is an nc rational mapping that is regular at 0 with $q(0) = 0$, then $\varphi = r \circ q$ is plush on some free ball.*

Proof. We assume r is given in (1.7) and is plush on $\mathbb{B}_\varepsilon \subseteq M(\mathbb{C})^{\mathtt{h}}$. By (2.3) and the chain rule for $X, H \in M_n(\mathbb{C})^{\mathtt{h}}$,

$$\begin{aligned}
&\varphi_{x,x^*}(X, X^*)[H, H^*] \\
&\quad = C^*\Delta(q(X))\Lambda_B(q_x(X)[H])^*\Delta(q(X))\Lambda_B(q_x(X)[H])\Delta(q(X))C \\
&\qquad + C^*\Delta(q(X))\Lambda_B(q_x(X)[H])\Delta(q(X))\Lambda_B(q_x(X)[H])^*\Delta(q(X))C \\
&\quad = C^*\Delta(Y)\Lambda_B(E)^*\Delta(Y)\Lambda_B(E)\Delta(Y)C + C^*\Delta(Y)\Lambda_B(E)\Delta(Y)\Lambda_B(E)^*\Delta(Y)C \\
&\quad = r_\downarrow(Y, Y)[E] + r_\uparrow(Y, Y)[E],
\end{aligned}$$

where $Y = q(X)$ and $E = q_x(X)[H]$ and $C = c \otimes I_n$. Since $q(0) = 0$, there is a $\delta > 0$ such that for each n and each $X \in \mathbb{B}_\delta(n) \subseteq M_n(\mathbb{C})^{\mathtt{g}}$, we have $Y = q(X) \in \mathbb{B}_\varepsilon(n) \subseteq M_n(\mathbb{C})^{\mathtt{h}}$. By Proposition 2.1, $r_\downarrow(Y, Y)[E], r_\uparrow(Y, Y)[E] \succeq 0$ and hence $\varphi_{x,x^*}(X, X^*)[H, H^*] \succeq 0$ for all n, $X \in \mathbb{B}_\delta(n)$ and $H \in M_n(\mathbb{C})^{\mathtt{g}}$. Thus φ is plush on $\mathbb{B}_\delta$. □

3. A realization theoretic characterization of plush nc rational functions

This section is devoted to the proof of Theorem 1.4, restated as Theorem 3.1 below. A **free neighborhood of** 0 in $M(\mathbb{C})^{\mathtt{g}}$ is a sequence $\Omega = (\Omega(n))_n$, where $\Omega(n) \subseteq M_n(\mathbb{C})^{\mathtt{g}}$ is open and that contains some free ball. In particular, a free ball is a free neighborhood of 0.

Throughout this section r is a symmetric descriptor realization (of size d) as in (1.7) and P and P_* are the orthogonal projections onto $\operatorname{rng} B$ and $\operatorname{rng} B^*$ respectively.

Theorem 3.1. *If P_*KP_* and PKP are positive semidefinite, then r is plush on a free ball; that is, there is an $\varepsilon > 0$ such that $r_{x,x^*}(X, X^*)[H, H^*] \succeq 0$ for all n, $X \in \mathbb{B}_\varepsilon(n)$ and $H \in M_n(\mathbb{C})$.*

Conversely, if r is plush on a free ball and the realization (1.7) is minimal, then P_*KP_* and PKP are both positive semidefinite.

Theorem 3.1 follows by combining Propositions 3.4 and 3.5 below. Recall the notations $\Delta(x)$ and $\Delta(X)$ from (2.1) and (2.2).

Lemma 3.2. *If the realization (1.7) is minimal, then for every $\varepsilon > 0$ there exists an n, an $X \in \mathbb{B}_\varepsilon(n)$ and a vector $v \in \mathbb{C}^n$ such that*

$$z = \Delta(X)(c \otimes v) \in \mathbb{C}^d \otimes \mathbb{C}^n$$

has d linearly independent components in $\mathbb{C}^n$; that is, writing $z = \sum_{j=1}^d e_j \otimes z_j$, the set $\{z_1, \dots, z_d\} \subseteq \mathbb{C}^n$ is linearly independent.

Proof. Substitute $x_j = y_j + iy'_j$ to obtain the matrix-valued symmetric nc rational function $\widetilde{\Delta}(y, y') = \Delta(x)$ in $2\mathtt{g}$ symmetric variables and apply a hermitian version of [HMV06, Lemmas 7.2 and 7.4] (which hold because the local-global principle of linear dependence also works in hermitian settings, cf. [BK13]) to obtain the desired conclusion. □

Lemma 3.3. *Let $\{e_1, \dots, e_d\}$ denote a basis for $\mathbb{C}^d$ and $\mathtt{k}$ be a positive integer. If $z = \sum_{i=1}^d e_i \otimes z_i \in \mathbb{C}^d \otimes \mathbb{C}^n$ and $\{z_1, \dots, z_d\}$ is a linearly independent set of vectors in $\mathbb{C}^n$, then for any $E \in M_d(\mathbb{C})^{\mathtt{k}}$,*

$$\left\{\Lambda_E(H)z \colon H \in M_n(\mathbb{C})^{\mathtt{k}}\right\} = \operatorname{rng} E \otimes \mathbb{C}^n.$$

Proof. We have

$$\Lambda_E(H)z = \sum_{j=1}^{\mathtt{k}} (E_j \otimes H_j)z = \sum_{j=1}^{\mathtt{k}} \sum_{i=1}^{d} E_j e_i \otimes H_j z_i.$$

Fix $1 \le i_0 \le d$, $1 \le j_0 \le \mathtt{k}$ and an $f \in \mathbb{C}^n$. Let $H_j = 0$ for $j \neq j_0$ and let H_{j_0} be such that $H_{j_0} z_i = 0$ for $i \neq i_0$ and $H_{j_0} z_{i_0} = f$. Then

$$\Lambda_E(H)z = E_{j_0} e_{i_0} \otimes f.$$

Since $\mathcal{S} = \{\Lambda_E(H)z \colon H \in M_n(\mathbb{C})^{\mathtt{k}}\} \subseteq \mathbb{C}^d \otimes \mathbb{C}^n$ is a subspace, it follows that $\mathcal{S} \supseteq [\operatorname{rng} E_{j_0}] \otimes \mathbb{C}^n$ and finally that $\mathcal{S} \supseteq [\operatorname{rng} E] \otimes \mathbb{C}^n$. Since the reverse inclusion is evident, the proof is complete. □

Proposition 3.4 (Necessity)**.** *Suppose r as in (1.7) is a minimal realization. If there is an $\varepsilon > 0$ such that $r_{\downarrow}(X, 0)[H] \succeq 0$ for all n, all $X \in \mathbb{B}_\varepsilon(n)$, and all $H \in M_n(\mathbb{C})^{\mathtt{g}}$, then $PKP \succeq 0$. In particular, if r is plush on some free ball, then PKP and $P_* K P_*$ are both positive semidefinite.*

Proof. Since the realization (1.7) is assumed minimal, Lemma 3.2 implies there exists an n, a tuple $X \in \mathbb{B}_\varepsilon(n)$, and a vector v such that $z = \Delta(X)(c \otimes I)v \in \mathbb{C}^d \otimes \mathbb{C}^n$ has d linearly independent components in $\mathbb{C}^n$. By assumption, for this X and v and all H,

$$v^* r_{\downarrow}(X, 0)[H]v = z^* \Lambda_B(H)^* (K \otimes I) \Lambda_B(H) z \ge 0. \tag{3.1}$$

By Lemma 3.3, $\{\Lambda_B(H)z : H \in M_n(\mathbb{C})^{\mathtt{g}}\} = [\operatorname{rng} B] \otimes \mathbb{C}^n$. Thus $PKP \succeq 0$ by (3.1). □

Proposition 3.5 (Sufficiency)**.** *Let Q and R denote the inclusions of* $\operatorname{rng} B$ *and* $\operatorname{rng} B^*$ *into $\mathbb{C}^d$ respectively. If PKP and $P_* K P_*$ are both positive semidefinite, then there is an $\varepsilon > 0$ such that, for each n and $X \in \mathbb{B}_\varepsilon(n)$, both $(Q \otimes I_n)^* \Delta(X)(Q \otimes I_n)$ and $(R \otimes I_n)^* \Delta(X)(R \otimes I_n)$ are positive semidefinite and r is plush on $\mathbb{B}_\varepsilon$.*

Proposition 3.5 can be deduced as a consequence of the construction in Section 5. A direct proof follows and starts with some geometric definitions.

