REGULAR AND POSITIVE NONCOMMUTATIVE RATIONAL FUNCTIONS

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Abstract. Call a noncommutative rational function $\mathbb{r}$ regular if it has no singularities, i.e., $\mathbb{r}(X)$ is defined for all tuples of self-adjoint matrices X. In this article regular noncommutative rational functions $\mathbb{r}$ are characterized via the properties of their (minimal size) linear systems realizations $\mathbb{r} = \mathbf{b}^* L^{-1}\mathbf{c}$. It is shown that $\mathbb{r}$ is regular if and only if $L = A_0 + \sum_j A_j x_j$ is free elliptic. Roughly speaking, a linear pencil L is free elliptic if, after a finite sequence of basis changes and restrictions, the real part of A_0 is positive definite and the other A_j are skew-adjoint. The second main result is a solution to a noncommutative version of Hilbert's 17th problem: a positive regular noncommutative rational function is a sum of squares.

1. Introduction

Let $\mathbb{k}$ be the field of real or complex numbers and $\boldsymbol{x} = (x_1, \dots, x_g)$ a tuple of freely noncommuting variables. By the theory of division rings [Ami66, Coh95, Reu96], the free algebra $\mathbb{k}\langle\boldsymbol{x}\rangle$ of noncommutative polynomials admits a universal skew field of fractions $\mathbb{k}(\!\langle\boldsymbol{x}\rangle\!)$, whose elements are called noncommutative rational functions. They are usually represented with syntactically valid expressions involving $x_1, \dots, x_g, +, \cdot, (,), ^{-1}$ and elements from $\mathbb{k}$. Noncommutative rational functions play a prominent role in a wide range of areas. In ring theory, they appear as quasideterminants of matrices over noncommutative rings [GKLLRT95] and in the context of rings satisfying rational identities [Ber76]. In theoretical computer science, recognizable series of weighted automata are precisely formal power series expansions of noncommutative rational functions [BR11]. For similar reasons they emerge as transfer functions of linear systems evolving along free semigroups in control theory [BGM05]. These linear systems techniques are also applied in free probability for computing asymptotic eigenvalue distributions of noncommutative rational function evaluations on random matrices [BMS16]. In free analysis they are noncommutative analogs of meromorphic functions and are endowed with the difference-differential calculus [K-VV12, AM15]. Finally, ensembles of noncommutative rational functions are natural maps between noncommutative semialgebraic domains in free real algebraic geometry [HMV06, BPT13].

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While the interest in noncommutative rational functions originated from the universal property of the free skew field $\Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$, their importance in aforementioned areas derives from properties of their matrix evaluations. Here one evaluates a given noncommutative rational function $\mathbb{r}$ on tuples of self-adjoint matrices, which leads to the notion of the domain of $\mathbb{r}$. Minimal factorizations [K-VV09] and the extent of matrix convexity [HMV06] of a noncommutative rational function $\mathbb{r}$ are examples of problems that directly depend on knowing the domain of $\mathbb{r}$. On a more applied side, understanding of the singularities of (non)commutative rational functions is also important in control theory, e.g. for stability questions [GK-VVW16] or controllability and observability of linear systems [WSCP91]. In this paper we analyze two properties of noncommutative rational functions arising from their evaluations, namely regularity and positivity.

A noncommutative rational function $\mathbb{r}$ is *regular* if its domain contains all tuples of like sized self-adjoint matrices. For example, $(1+x_1^2)^{-1}$ and $(1+x_1^2x_2^2)^{-1}$ are regular functions[1], as well as are all noncommutative polynomials. In general, checking the regularity of $\mathbb{r}$ is harder if the representatives of $\mathbb{r}$ are complicated, e.g. if they contain numerous nested inverses. Furthermore, we note that, as in the commutative case, further difficulties arise because singularities of a given rational expression might be removable. The proper tool for investigating regularity comes from automata and control theory: every noncommutative rational function admits a *linear systems realization*

$$\mathbb{r} = \mathbf{c}^* L^{-1} \mathbf{b}, \tag{1.1}$$

where $\mathbf{b}, \mathbf{c} \in \Bbbk^d$ and L is a *linear matrix pencil* of size d: $L = A_0 + \sum_j A_j x_j$ with $A_j \in \mathrm{M}_d(\Bbbk)$. For the existence we refer to [Coh95, Sections 4.2 and 6.2] or [BGM05] for the case $A_0 = I$. Linear pencils give rise to linear matrix inequalities $L(x) \succeq 0$ and are thus ubiquitous in optimization [WSV12], systems engineering [SIG97] and in real algebraic geometry, see e.g. determinantal representations of polynomials [Brä11, NT12], the solution of the Lax conjecture [HV07], and the solution of the Kadison-Singer paving conjecture [MSS15]. If the linear pencil L is of minimal size satisfying (1.1), then the "no hidden singularities theorem" [K-VV09, Theorem 3.1] implies that $\mathbb{r}$ is regular if and only if every evaluation of L on a tuple of self-adjoint matrices is nonsingular. Characterization of regular functions thus turns into a problem of recognizing everywhere invertible pencils.

After describing regular functions we address their positivity. We say that a noncommutative rational function $\mathbb{r}$ is *positive* if $\mathbb{r}(X)$ is positive semidefinite for every tuple of self-adjoint matrices X in the domain of $\mathbb{r}$. For example, x_1^{-2} and $x_2^2 - x_2x_1(1+x_1^2)^{-1}x_1x_2$ are positive functions. We solve the analog of Hilbert's 17th problem for regular noncommutative rational functions. The original solution by Artin, stating that a nonnegative commutative polynomial is a sum of squares of rational functions (see e.g. [BCR98, Mar08, DAn11]), has been extended to the noncommutative setting in various ways [PS76, Hel02, McC01]. For example, Helton [Hel02, Theorem 1.1] showed that every positive noncommutative polynomial is a sum of hermitian squares $\sum_k q_k^* q_k$, where q_k are noncommutative polynomials. More general results about noncommutative polynomials that are positive semidefinite on certain free semialgebraic sets are now commonly known as noncommutative

[1] If S_1 and S_2 are positive semidefinite matrices, then the eigenvalues of S_1S_2 are real and nonnegative, so $I + S_1S_2$ is invertible.

Positivstellensätze [HMP07, Nel, HKM16]; for positivity results on free analytic functions see e.g. [PT-D, BK-V15].

1.1. **Main results and reader's guide.** In Section 2 we characterize everywhere invertible pencils. For example, if A_j are skew-adjoint matrices for $1 \le j \le g$ and A_0 is a sum of a positive definite and a skew-adjoint matrix, then $\Lambda(X) = A_0 \otimes I + \sum_{j>0} A_j \otimes X_j$ is clearly nonsingular for every tuple X of self-adjoint matrices. The condition on the coefficients of Λ can be also stated as

$$\operatorname{Re}(\Lambda) = \operatorname{Re}(\Lambda(0)) \succ 0,$$

where $\operatorname{Re} Y = \frac{1}{2}(Y + Y^*)$ denotes the real part of a matrix. More generally, we say that a linear pencil L is *free elliptic* if there exist constant matrices $D_1, \dots, D_\ell, V_1, \dots, V_{\ell-1}$ and $V_\ell = 0$ of appropriate sizes such that

$$\operatorname{Re}(D_k L V_1 \cdots V_{k-1}) = \operatorname{Re}(D_k L(0) V_1 \cdots V_{k-1}) \succeq 0$$

for $1 \le k \le \ell$ and the columns of V_k form a basis of $\ker \operatorname{Re}(D_k L(0) V_1 \cdots V_{k-1})$. See also Definition 2.1 for a recursive version. The pencil Λ described previously is free elliptic with $\ell = 1$ and $D_1 = I$. The name refers to elliptic systems of partial differential equations [Mir70, GB83] and is justified in our main result on linear pencils.