For the $d \times d$ signature matrix K, a subspace $\mathcal{N} \subseteq \mathbb{C}^d$ is **K-nonnegative** if

$$\langle Kh, h \rangle \ge 0$$

for all $h \in \mathcal{N}$. Note that the hypothesis PKP is positive semidefinite in Proposition 3.5 is equivalent to $Q^*KQ \succeq 0$ and to the condition that the range of B is K-nonnegative. If $\mathcal{N}$ is K-nonnegative, then $[h,g] = \langle Kh, g\rangle$ defines a semi-inner product on $\mathcal{N}$. In particular, if $h \in \mathcal{N}$ and $[h,h] = 0$, then $[h,f] = 0$ for all $f \in \mathcal{N}$ and hence

$$\mathcal{N}^0 = \{h \in \mathcal{N} : \langle Kh, h\rangle = 0\} \subseteq \mathcal{N}$$

is a subspace, called the **K-neutral subspace** of $\mathcal{N}$. Now suppose $\mathcal{N}^+ \subseteq \mathcal{N}$ is a complementary subspace to $\mathcal{N}^0$; that is $\mathcal{N}^+ \cap \mathcal{N}^0 = \{0\}$ and $\mathcal{N} = \mathcal{N}^+ + \mathcal{N}^0$. If $h \in \mathcal{N}^+$ and $h \neq 0$, then

$$\langle Kh, h\rangle > 0.$$

Because $\mathcal{N}^+$ is finite dimensional, it follows that there is an $\eta > 0$ such that

$$\langle Kh, h\rangle \geq \eta \|h\|^2,$$

for $h \in \mathcal{N}^+$. Thus, letting $V : \mathcal{N}^+ \to \mathbb{C}^d$ denote the inclusion, we have $V^*KV \succeq \eta I_{\mathcal{N}^+} > 0$.

Proof of Proposition 3.5. For notational purposes, let $\mathcal{R}$ and $\mathcal{R}_*$ denote $\operatorname{rng} B$ and $\operatorname{rng} B^*$ respectively. Let $\mathcal{R}^0$ denote the K-neutral subspace of $\mathcal{R}$. There is a $1 \geq \eta > 0$ and a subspace $\mathcal{R}^+ \subseteq \mathcal{R}$ such that,

(Q1) $\mathcal{R}^0 + \mathcal{R}^+ = \mathcal{R}$ and $\mathcal{R}^0 \cap \mathcal{R}^+ = \{0\}$;
(Q2) $Q_+^* K Q_+ \succeq \eta I_{\mathcal{R}^+}$, where Q_+ denotes the inclusion of $\mathcal{R}^+$ into $\mathbb{C}^d$.

Likewise (after changing $1 \geq \eta > 0$ if needed) there exists a subspace $\mathcal{R}_*^+ \subseteq \mathcal{R}_*$ such that

(R1) $\mathcal{R}_*^0 + \mathcal{R}_*^+ = \mathcal{R}_*$ and $\mathcal{R}_*^0 \cap \mathcal{R}_*^+ = \{0\}$;
(R2) $R_+^* K R_+ \succeq \eta I_{\mathcal{R}_*^+}$, where R_+ denotes the inclusion of $\mathcal{R}_*^+$ into $\mathbb{C}^d$.

Let $\Phi(x,x^*) = \Lambda_B(x) + \Lambda_B(x)^*$. There is an $\varepsilon > 0$ such that if $X \in \mathbb{B}_\varepsilon$, then $\sum_{j=1}^\infty \|\Phi(X,X^*)\|^j < \frac{\eta}{2}$. It suffices to prove, if $X \in \mathbb{B}_\varepsilon(n)$, then $(Q \otimes I_n)^* \Delta(X)(Q \otimes I_n) \succeq 0$ and $(R \otimes I_n)^* \Delta(X)(R \otimes I_n) \succeq 0$.

Suppose $X \in \mathbb{B}_\varepsilon(n)$ and thus $\|\Phi(X,X^*)\| < \frac{\eta}{2} \leq \frac{1}{2}$. In particular, $I_{dn} - \Phi(X,X^*)(K \otimes I_n)$ is invertible and

$$\Delta(X) = (K \otimes I_n - \Phi(X,X^*))^{-1} = [K \otimes I_n](I - \Phi(X,X^*)[K \otimes I_n])^{-1}. \tag{3.2}$$

Note, if $\gamma \in \mathcal{R}^0$ and $\delta \in \mathbb{C}^d$, then $B_j\delta \in \mathcal{R}$ and hence $\delta^* B_j^* K\gamma = 0$. Thus $B_j^* K \gamma = 0$ and hence, for $z \in \mathbb{C}^n$,

$$\begin{aligned}\Phi(X,X^*)(K \otimes I_n)(\gamma \otimes z) &= \Phi(X,X^*)(K\gamma \otimes z) \\ &= \sum B_j K\gamma \otimes X_j z + \sum B_j^* K\gamma \otimes X_j^* z \\ &= \sum B_j K\gamma \otimes X_j z \in \mathcal{R} \otimes \mathbb{C}^n.\end{aligned}$$

It follows that

$$[I_{dn} - \Phi(X,X^*)(K \otimes I_n)](\gamma \otimes z) \in \mathcal{R} \otimes \mathbb{C}^n.$$

Hence

$$\mathcal{S}_X := [I_{dn} - \Phi(X,X^*)(K\otimes I_n)]\mathcal{R}^0 \otimes \mathbb{C}^n \subseteq \mathcal{R}\otimes\mathbb{C}^n.$$

Since $I_{dn} - \Phi(X,X^*)(K\otimes I_n)$ is invertible, $\dim \mathcal{S}_X = n\,\dim\mathcal{R}^0$. Furthermore, using (3.2) and $(Q\otimes I_n)\mathcal{S}_X = \mathcal{S}_X$,

$$\begin{aligned}&(Q^*\otimes I_n)\Delta(X)(Q\otimes I_n)\mathcal{S}_X\\ &\quad= (Q^*\otimes I_n)\Delta(X)[I_{dn} - \Phi(X,X^*)(K\otimes I_n)]\mathcal{R}^0\otimes\mathbb{C}^n\\ &\quad= (Q^*K\otimes I_n)\mathcal{R}^0\otimes\mathbb{C}^n = [Q^*K\mathcal{R}^0]\otimes\mathbb{C}^n = \{0\},\end{aligned}$$

since $\mathcal{R}^0$ is the K-neutral subspace of $\mathcal{R}$. Thus

$$\mathcal{S}_X \subseteq \ker(Q^*\otimes I)\,\Delta(X)\,(Q\otimes I) \subseteq \mathcal{R}\otimes\mathbb{C}^n.$$

Using $\|\Phi(X,X^*)\| < \frac{\eta}{2}$ and $(Q_+\otimes I_n)^*(K\otimes I_n)(Q_+\otimes I_n) \succeq \eta I_{\mathcal{R}^+}$, as well as since $\eta < 1$,

(3.3)

$$\begin{aligned}(Q_+^*\otimes I_n)\,\Delta(X)\,(Q_+\otimes I_n) &= (Q_+^*\otimes I_n)\,(K - \Phi(X,X^*))^{-1}\,(Q_+\otimes I_n)\\ &= (Q_+^*\otimes I_n)K\ (I - \Phi(X,X^*)K)^{-1}\ (Q_+^*\otimes I_n)\\ &= (Q_+^*\otimes I_n)K(Q_+^*\otimes I_n) + (Q_+^*\otimes I_n)\left[\sum_{n=1}^{\infty} K(\Phi(X,X^*)K)^n\right](Q_+^*\otimes I_n)\\ &\succeq Q_+^*KQ_+\otimes I_n - \sum_{k=1}^{\infty}\|\Phi(X,X^*)\|^k I_{\mathcal{R}^+} \succeq \frac{\eta}{2} I_{\mathcal{R}^+}.\end{aligned}$$

In particular, $\mathcal{S}_X \cap (\mathcal{R}^+\otimes\mathbb{C}^n) = \{0\}$.