Theorem A. *A pencil L is free elliptic if and only if $L(X)$ is of full rank for every self-adjoint tuple X.*

More precise statements involving size bounds are given in Proposition 2.4 and Theorem 2.6. For square pencils L, $L(X)$ is always invertible if and only if the free locus of L, defined in [KV], does not contain any self-adjoint tuples. Theorem A can be therefore seen as a weak real Nullstellensatz for linear pencils. In Section 3 we apply Theorem A to regular noncommutative rational functions via the realization theory. Among regular functions we also describe *strongly bounded functions* $\mathtt{r}$, i.e., those for which there exist $\varepsilon, M > 0$ such that for every (not necessarily self-adjoint) tuple X satisfying $\|X^* - X\| < \varepsilon$ we have $\|\mathtt{r}(X)\| \le M$.

Theorem B. *Let $\mathtt{r} \in \Bbbk\mathopen{<\!\!(}\boldsymbol{x}\mathclose{)\!\!>}$. Then $\mathtt{r}$ is regular if and only if $\mathtt{r} = \mathbf{c}^* L^{-1} \mathbf{b}$ for some free elliptic pencil L. Furthermore, $\mathtt{r}$ is strongly bounded if and only if $\mathtt{r} = \mathbf{c}^*(A_0 + \sum_j A_j x_j)^{-1}\mathbf{b}$, where $\operatorname{Re} A_0$ is positive definite and A_j are skew-adjoint for $j > 0$.*

See Theorem 3.7 for the proof. In Section 4 we address positivity of noncommutative rational functions. We prove the following analog of Helton's sum of squares theorem [Hel02] for regular noncommutative rational functions.

Theorem C. *Let $\mathtt{r} \in \Bbbk\mathopen{<\!\!(}\boldsymbol{x}\mathclose{)\!\!>}$ be regular. Then $\mathtt{r}(X)$ is positive semidefinite for every tuple X of self-adjoint matrices if and only if $\mathtt{r}$ is a sum of hermitian squares of regular functions in $\Bbbk\mathopen{<\!\!(}\boldsymbol{x}\mathclose{)\!\!>}$.*

This statement is proved as Theorem 4.5 using a Hahn-Banach separation argument for a convex cone in a finite-dimensional vector space constructed from a noncommutative rational function. Lastly, we discuss the algorithmic perspective and present examples in Section 5.

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2. Full rank pencils

In this section we prove that a linear pencil is free elliptic if and only if it is of full rank on all self-adjoint tuples (Theorem 2.6).

2.1. **Basic notation.** Throughout the paper let $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$ and fix $g \in \mathbb{N}$. Let $\Bbbk\langle \boldsymbol{x}\rangle = \Bbbk\langle x_1, \dots, x_g\rangle$ be the free $\Bbbk$-algebra of noncommutative polynomials in freely noncommuting variables $x_1, \dots, x_g$. We endow $\Bbbk\langle \boldsymbol{x}\rangle$ with the involution $*$ satisfying $x_j^* = x_j$ and $\alpha^* = \bar{\alpha}$ for $\alpha \in \Bbbk$.

Let $*$ be the involution on $\mathrm{M}_n(\Bbbk)$ given by the transpose (if $\Bbbk = \mathbb{R}$) or the conjugate transpose (if $\Bbbk = \mathbb{C}$) and let $\mathrm{M}_n(\Bbbk)_{\mathrm{sa}} \subseteq \mathrm{M}_n(\Bbbk)$ be the $\mathbb{R}$-subspace of self-adjoint matrices. The notation $A \succ 0$ or $A \succeq 0$ for $A \in \mathrm{M}_n(\Bbbk)_{\mathrm{sa}}$ means that A is positive definite or positive semidefinite, respectively, while $\|\cdot\|$ always refers to the operator norm. Furthermore denote

$$\mathcal{M}^g = \bigcup_n \mathrm{M}_n(\Bbbk)^g, \qquad \mathcal{M}^g_{\mathrm{sa}} = \bigcup_n \mathrm{M}_n(\Bbbk)^g_{\mathrm{sa}}$$

and

$$\mathrm{Re}\, X = \frac{1}{2}(X + X^*), \qquad \mathrm{Im}\, X = \begin{cases} \frac{1}{2}(X - X^*) & \text{if } \Bbbk = \mathbb{R} \\ \frac{1}{2i}(X - X^*) & \text{if } \Bbbk = \mathbb{C} \end{cases}$$

for $X \in \mathcal{M}^g$.