Summarizing,

(1) $\mathcal{S}_X$, $\mathcal{R}^+\otimes\mathbb{C}^n \subseteq \mathcal{R}\otimes\mathbb{C}^n$;
(2) $(Q^*\otimes I_n)\,\Delta(X)(Q\otimes I_n)\mathcal{S}_X = \{0\}$;
(3) $\dim\mathcal{S}_X = n\,\dim\mathcal{R}^0$ and $\dim\mathcal{R}^+\otimes\mathbb{C}^n = n\,(\dim\mathcal{R} - \dim\mathcal{R}^0)$;
(4) $(Q_+^*\otimes I_n)\Delta(X)(Q_+\otimes I_n) \succeq \frac{\eta}{2} I_{\mathcal{R}^+}$ (see (3.3));
(5) $\mathcal{S}_X \cap (\mathcal{R}^+\otimes\mathbb{C}^n) = \{0\}$.

It follows that $\mathcal{S}_X + [\mathcal{R}^+\otimes\mathbb{C}^n] = \mathcal{R}$ and if $\delta\in\mathcal{S}_X$ and $\gamma\in\mathcal{R}^+\otimes\mathbb{C}^n$, then

$$\begin{aligned}&\langle (Q^*\otimes I_n)\Delta(X)(Q\otimes I_n)(\delta+\gamma), \delta+\gamma\rangle\\ &\qquad= \langle (Q^*\otimes I_n)\Delta(X)(Q\otimes I_n)\gamma,\gamma\rangle \geq \frac{\eta}{2}\|\gamma\|^2 \geq 0.\end{aligned}$$

Hence $(Q^*\otimes I_n)\Delta(X)(Q\otimes I_n) \succeq 0$ as desired. By symmetry, $(R^*\otimes I_n)\Delta(X)(R\otimes I_n) \succeq 0$. Thus $Q^*\Delta(x)Q$ and $R^*\Delta(x)R$ are both positive semidefinite in a neighborhood of 0. Thus r is plush by Corollary 2.2. ☐

4. Convex nc rational functions

Recall that, by definition, a symmetric rational function f is convex on a set $\Omega \subseteq M(\mathbb{C})^g$ if $f''(X,X^*)[H,H^*] \succeq 0$ for all n, $X \in \Omega(n)$ and $H \in M_n(\mathbb{C})^g$. (See equation (1.4).) The main result of this section is Theorem 1.6, restated and proved as Proposition 4.1 below. An immediate consequence is the fact that if a symmetric nc rational function is convex in a free ball, then it is plush in a free ball. Thus, combined with Corollary 2.3, Theorem 1.6 establishes one-half of Theorem 1.3.

Throughout this section, f denotes the symmetric descriptor realization,

$$f(x) = v^*(J - \Lambda_A(x) - \Lambda_A(x)^*)^{-1}v, \tag{4.1}$$

where $\mathtt{h}$ is a positive integer, $A \in M_d(\mathbb{C})^{\mathtt{h}}$ and $0 \neq v \in \mathbb{C}^d$.

Proposition 4.1. *If* $\operatorname{rng} A + \operatorname{rng} A^*$ *is a* J*-nonnegative subspace of* $\mathbb{C}^d$*, then* f *is convex in a neighborhood of* 0*.*

Conversely, if the realization (4.1) *is minimal and* f *is convex in a neighborhood of* 0*, then* $\operatorname{rng} A + \operatorname{rng} A^*$ *is a* J*-nonnegative subspace of* $\mathbb{C}^d$*.*

Corollary 4.2. *If* f *is convex, then* f *is plush.*

Proof. By Proposition 4.1 both $\operatorname{rng} A$ and $\operatorname{rng} A^*$ are J-nonnegative subspaces. An application of Theorem 3.1 completes the proof. □

Corollary 4.3. *Suppose* f *is a symmetric nc rational function in* $\mathtt{h}$ *variables, and* $q : M(\mathbb{C})^{\mathtt{g}} \dashrightarrow M(\mathbb{C})^{\mathtt{h}}$ *is an analytic nc rational mapping. If* f *is convex in a neighborhood of* 0*, then* $r = f \circ q$ *is plush in a neighborhood of* 0*.*

Proof. By Corollary 4.2, since f is convex it is plush. The result now follows from Corollary 2.3. □

The proof of Proposition 4.1 uses Lemma 4.4 below.

Lemma 4.4. *Let* $\mathcal{J} \in M_d(\mathbb{C})$ *be a signature matrix. If* $\mathcal{N} \subseteq \mathbb{C}^d$ *is a* $\mathcal{J}$*-nonnegative subspace, then there is a* $\delta > 0$ *such that if* n *is a positive integer,* $T \in M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$ *is selfadjoint,* $\operatorname{rng} T \subseteq \mathcal{N} \otimes \mathbb{C}^n$ *and* $\|T\| < \delta$*, then*

$$(\mathcal{P} \otimes I_n)\,(\mathcal{J} \otimes I_n - T)^{-1}\,(\mathcal{P} \otimes I_n) \succeq 0,$$

where $\mathcal{P}$ *is the orthogonal projection onto* $\mathcal{N}$*.*

Proof. Let $\mathcal{N}_0$ denote the $\mathcal{J}$-neutral subspace of $\mathcal{N}$. In particular, $\mathcal{P}\mathcal{J}w_0 = 0$ for $w_0 \in \mathcal{N}_0$. Let $\mathcal{N}_+$ denote the orthogonal complement of $\mathcal{N}_0$ in $\mathcal{N}$. Hence $\mathcal{N}_0 \oplus \mathcal{N}_+ = \mathcal{N}$ and $\mathcal{N}_+$ is a $\mathcal{J}$-strictly positive subspace. In particular, there is an $\eta > 0$ such that if $w \in \mathcal{N}_+$, then $\langle \mathcal{J}w, w\rangle \geq \eta\langle w, w\rangle$. Choose $\delta = \frac{\eta}{1+\eta} < 1$ and note $\sum_{j=1}^{\infty} \delta^j = \eta$.

Now let n be given. Let $\widetilde{\mathcal{J}} = \mathcal{J} \otimes I_n$ and note that $\widetilde{\mathcal{N}} := \mathcal{N} \otimes \mathbb{C}^n$ is $\widetilde{\mathcal{J}}$-nonnegative and $\widetilde{\mathcal{N}}_0 := \mathcal{N}_0 \otimes \mathbb{C}^n$ is its $\widetilde{\mathcal{J}}$-neutral subspace. Since $\widetilde{\mathcal{P}} := \mathcal{P} \otimes I_n$ is the orthogonal projection onto the $\widetilde{\mathcal{J}}$-nonnegative subspace $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}_0$ is neutral, $\widetilde{\mathcal{P}}\widetilde{\mathcal{J}}w_0 = 0$ for $w_0 \in \mathcal{N}_0 \otimes \mathbb{C}^n$. Moreover, if $w \in \widetilde{\mathcal{N}}_+$, then $\langle \widetilde{\mathcal{J}}w, w\rangle \geq \eta\langle w, w\rangle$.

Fix T as in the statement of the lemma. Since $\delta < 1$, $\widetilde{\mathcal{J}} - T$ is invertible with the inverse given by the convergent series

$$(\widetilde{\mathcal{J}} - T)^{-1} = \widetilde{\mathcal{J}} + \widetilde{\mathcal{J}} \sum_{j=1}^{\infty} (T\widetilde{\mathcal{J}})^j.$$