2.1.1. *Linear matrix pencils.* If $A_0, \dots, A_g \in \mathrm{M}_{d\times e}(\Bbbk)$, then

$$L = A_0 + \sum_{j=1}^{g} A_j x_j \in \mathrm{M}_{d\times e}(\Bbbk) \otimes \Bbbk\langle \boldsymbol{x}\rangle \tag{2.1}$$

is a (rectangular) **pencil** of size $d \times e$. It can be naturally evaluated on $\mathcal{M}^g$ as

$$L(X) = A_0 \otimes I + \sum_{j=1}^{g} A_j \otimes X_j \in \mathrm{M}_{dn\times en}(\Bbbk)$$

for $X \in \mathrm{M}_n(\Bbbk)^g$.

2.2. **Free elliptic pencils.**

Definition 2.1. Let $d \ge e$ and $L = A_0 + \sum_j A_j x_j$ with $A_j \in \mathrm{M}_{d\times e}(\Bbbk)$.

(1) L is **strongly free elliptic** if there exists $D \in \mathrm{M}_{e\times d}(\Bbbk)$ such that

$$\mathrm{Re}(DA_0) \succ 0, \qquad \mathrm{Re}(DA_j) = 0 \quad \text{for } j > 0.$$

(2) With respect to e we recursively define L to be **free elliptic** if
   (a) it is strongly free elliptic; or
   (b) there exists $D \in \mathrm{M}_{e\times d}(\Bbbk)$ such that

$$0 \ne \mathrm{Re}(DA_0) \succeq 0, \qquad \mathrm{Re}(DA_j) = 0 \quad \text{for } j > 0$$

   and LV is free elliptic, where columns of V form a basis for $\ker \mathrm{Re}(DA_0)$. Note that LV is a pencil of size $d \times e'$ with $e' < e$.

A pencil of size $d \times e$ with $d < e$ is (strongly) free elliptic if and only if L^* is (strongly) free elliptic.

Example 2.2. Let $\Bbbk = \mathbb{R}$, $g = 2$ and

$$L = \begin{pmatrix} 1 & x_1 - x_2 & x_1 - 1 \\ x_2 - x_1 & 1 & 1 \\ 1 - x_1 & -1 & 0 \end{pmatrix} = A_0 + A_1 x_1 + A_2 x_2.$$

It is easy to check that every 3×3 matrix D satisfying $\operatorname{Re}(DA_1) = \operatorname{Re}(DA_2) = 0$ is a scalar multiple of I, so L is not strongly free elliptic. However, we have

$$\operatorname{Re}(A_0) = \operatorname{diag}(1,1,0), \qquad \operatorname{Re}(A_1) = \operatorname{Re}(A_2) = 0.$$

Restricting to the kernel of $\operatorname{Re}(A_0)$ we obtain

$$V = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad L' = LV = \begin{pmatrix} x_1 - 1 \\ 1 \\ 0 \end{pmatrix}.$$

Choosing $D' = (0\ 1\ 0)$ we get $D'L' = 1$, hence L' is strongly free elliptic and L is free elliptic.