If $w_0 \in \widetilde{\mathcal{N}}_0$ and $w_+ \in \widetilde{\mathcal{N}}_+$, then, since $\langle \widetilde{\mathcal{J}}w_0, v\rangle = 0 = \langle w_0, \widetilde{\mathcal{J}}v\rangle$ for $v \in \widetilde{\mathcal{N}}$ and since $\operatorname{rng} T \subseteq \widetilde{\mathcal{N}}$,

$$\langle w_0, \widetilde{\mathcal{J}}(T\widetilde{\mathcal{J}})^j w_+\rangle = 0 = \langle w_0, \widetilde{\mathcal{J}}(T\widetilde{\mathcal{J}})^j w_0\rangle,$$

for all nonnegative integers j. Hence

$$\begin{aligned}\langle (\widetilde{\mathcal{J}} - T)^{-1}(w_0 + w_+), w_0 + w_+\rangle &= \langle \widetilde{\mathcal{J}}w_+, w_+\rangle + \langle \sum_{j=1}^{\infty} \widetilde{\mathcal{J}}(T\widetilde{\mathcal{J}})^j w_+, w_+\rangle \\ &\geq (\eta - \sum_{j=1}^{\infty} \|T\|^j)\|w_+\|^2 \\ &\geq (\eta - \eta)\|w_+\|^2 = 0\end{aligned}$$

and the conclusion of the lemma follows. □

Proof of Proposition 4.1. Let $\Phi(x) = \Lambda_A(x) + \Lambda_A(x)^*$ and let

$$\Gamma(x) = (J - \Lambda_A(x) - \Lambda_A(x)^*)^{-1},$$

and for $X \in M_n(\mathbb{C})^{\mathtt{h}}$ for which the inverse exists,

$$\Gamma(X) = (J \otimes I_n - \Lambda_A(X) - \Lambda_A(X)^*)^{-1}.$$

By (2.4),

$$f''(x, x^*)[h, h^*] = 2v^*\Gamma(x)\Phi(h)\Gamma(x)\Phi(h)\Gamma(x)v.$$

Moreover, f is convex in a neighborhood of 0 if and only if there is a $\eta > 0$ such that for all n, all $X \in \mathbb{B}_\eta(n)$ and all $H \in M_n(\mathbb{C})^g$,

$$f''(X, X^*)[H, H^*] = 2(v^* \otimes I_n)\Gamma(X)\Phi(H)\,\Gamma(X)\,\Phi(H)\Gamma(X)(v \otimes I_n) \succeq 0,$$

by [HMV06, Proposition 5.1], Remark 1.7, and equations (1.4) and (2.4).

Now suppose $\mathcal{N} = \operatorname{rng} A + \operatorname{rng} A^*$ is J-nonnegative. By Lemma 4.4, there is a $\delta > 0$ such that for each n and each tuple $X \in \mathbb{B}_\delta(n)$,

$$\begin{aligned}&(P \otimes I_n)\,\Gamma(X)\,(P \otimes I_n) \\ &\quad=(P^* \otimes I_n)\,(J \otimes I_n - \Lambda_A(X) - \Lambda_A(X)^*)^{-1}(P \otimes I_n) \succeq 0,\end{aligned}$$

where P is the orthogonal projection onto $\mathcal{N}$. Since $\Phi(H)$ maps into the range of $P \otimes I_n$, it follows that $f''(X, X^*)[H, H^*]$ is positive semidefinite for $X \in \mathbb{B}_\delta$. Thus f is convex on $\mathbb{B}_\delta$.

Conversely, suppose there is an $\varepsilon > 0$ such that f is convex on $\mathbb{B}_\varepsilon \subseteq M(\mathbb{C})^{\mathtt{h}}$.

For $H, \widetilde{H} \in M(\mathbb{C})^{\mathtt{h}}$ let

$$\Psi(H, \widetilde{H}) = \Lambda_A(H) + \Lambda_A(\widetilde{H})^*.$$

Given $X, \widetilde{X}, H, \widetilde{H} \in M_n(\mathbb{C})^{\mathtt{h}}$, let

$$\widehat{X} = \begin{bmatrix} X & 0 \\ 0 & \widetilde{X} \end{bmatrix}, \quad \widehat{H} = \begin{bmatrix} 0 & H \\ \widetilde{H} & 0 \end{bmatrix},$$

let

$$\begin{aligned} f_{\downarrow}(X, \widetilde{X})[H, \widetilde{H}] &= \Gamma(X)\Psi(H, \widetilde{H})\Gamma(\widetilde{X})\Psi(H, \widetilde{H})^*\Gamma(X), \\ f_{\uparrow}(X, \widetilde{X})[H, \widetilde{H}] &= \Gamma(\widetilde{X})\Psi(H, \widetilde{H})^*\Gamma(X)\Psi(H, \widetilde{H})\Gamma(\widetilde{X}), \end{aligned}$$

and observe

$$\Phi(\widehat{H}) = \begin{bmatrix} 0 & \Psi(H, \widetilde{H}) \\ \Psi(H, \widetilde{H})^* & 0 \end{bmatrix}$$

and therefore

$$f''(\widehat{X}, \widehat{X}^*)[\widehat{H}, \widehat{H}^*] = 2(v \otimes I_{2n})^* \begin{bmatrix} f_{\downarrow}(X, \widetilde{X})[H, \widetilde{H}] & 0 \\ 0 & f_{\uparrow}(X, \widetilde{X})[H, \widetilde{H}] \end{bmatrix} (v \otimes I_{2n}).$$

Hence, since $f''(\widehat{X}, \widehat{X}^*)[\widehat{H}, \widehat{H}^*]$ is positive semidefinite for $\widehat{X} \in \mathbb{B}_\varepsilon(2n)$ and $\widehat{H} \in M_{2n}(\mathbb{C})^{\mathtt{h}}$,

$$\begin{aligned} (v \otimes I_n)^* \, f_{\downarrow}(X, \widetilde{X})[H, \widetilde{H}] \, (v \otimes I_n) &\succeq 0, \\ (v \otimes I_n)^* \, f_{\uparrow}(X, \widetilde{X})[H, \widetilde{H}] \, (v \otimes I_n) &\succeq 0, \end{aligned}$$

for all $X, \widetilde{X} \in \mathbb{B}_\varepsilon(n)$ and $H, \widetilde{H} \in M_n(\mathbb{C})^{\mathtt{h}}$. In particular, for each $X \in \mathbb{B}_\varepsilon(n)$ and $H, \widetilde{H} \in M_n(\mathbb{C})^{\mathtt{g}}$,

$$\begin{aligned} 0 &\preceq (v \otimes I_n)^* f_{\downarrow}(X, 0)[H, \widetilde{H}](v \otimes I_n) \\ &= (v \otimes I_n)^* \Gamma(X)\Psi(H, \widetilde{H})(J \otimes I_n)\Psi(H, \widetilde{H})^*\Gamma(X)(v \otimes I_n). \end{aligned}$$

Using minimality of the realization for f, by Lemmas 3.2 and 3.3 there exist $X \in \mathbb{B}_\varepsilon(n)$ and $u \in \mathbb{C}^n$ such that the set

$$\{\Psi(H, \widetilde{H})\Gamma(X)(v \otimes u) : H, \widetilde{H} \in M_n(\mathbb{C})^{\mathtt{h}}\}$$

spans $(\operatorname{rng} A + \operatorname{rng} A^*) \otimes \mathbb{C}^n$. Hence $PJP \succeq 0$, where P is the orthogonal projection onto $\operatorname{rng} A + \operatorname{rng} A^*$. $\square$

5. Plush rationals are composite of a convex with an analytic

In this section we prove Theorem 1.3, restated as Theorem 5.1 below. It is the main result of this paper.

Theorem 5.1. *Suppose r is a symmetric nc rational function. If r is plush in a neighborhood of the origin, then there exists a positive integer $\mathtt{h}$, a convex nc rational function f in $\mathtt{h}$ variables, and an analytic nc rational mapping $q : M(\mathbb{C})^{\mathtt{g}} \dashrightarrow M(\mathbb{C})^{\mathtt{h}}$ such that $r = f \circ q$. Moreover, a choice of f and q is explicitly constructed from a minimal realization of r. See formulas (5.11) and (5.13) and Subsection 5.3.3.*

5.1. **A formal recipe for f and q.** We may assume r is a minimal descriptor realization as in formula (1.7). There exist nonnegative integers a and b such that

$$K = \begin{bmatrix} I_a & 0 \\ 0 & -I_b \end{bmatrix}.$$

Since r is, by assumption, plush in a neighborhood of 0, both $\operatorname{rng} B$ and $\operatorname{rng} B^*$ are K-nonnegative by Theorem 3.1. Hence we may assume $a \geq 1$ (as otherwise r is constant). Likewise, we may assume $b \geq 1$ as otherwise r is convex in a neighborhood of 0 by Proposition 4.1, and therefore plush by Corollary 4.2, and the conclusion of the theorem follows upon choosing $q(x) = x$. and $f = r$.