Remark 2.3. The terminology of Definition 2.1 refers to the ellipticity of partial differential equations [Nir59, Mir70, GB83]. For example, a first order system

$$\sum_{j=1}^{g} P_j(x) \frac{\partial u}{\partial x_j} = f(x,u),$$

where P_j are $d \times e$ matrices with $d \geq e$, is elliptic at the point x if the matrix

$$P(x,\xi) = \sum_{j}^{g} P_j(x)\xi_j$$

has rank e for all $\xi \in \mathbb{R}^g \setminus \{0\}$; see [GB83, Section 4.7]. The analogy between free elliptic pencils and elliptic systems becomes clear in Theorem 2.6 where we prove that a pencil L is free elliptic if and only if $L(X)$ is of full rank for all $X \in \mathcal{M}_{\mathrm{sa}}^g$.

Proposition 2.4. *A pencil L of size $d \times e$ with $d \geq e$ is strongly free elliptic if and only if for some $\varepsilon > 0$,*

$$L(X)^* L(X) \succ \varepsilon I \qquad \text{for all} \quad X \in \mathcal{M}_{\mathrm{sa}}^g. \tag{2.2}$$

Furthermore, it suffices to test (2.2) for X of size at most $(g+1)e^2$.

Proof. Let $L = A_0 + \sum_j A_j x_j$ with $A_j \in \mathrm{M}_{d\times e}(\Bbbk)$ and $d \geq e$.

($\Rightarrow$) Let D be a matrix with $\operatorname{Re}(DA_0) = R^*R$ for $R \in \mathrm{GL}_d(\Bbbk)$ and $\operatorname{Re}(DA_j) = 0$ for $j > 0$. Denote $K = R^{-*}\operatorname{Im}(DL)R^{-1}$ and let $\kappa = 1$ if $\Bbbk = \mathbb{R}$ and $\kappa = i$ if $\Bbbk = \mathbb{C}$. If $X \in \mathrm{M}_n(\Bbbk)_{\mathrm{sa}}^g$ and $\mathbf{v} \in \Bbbk^{en}$, then

$$\begin{aligned}
\|D\|^2 \langle L(X)\mathbf{v}, L(X)\mathbf{v}\rangle &\geq \langle (DL)(X)\mathbf{v}, (DL)(X)\mathbf{v}\rangle \\
&= \langle (R^*R + \kappa R^*KR)(X)\mathbf{v}, (R^*R + \kappa R^*KR)(X)\mathbf{v}\rangle \\
&\geq \|R^{-*}\|^{-2} \langle (I+\kappa K)(X)(R\otimes I)\mathbf{v}, (I+\kappa K)(X)(R\otimes I)\mathbf{v}\rangle \\
&\geq \|R^{-*}\|^{-2} \langle (R\otimes I)\mathbf{v}, (R\otimes I)\mathbf{v}\rangle \\
&\geq \|R^{-*}\|^{-4} \langle \mathbf{v}, \mathbf{v}\rangle
\end{aligned}$$

since κK is skew-adjoint on $\mathcal{M}_{\mathrm{sa}}^g$. Hence we can take $\varepsilon = \|R^{-1}\|^{-4}\|D\|^{-2}$.

($\Leftarrow$) Since A_0 is of full rank, there exists $R \in \mathrm{GL}_d(\Bbbk)$ such that RA_0 is an isometry, i.e., $(RA_0)^*(RA_0) = I$. Since

$$\langle (RL)(X)\mathbf{v}, (RL)(X)\mathbf{v}\rangle \geq \|R^{-1}\|^{-2}\langle L(X)\mathbf{v}, L(X)\mathbf{v}\rangle$$

for every $X \in \mathrm{M}_n(\Bbbk)^g_{\mathrm{sa}}$ and $\mathbf{v} \in \Bbbk^{en}$, we have $(RL)(X)^*(RL)(X) \succ \varepsilon\|R^{-1}\|^{-2}I$.