A subspace $\mathcal{P}$ is a **maximal K-nonnegative subspace** if $\mathcal{P}$ is K-nonnegative and if $\mathcal{N}$ is nonnegative with $\mathcal{P} \subseteq \mathcal{N}$, then $\mathcal{N} = \mathcal{P}$. It is well known that, in this case, the dimension of $\mathcal{P}$ is a and moreover, there is a contraction $\rho : \mathbb{C}^a \to \mathbb{C}^b$, known as the **angular operator** for $\mathcal{P}$ [And79], such that $\mathcal{P}$ is the range of the map

$$\begin{bmatrix} I_a \\ \rho \end{bmatrix} : \mathbb{C}^a \to \mathbb{C}^a \oplus \mathbb{C}^b.$$

Let $\rho, \rho_* : \mathbb{C}^a \to \mathbb{C}^b$ denote the angular operators for maximal K-nonnegative subspaces $\mathcal{P}$ and $\mathcal{P}_*$ containing $\operatorname{rng} B$ and $\operatorname{rng} B^*$ respectively. Let P, P_* denote the orthogonal projections onto $\mathcal{P}$ and $\mathcal{P}_*$ respectively. Let $\langle x \rangle$ denote the set of words in $x_1, \dots, x_{\mathtt{g}}$ and $\langle x \rangle_+ = \langle x \rangle \setminus \{1\}$; these are analytic words (no x_j^*s).

If Q is a positive semidefinite matrix, then, up to unitary equivalence, it is of the form $Q_+ \oplus 0$, where Q_+ is positive definite. Hence, again up to unitary equivalence, the Moore-Penrose pseudoinverse $Q^\dagger$ of Q takes the form $Q_+^{-1} \oplus 0$. In particular, the ranges of Q and $Q^\dagger$ are the same. Let D and D_* denote the positive (semidefinite) square roots of $I_a - \rho^*\rho$ and $I_a - \rho_*^*\rho_*$, respectively. Define $\psi : \mathbb{C}^{a+b} \to \mathbb{C}^a \oplus \mathbb{C}^a \oplus \mathbb{C} \oplus \mathbb{C}$ and $\psi_* : \mathbb{C}^a \oplus \mathbb{C}^a \oplus \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}^{a+b}$ by

$$\psi := \begin{bmatrix} D^\dagger \begin{bmatrix} I_a & \rho^* \end{bmatrix} \\ 0_{a \times (a+b)} \\ c^* \\ c^* \end{bmatrix} \qquad \text{and} \qquad \psi_* := \begin{bmatrix} 0_{(a+b)\times a} & \begin{bmatrix} I_a \\ \rho_* \end{bmatrix} D_*^\dagger & c & c \end{bmatrix}.$$

The definition of the formal representation $(J, \mathbb{A}, v)$ of $\mathtt{f}$ is as follows.

(1) Define, for each $w \in \langle x \rangle_+$,

$$\mathbb{A}_w := \psi\, w(KB)\, P_* K \psi_* = \psi\, w(KB)\, K\psi_* \in M_{2a+2}(\mathbb{C}).$$

(2) Let

$$J = I_a \oplus I_a \oplus 1 \oplus -1 \in M_{2a+2}(\mathbb{C}) \tag{5.1}$$

and

$$v = \begin{bmatrix} 0 \\ 0 \\ s \\ t \end{bmatrix} \in \mathbb{C}^a \oplus \mathbb{C}^a \oplus \mathbb{C} \oplus \mathbb{C} = \mathbb{C}^{2a+2}.$$

Here we take $s, t \in \mathbb{C}$ such that $s - t = 1$ and $s + t = c^* K c$ (hence $s^2 - t^2 = c^* K c$).

The expression

$$\begin{aligned} \mathtt{f}(y) &:= v^* \left(J - \sum_{w \in \langle x \rangle_+} (\mathbb{A}_w y_w + \mathbb{A}_w^* y_w^*) \right)^{-1} v \\ &:= v^* \left(J - \sum_{n=1}^{\infty} \sum_{|w|=n} (\mathbb{A}_w y_w + \mathbb{A}_w^* y_w^*) \right)^{-1} v \end{aligned}$$

defines a formal power series in infinitely many variables y_w, y_w^*; more precisely, it is an element of the completion of $\mathbb{C}\langle y_w, y_w^* : w \in \langle x \rangle_+ \rangle$ with respect to the descending chain of ideals

$$J_n = \Big(y_{w_1} \cdots y_{w_\ell} \colon \sum_{k=1}^{\ell} |w_k| = n \Big).$$

In the spirit of Proposition 4.1 one could say that $\mathtt{f}$ is *formally convex.* Let

$$\mathtt{q}(x_1, \dots, x_{\mathtt{g}}) = \mathtt{q}(x) = (w)_{w \in \langle x \rangle_+}.$$

Thus $\mathtt{q}$ is an analytic polynomial mapping with infinitely many outputs.

Theorem 5.2. *Viewing $y_w = q_w(x) = w(x)$ and composing $\mathtt{f}$ with $\mathtt{q}$ gives*

$$r(x) = \mathtt{f}(\mathtt{q}(x)),$$

in the ring of formal power series.

Theorem 5.2 is proved in Subsection 5.2 and it is used in the proof of Theorem 5.1 appearing in Subsection 5.3. Referring to the variables y_w, y_w^* as **intermediate variables**, $\mathtt{f}$ depends on infinitely many intermediate variables and $\mathtt{q}$, while a function of the variables x, outputs the intermediate variables. In Subsection 5.3 as part of the proof of Theorem 5.1, rational f and q are constructed using only finitely many intermediate variables.

5.2. **Proof of Theorem 5.2.** Let $\mathfrak{S}^-$ denote the set of strictly alternating words in two letters $\{\mathtt{x}, \mathtt{y}\}$. Hence, $\mathtt{xyxy}, \mathtt{xyxyx}$, and $\mathtt{yxyxy}$ are examples of such words. We do not include the empty word in $\mathfrak{S}^-$. Using the fact that $\psi_* J\psi = 0$, and hence $\mathbb{A}_w J \mathbb{A}_u = 0$ for $u, w \in \langle x \rangle_+$, compute

$$
\begin{aligned}
\mathtt{f}(\mathtt{q}(x)) &= v^* \left(I - \Lambda_{J\mathbb{A}}(\mathtt{q}(x)) - \Lambda_{J\mathbb{A}^*}(\mathtt{q}(x)^*)\right)^{-1} Jv \\
&= v^* \left[\sum_{k=1}^{\infty} \left(\Lambda_{J\mathbb{A}}(\mathtt{q}(x)) + \Lambda_{J\mathbb{A}^*}(\mathtt{q}(x)^*)\right)^n\right] Jv + v^* Jv \\
&= v^* \left[\sum_{w \in \mathfrak{S}^-} w(\Lambda_{J\mathbb{A}}(\mathtt{q}(x)), \Lambda_{J\mathbb{A}^*}(\mathtt{q}(x)^*))\right] Jv + v^* Jv.
\end{aligned}
\tag{5.2}
$$

The next and longest part of the argument simplifies $w(\Lambda_{JA}(q(x)), \Lambda_{JA^*}(q(x)^*))$ for $w \in \mathfrak{S}^-$.

Lemma 5.3. *For* $1 \le j, \ell \le \mathtt{g}$,

$$
\begin{aligned}
B_\ell^* K \left[\psi^* J \psi\right] K B_j &= B_\ell^* K B_j \\
B_\ell K \left[\psi_* J \psi_*^*\right] K B_j^* &= B_\ell K B_j^*.
\end{aligned}
$$

The proof of Lemma 5.3 uses the following construction. First note that the orthogonal projection P onto $\mathcal{P}$ is given by

$$
P = \begin{bmatrix} I \\ \rho \end{bmatrix} (I + \rho^* \rho)^{-1} \begin{bmatrix} I & \rho^* \end{bmatrix}
$$

and a similar formula holds for P_*, the orthogonal projection onto $\mathcal{P}_*$. Set

$$
E_j = (I + \rho^* \rho)^{-1} \begin{bmatrix} I & \rho^* \end{bmatrix} B_j \begin{bmatrix} I \\ \rho_* \end{bmatrix} (I + \rho_*^* \rho_*)^{-1} \in M_a(\mathbb{C}).
$$

Thus,

$$
P B_j P_* = \begin{bmatrix} I \\ \rho \end{bmatrix} E_j \begin{bmatrix} I & \rho_*^* \end{bmatrix}.
$$

Finally, since $(\ker B_j)^\perp = \operatorname{rng} B_j^* \subseteq \mathcal{P}_*$, it follows that $P B_j P_* = B_j$. Hence,

$$
B_j = \begin{bmatrix} I \\ \rho \end{bmatrix} E_j \begin{bmatrix} I & \rho_*^* \end{bmatrix}.
\tag{5.3}
$$

Proof of Lemma 5.3. Compute,

$$
\psi^* J \psi = \begin{bmatrix} I \\ \rho \end{bmatrix} (D^\dagger)^2 \begin{bmatrix} I & \rho^* \end{bmatrix}.
$$

Thus, using formula (5.3), $(I - \rho^* \rho)(D^\dagger)^2 (I - \rho^* \rho) = D^2 (D^\dagger)^2 D^2 = I - \rho^* \rho$ and

$$
\begin{bmatrix} I_a & \rho^* \end{bmatrix} K \begin{bmatrix} I_a \\ \rho \end{bmatrix} = I - \rho^* \rho,
$$

it follows that

$$\begin{aligned} B_\ell^* K\psi^* J\psi K B_j &= \begin{bmatrix} I \\ \rho_* \end{bmatrix} E_\ell^*(I-\rho^*\rho)(D^\dagger)^2(I-\rho^*\rho)E_j \begin{bmatrix} I & \rho_*^* \end{bmatrix} \\ &= \begin{bmatrix} I \\ \rho_* \end{bmatrix} E_\ell^*(I-\rho^*\rho)E_j \begin{bmatrix} I & \rho_*^* \end{bmatrix} \\ &= B_\ell^* K B_j. \end{aligned}$$

The other identity can be proved in a similar fashion. We omit the details. □

For notational purposes, let $\widetilde{\Omega}$ and $\widetilde{\Gamma}$ denote the formal power series

$$\widetilde{\Omega}(x) = \sum_{n=1}^{\infty}(K\Lambda_B(x))^n = \sum_{w\in\langle x\rangle_+} w(KB)\, w(x) = \sum_{j=1}^{\mathtt{g}} KB_j \sum_{w\in\langle x\rangle} w(KB)\, x_j w(x)$$

and

$$\widetilde{\Gamma}(x^*) = \sum_{n=1}^{\infty}(K(\Lambda_B(x))^*)^n = \sum_{w\in\langle x\rangle_+} Kw(KB)^*K\, w(x)^* = K\widetilde{\Omega}(x)^*K.$$

With these notations,

$$\begin{aligned} \Lambda_{J\mathbb{A}}(\mathfrak{q}(x)) &= \sum_{w\in\langle x\rangle_+} J\mathbb{A}_w \mathfrak{q}_w(x) = \sum_w J\psi\, w(KB)P_*K\,\psi_*\, w(x) \\ &= J\psi\big[\sum_w w(KB)w(x)\big]\ P_*K\,\psi_* = J\psi\ \widetilde{\Omega}(x)\ P_*K\psi_* \\ &= J\psi\ \widetilde{\Omega}(x)\ K\psi_* \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \Lambda_{J\mathbb{A}^*}(\mathfrak{q}(x)^*) &= \sum_{w\in\langle x\rangle_+} J\mathbb{A}_w^* \mathfrak{q}_w(x)^* = \sum_w J\psi_*^* KP_*\, w(KB)^*\,\psi^*\ w(x)^* \\ &= J\psi_*^* KP_*\ \widetilde{\Omega}(x)^*\ \psi^* \\ &= J\psi_*^* K\ \widetilde{\Omega}(x)^*\ \psi^* = J\psi_*^*\ \widetilde{\Gamma}(x^*)K\psi^*. \end{aligned} \tag{5.5}$$

Further, using Lemma 5.3,

$$\widetilde{\Gamma}(x^*)K\,[\psi^* J\psi]\widetilde{\Omega}(x) = \widetilde{\Gamma}(x^*)\,\widetilde{\Omega}(x). \tag{5.6}$$

Combining (5.4), (5.5) and (5.6) gives

$$\begin{aligned} \Lambda_{J\mathbb{A}^*}(\mathfrak{q}(x)^*)\Lambda_{J\mathbb{A}}(\mathfrak{q}(x)) &= J\psi_*^*\ [\widetilde{\Gamma}(x^*)K\psi^*J\psi\ \widetilde{\Omega}(x)]\ K\psi_* \\ &= J\psi_*^*\ [\widetilde{\Gamma}(x^*)\widetilde{\Omega}(x)]\ K\psi_*. \end{aligned}$$

Similarly,

$$\widetilde{\Omega}(x)K\psi J\psi^*\widetilde{\Gamma}(x^*) = \widetilde{\Omega}(x)\widetilde{\Gamma}(x^*).$$

Thus,

$$\begin{aligned}\Lambda_{J\mathbb{A}}(\mathtt{q}(x))\Lambda_{J\mathbb{A}^*}(\mathtt{q}(x)^*) &= J\psi\ [\widetilde{\Omega}(x)\psi_* J\psi_*^*\widetilde{\Gamma}(x^*)]\ K\psi^* \\ &= J\psi\ [\widetilde{\Omega}(x)\ \widetilde{\Gamma}(x^*)]\ K\psi^*.\end{aligned}$$

Next turn to an alternating word, say $w(\mathbb{x},\mathbb{y}) = \mathbb{x}\mathbb{y}\cdots\mathbb{x}\mathbb{y}$ where $\mathbb{x}$ and $\mathbb{y}$ each appear N times. Writing $\widetilde{\Omega}$ and $\widetilde{\Gamma}$ instead of $\widetilde{\Omega}(x)$ and $\widetilde{\Gamma}(x^*)$ and computing as above,

$$\begin{aligned}w(\Lambda_{J\mathbb{A}}(\mathtt{q}(x)),\Lambda_{J\mathbb{A}^*}(\mathtt{q}(x)^*)) &= J\psi\ \widetilde{\Omega}\ \widetilde{\Gamma}\ \widetilde{\Omega}\cdots\widetilde{\Gamma}\ \widetilde{\Omega}\ \widetilde{\Gamma}\ K\psi^* \\ &= J\psi\ w(\widetilde{\Omega},\widetilde{\Gamma})\ K\psi^*.\end{aligned} \tag{5.7}$$

The last step in the proof of Theorem 5.2 is to match moments as follows. The right hand side of (5.7) is the sum over all terms of the form

$$T = J\psi\ (K\Lambda_B(x))^{n_1}(K\Lambda_B(x)^*)^{m_1}\cdots(K\Lambda_B(x))^{n_N}(K\Lambda_B(x)^*)^{m_N}\ K\psi^*,$$

for positive integers n_j, m_j. Further,

$$v^*TJv = c^*(K\Lambda_B(x))^{n_1}(K\Lambda_B(x)^*)^{m_1}\cdots(K\Lambda_B(x))^{n_N}(K\Lambda_B(x)^*)^{m_N}\ Kc$$

and

$$v^*Jv = s^2 - t^2 = c^*Kc.$$

Hence, letting $\mathcal{T}$ denote all possible products of the form T (save for the empty product) and $\langle x, x^*\rangle_+$ the nonempty words in (x, x^*),

$$\begin{aligned}v^*[\sum_{w\in\mathfrak{S}^-} w(\Lambda_{J\mathbb{A}}(\mathtt{q}(x)),\Lambda_{J\mathbb{A}^*}(\mathtt{q}(x)^*))]Jv + v^*Jv &= \sum_{T\in\mathcal{T}} v^*TJv + v^*Jv \\ &= c^*\,[\sum_{u\in\langle x,x^*\rangle_+} u(KB, KB^*)]\,Kc + c^*Kc \\ &= c^*(K - \Lambda_B(x) - \Lambda_B(x)^*)^{-1}c,\end{aligned} \tag{5.8}$$

since the sum over $w \in \mathfrak{S}^-$ gives all possible products of $K\Lambda_B(x), K\Lambda_B(x)^*$ save for the empty product (I). Combining (5.8) and (5.2) completes the proof of Theorem 5.2.

5.3. **Proof of Theorem 5.1.** In this section Theorem 5.1 is deduced from Theorem 5.2. It is possible to prove Theorem 5.1 directly.

5.3.1. *A recipe for f and q having finitely many intermediate variables.* In the construction of $\mathtt{f}$ and $\mathtt{q}$ in Subsection 5.1 the intermediate space has infinitely many variables. In this subsection that construction is refined, under the additional assumption that $\{B_1,\dots,B_{\mathtt{g}}\}$ is linearly independent, to produce rational convex f and analytic q having an intermediate space with finitely many variables that are shown, in Subsection 5.3.2, to satisfy the conclusion of Theorem 5.1. Finally, Subsection 5.3.3 shows how to pass from linear dependence to independence of the set $\{B_1,\dots,B_{\mathtt{g}}\}$.

To construct f and q, let $\{C_1, \dots, C_{\mathtt{h}}\}$ denote a basis for the algebra generated by $\{KB_1, \dots, KB_{\mathtt{g}}\}$ and, without loss of generality, assume $C_j = KB_j$ for $1 \le j \le \mathtt{g}$ (since we are assuming $\{B_1, \dots, B_{\mathtt{g}}\}$ is linearly independent) and for $\mathtt{g}+1 \le j \le \mathtt{h}$, that

$$C_j = \mathbb{w}_j(KB)$$

for some (non-empty) word $\mathbb{w}_j$. Note that $\mathtt{h} \le (a+b)^2$ as $KB_j \in M_{a+b}(\mathbb{C})$. In particular, we can set $\mathbb{w}_j = x_j$ for $1 \le j \le \mathtt{g}$.

There is an $\mathtt{h}$-tuple $\Xi \in M_{\mathtt{h}}(\mathbb{C})^{\mathtt{h}}$ such that for each $1 \le j, k \le \mathtt{h}$,

$$C_j C_k = \sum_{s=1}^{\mathtt{h}} (\Xi_k)_{j,s} C_s, \tag{5.9}$$

though we will be mostly interested in $1 \le j, k \le \mathtt{g}$. Moreover, for $1 \le j \le \mathtt{h}$ and a word w in $(x_1, \dots, x_{\mathtt{h}})$,

$$C_j\, w(C) = \sum_{s=1}^{\mathtt{h}} w(\Xi)_{j,s} C_s, \tag{5.10}$$

by [HKMV20, Lemma 2.5].

Define f and q as follows.

(1) Let J denote the signature matrix defined in (5.1) and, for $1 \le s \le \mathtt{h}$, define

$$A_s := \mathbb{A}_{\mathbb{w}_s} = \psi\ \mathbb{w}_s(KB)\ P_* K\psi_* = \psi\ \mathbb{w}_s(KB)\ K\psi_*.$$

Set

$$f(y) = v^*(J - \Lambda_A(y) - \Lambda_A(y)^*)^{-1} v, \tag{5.11}$$

where $A = (A_1, \dots A_{\mathtt{h}}) \in M_{2a+2}(\mathbb{C})^{\mathtt{h}}$ and $y = (y_1, \dots, y_{\mathtt{h}})$.
Since

$$\operatorname{rng} A + \operatorname{rng} A^* \subseteq \left\{ \begin{bmatrix} D^\dagger \begin{bmatrix} I_a & \rho^* \end{bmatrix} u \\ D_*^\dagger \begin{bmatrix} I_a & \rho_*^* \end{bmatrix} z \\ c^*(u+z) \\ c^*(u+z) \end{bmatrix} : u, z \in \mathbb{C}^a \oplus \mathbb{C}^b \right\},$$

it follows that $\operatorname{rng} A + \operatorname{rng} A^*$ is J-nonnegative and therefore f is convex, by Proposition 4.1.

(2) Let $b(y) = \begin{bmatrix} b_1(y) & \dots & b_{\mathtt{h}}(y) \end{bmatrix}$ denote the map associated to Ξ by

$$b(y) = y(I - \Lambda_\Xi(y))^{-1}. \tag{5.12}$$

For $1 \le s \le \mathtt{h}$, let

$$q_s(x) = b_s(x_1, \dots, x_{\mathtt{g}}, 0, \dots, 0) = \sum_{j=1}^{\mathtt{g}} \sum_{w \in \langle x \rangle} (w(\Xi))_{j,s} x_j w. \tag{5.13}$$

Evidently $q = \begin{bmatrix} q_1 & \cdots & q_{\mathtt{h}} \end{bmatrix}$ is analytic and rational.

Remark 5.4. The nc rational mapping $b(y)$ of (5.12), associated to a tuple Ξ satisfying (5.9), is a **convexotonic** map, see [HKMV20, Section 1.1 and Lemma 2.5]. Up to linear change of variables and an irreducibility assumption, convexotonic maps are the only bianalytic maps between free spectrahedra [AHKM18, HKMV20].

5.3.2. *Proof that $r = f \circ q$.* Since $r = \mathtt{f} \circ \mathtt{q}$ by Theorem 5.2, both f and q are rational, f is convex and q is analytic, Theorem 5.1 in the case that $\{B_1, \dots, B_{\mathtt{g}}\}$ is linearly independent is a consequence of Proposition 5.5.

Proposition 5.5. $\mathtt{f}(\mathtt{q}(x)) = f(q(x))$.

Proof. Since

$$\mathtt{f}(\mathtt{q}(x)) = v^* \left(I - \Lambda_{J\mathbb{A}}(\mathtt{q}(x)) - \Lambda_{J\mathbb{A}^*}(\mathtt{q}(x)^*)\right)^{-1} Jv$$

and

$$f(q(x)) = v^* \left(I - \Lambda_{JA}(q(x)) - \Lambda_{JA^*}(q(x)^*)\right)^{-1} Jv,$$

the conclusion follows from Lemma 5.6 below. ☐

Lemma 5.6. *With notations as above,*

$$\Lambda_A(q(x)) = \Lambda_{\mathbb{A}}(\mathtt{q}(x)).$$

Recall the notation $C_j = \mathtt{w}_j(KB)$ for $1 \le j \le \mathtt{h}$ and that $C_j = KB_j$ for $1 \le j \le \mathtt{g}$. Thus, by (5.10), for $1 \le j \le \mathtt{g}$ and $w \in \langle x \rangle$,

$$KB_j \; w(KB) = \sum_{s=1}^{\mathtt{h}} w(\Xi)_{j,s} \; \mathtt{w}_s(KB) = \sum_{s=1}^{\mathtt{h}} w(\Xi)_{j,s} C_s. \tag{5.14}$$

Proof. Using the identity in (5.14) in the fourth equality

$$\begin{aligned}
\Lambda_{\mathbb{A}}(\mathtt{q}(x)) &= \psi\ \Lambda_{\big(w(KB)\big)_{\langle x\rangle_+}}(\mathtt{q}(x))\ P_*K\psi_* \\
&= \psi \sum_{w\in\langle x\rangle_+} w(KB)\ \mathtt{q}_w(x)\ P_*K\psi_* \\
&= \psi \sum_{u=1}^{\mathtt{g}} \sum_{w\in\langle x\rangle} [KB_u w(KB)]\ x_u w(x)\ P_*K\psi_* \\
&= \psi \sum_{u=1}^{\mathtt{g}} \sum_{w\in\langle x\rangle} [\sum_{j=1}^{\mathtt{h}} (w(\Xi))_{u,j} C_j\ ]\ x_u w(x)\ P_*K\psi_* \\
&= \sum_{j=1}^{\mathtt{h}} \psi\ C_j\ P_*K\psi_*\ [\sum_{u=1}^{\mathtt{g}} \sum_{w\in\langle x\rangle} (w(\Xi))_{u,j} x_u\, w(x)] \\
&= \sum_{j=1}^{\mathtt{h}} \psi\ [\mathbb{W}_j(KB)\ P_*K\psi_*]\ q_j(x) \\
&= \sum_{j=1}^{\mathtt{h}} A_j q_j(x) = \Lambda_A(q(x)).
\end{aligned}$$

□

5.3.3. *Linearly dependent B_j.* To complete the proof of Theorem 1.3, suppose, without loss of generality, that $1 \le \mathtt{k} \le \mathtt{g}$ and $\{\widehat{B}_1, \dots, \widehat{B}_{\mathtt{k}}\}$ is a basis for the span of $\{B_1, \dots, B_{\mathtt{g}}\}$. Let

$$\widehat{r}(y) = c\left(K - \sum_{j=1}^{\mathtt{k}} \widehat{B}_j y_j - \sum \widehat{B}_j^* y_j^*\right)^{-1} c.$$

Thus $\widehat{r}$ is a symmetric descriptor realization. There is a $\mathtt{g}\times\mathtt{k}$ matrix M such that $r(x) = \widehat{r}(Mx)$. Moreover, since $\operatorname{rng}\widehat{B} = \operatorname{rng} B$ and $\operatorname{rng}\widehat{B}^* = \operatorname{rng} B^*$ and since r is assumed plush, Theorem 3.1 implies $\widehat{r}$ is also plush. Thus, by what has already been proved, there exists a positive integer $\mathtt{h}$, an analytic nc rational mapping $\widehat{q} : M(\mathbb{C})^{\mathtt{k}} \dashrightarrow M(\mathbb{C})^{\mathtt{h}}$ and a convex nc rational function f (in $\mathtt{h}$ variables) such that $\widehat{r}(y) = (f\circ\widehat{q})(y)$. Set $q(x) = \widehat{q}(Mx)$. Thus q is an analytic nc rational mapping and $r = f\circ q$.

References

[And79] T. Ando, *Linear operators on Kreĭn spaces.* Hokkaido University, Research Institute of Applied Electricity, Division of Applied Mathematics, Sapporo, 1979. ii+59 pp. 17

[AM15a] J. Agler, J. McCarthy: *Global holomorphic functions in several non-commuting variables*, Canad. J. Math. 67 (2015) 241–285. 6

[AM15b] J. Agler, J.E. McCarthy: *Pick interpolation for free holomorphic functions*, Amer. J. Math. 137 (2015) 1685–1701. 1, 6

[AHKM18] M. Augat, J.W. Helton, I. Klep, S. McCullough: *Bianalytic Maps Between Free Spectrahedra*, Math. Ann. 371 (2018) 883–959. 6, 7, 23

[BGM05] J.A. Ball, G. Groenewald, T. Malakorn: Structured noncommutative multidimensional linear systems. SIAM J. Control Optim. 44 (2005) 1474–1528. 4

[BMV18] J.A. Ball, G. Marx, V. Vinnikov: *Interpolation and transfer-function realization for the noncommutative Schur–Agler class*, Operator theory in different settings and related applications, 23–116, Oper. Theory Adv. Appl. 262, Birkhäuser/Springer, Cham, 2018. 1, 6

[BR11] J. Berstel, C. Reutenauer: *Noncommutative rational series with applications*, Encyclopedia of Mathematics and its Applications 137, Cambridge University Press, Cambridge, 2011. 4

[BPR13] G. Blekherman, P.A. Parrilo, R.R. Thomas (editors): *Semidefinite optimization and convex algebraic geometry*, MOS-SIAM Series on Optimization 13, SIAM, 2013. 6

[BK13] M. Brešar, I. Klep: *A local-global principle for linear dependence of noncommutative polynomials.* Israel J. Math. 193 (2013) 71–82. 11

[Coh95] P.M. Cohn: *Skew fields. Theory of general division rings*, Encyclopedia of Mathematics and its Applications, 57. Cambridge University Press, 1995. 4

[DAn93] J.P. D'Angelo: *Several complex variables and the geometry of real hypersurfaces*, CRC Press, 1993. 1

[dOHMP09] M. de Oliveira, J.W. Helton, S. McCullough, M. Putinar: *Engineering systems and free semi-algebraic geometry*, in: Emerging applications of algebraic geometry (edited by M. Putinar, S. Sullivant), 17–61, Springer-Verlag, 2009. 6

[EW97] E.G. Effros, S. Winkler: *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, J. Funct. Anal. 144 (1997) 117–152. 6

[For17] F. Forstnerič: *Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis.* Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3rd Series, 56. Springer, Cham, 2017. 1

[Gre12] J.M. Greene: *Noncommutative plurisubharmonic polynomials part II: Local assumptions,* J. Math. Anal. Appl. 396 (2012) 481–496. 3

[GHV11] J.M. Greene, J.W. Helton, V. Vinnikov: *Noncommutative Plurisubharmonic Polynomials Part I: Global Assumptions*, J. Funct. Anal. 261 (2011) 3390–3417. 3

[HKM17] J.W. Helton, I. Klep, S. McCullough: *The tracial Hahn-Banach theorem, polar duals, matrix convex sets, and projections of free spectrahedra*, J. Eur. Math. Soc. 19 (2017) 1845–1897. 6

[HKMS19] J.W. Helton, I. Klep, S. McCullough, M. Schweighofer: *Dilations, Linear Matrix Inequalities, the Matrix Cube Problem and Beta Distributions*, Mem. Amer. Math. Soc. 257 (2019), no. 1232, 104 pp. 6

[HKMV20] J.W. Helton, I. Klep, S. McCullough and J. Volčič: *Bianalytic free maps between spectrahedra and spectraballs,* J. Funct. Anal. 278 (2020) 108472, 61pp. 7, 22, 23

[HM04] J.W. Helton, S.A. McCullough: *Convex polynomials have Degree Two or Less*, SIAM J. Matrix Anal. Appl. 25 (2004) 1124–1139. 3

[HM12] J.W. Helton, S. McCullough: *Every convex free basic semi-algebraic set has an LMI representation.* Ann. of Math. (2) 176 (2012) 979–1013. 6

[HM14] J.W. Helton, S. McCullough: *Free convex sets defined by rational expressions have LMI representations,* J. Convex Anal. 21 (2014) 425–448. 6, 7

[HMV06] J.W. Helton, S.A. McCullough, V. Vinnikov: *Noncommutative convexity arises from linear matrix inequalities*, J. Funct. Anal. 240 (2006) 105–191. 4, 5, 11, 15

[HV07] J.W. Helton, V. Vinnikov: *Linear matrix inequality representation of sets*, Comm. Pure Appl. Math. 60 (2007) 654–674. 6

[K-VV09] D.S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov: *Singularities of rational functions and minimal factorizations: the noncommutative and the commutative setting,* Linear Algebra Appl. 430 (2009) 869–889. 4

[MSS15] A.W. Marcus, D.A. Spielman, N. Srivastava: *Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem*, Ann. of Math. 182 (2015) 327–350. 6

[MT16] J.E. McCarthy, R.M. Timoney: *Non-commutative automorphisms of bounded non-commutative domains*, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016) 1037–1045. 1, 7

[MS08] P.S. Muhly, B. Solel: *Schur class functions and automorphism of Hardy algebras*, Doc. Math. 13 (2008) 365–411. 1, 7

[PT-D] J. Pascoe, R. Tully-Doyle: *The royal road to automatic noncommutative real analyticity, monotonicity, and convexity,* preprint `https://arxiv.org/abs/1907.05875` 5

[PT-D17] J. Pascoe, R. Tully-Doyle: *Free Pick functions: representations, asymptotic behavior and matrix monotonicity in several noncommuting variables,* J. Funct. Anal. 273 (2017) 283–328. 1, 5

[PSS18] B. Passer, O.M. Shalit, B. Solel: *Minimal and maximal matrix convex sets*, J. Funct. Anal. 274 (2018) 3197–3253. 6

[Pau02] V. Paulsen: *Completely bounded maps and operator algebras*, Cambridge Univ. Press, 2002. 6

[Pop08] G. Popescu: *Free pluriharmonic majorants and commutant lifting*, J. Func. Anal. 255 (2008) 891–939. 1

[Pop10] G. Popescu: *Free holomorphic automorphisms of the unit ball of* $B(H)^n$, J. reine angew. Math. 638 (2010) 119–168. 1, 7

[SIG96] R.E. Skelton, T. Iwasaki, K.M. Grigoriadis: *A Unified Algebraic Approach to Linear Control Design,* Taylor and Francis, 1996. 6

[SSS18] G. Salomon, O.M. Shalit, E. Shamovich: *Algebras of bounded noncommutative analytic functions on subvarieties of the noncommutative unit ball*, Trans. Amer. Math. Soc. 370 (2018) 8639–8690. 1, 7

[Vol18] J. Volčič: *Matrix coefficient realization theory of noncommutative rational functions,* J. Algebra 499 (2018) 397–437. 5, 7

Harry Dym, Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel

Email address: `Harry.dym@weizmann.ac.il`

J. William Helton, Department of Mathematics, University of California, San Diego

Email address: `helton@ucsd.edu`

Igor Klep, Department of Mathematics, Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

Email address: `igor.klep@fmf.uni-lj.si`

Scott McCullough, Department of Mathematics, University of Florida, Gainesville

Email address: `sam@math.ufl.edu`

Jurij Volčič, Department of Mathematics, Texas A&M University

Email address: `volcic@math.tamu.edu`

Index

Contents