Without loss of generality we can thus assume that A_0 is an isometry. Also let

$$\mathcal{K} = \Big(\sum_{j>0} \mathrm{im}\, A_j\Big)^{\perp}.$$

Because $L^*L - \varepsilon I$ is a positive polynomial (on matrices of size at most $(g+1)e^2$), it is a sum of hermitian squares of matrix-valued polynomials of degree at most 1 by [McC01, Theorem 0.2] or [MP05, Theorem 1.1]:

$$L^*L - \varepsilon I = \sum_{k=1}^{N}\Big(C_{k,0} + \sum_j C_{k,j}x_j\Big)^*\Big(C_{k,0} + \sum_j C_{k,j}x_j\Big), \qquad C_{k,j} \in \mathrm{M}_e(\Bbbk).$$

If $\mathbf{C} = (C_{k,j})_{j,k} \in \mathrm{M}_e(\Bbbk)^{N\times(g+1)}$ and $\ell^* = (1, x_1, \dots, x_g)$, then

$$L^*L - \varepsilon I = \ell^*\mathbf{C}^*\mathbf{C}\ell.$$

By looking at the coefficients of L we conclude that the positive semidefinite matrix $\mathbf{A} = \mathbf{C}^*\mathbf{C}$ is of the form

$$\mathbf{A} = \begin{pmatrix} I-\varepsilon I & B_1 & B_2 & \cdots & B_g \\ B_1^* & A_1^*A_1 & A_1^*A_2 & \cdots & A_1^*A_g \\ B_2^* & A_2^*A_1 & A_2^*A_2 & \cdots & A_2^*A_g \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_g^* & A_g^*A_1 & A_g^*A_2 & \cdots & A_g^*A_g \end{pmatrix} \tag{2.3}$$

where $B_j \in \mathrm{M}_e(\Bbbk)$ satisfy

$$B_j + B_j^* = A_0^*A_j + A_j^*A_0. \tag{2.4}$$

Let $\mathbf{v} \in \Bbbk^{gd}$ be arbitrary and set $\mathbf{w} = 0 \oplus \mathbf{v} \in \Bbbk^{(g+1)d}$. If $\mathbf{v}$ satisfies

$$\begin{pmatrix} A_1 & \cdots & A_g \end{pmatrix}\mathbf{v} = 0,$$

then $\mathbf{w}^*\mathbf{A}\mathbf{w} = 0$ and therefore $\mathbf{A}\mathbf{w} = 0$, so

$$\begin{pmatrix} B_1 & \cdots & B_g \end{pmatrix}\mathbf{v} = 0.$$

Hence the rows of a block matrix $(B_1 \cdots B_g)$ lie in the linear span of the rows of $(A_1 \cdots A_g)$, so there exists $T \in \mathrm{M}_{e\times d}(\Bbbk)$ such that

$$\begin{pmatrix} B_1 & \cdots & B_g \end{pmatrix} = T\begin{pmatrix} A_1 & \cdots & A_g \end{pmatrix}, \qquad T\mathcal{K} = 0. \tag{2.5}$$

Since $L(0)^*L(0) - \varepsilon I \succ 0$ and $\mathbf{A} \succeq 0$, the Schur complement of $I - \varepsilon I$ in $\mathbf{A}$

$$\begin{pmatrix} A_1^* \\ \vdots \\ A_g^* \end{pmatrix}\begin{pmatrix} A_1 & \cdots & A_g \end{pmatrix} - (1-\varepsilon)^{-1}\begin{pmatrix} B_1^* \\ \vdots \\ B_g^* \end{pmatrix}\begin{pmatrix} B_1 & \cdots & B_g \end{pmatrix} \tag{2.6}$$

is also positive semidefinite. Combining (2.5) and (2.6) yields

$$I - (1-\varepsilon)^{-1}T^*T \succeq 0. \tag{2.7}$$

Let $D = A_0^* - T$. Now (2.4) and (2.5) imply $\operatorname{Re}(DA_j) = 0$ for $j > 0$, while (2.7) together with $A_0^*A_0 = I$ yields

$$0 \preceq A_0^* \left((1-\varepsilon)I - T^*T\right) A_0 = 2\operatorname{Re}(DA_0) - (\varepsilon I + A_0^* D^* D A_0),$$

and therefore $\operatorname{Re}(DA_0) \succ 0$. □

Lemma 2.5. *Let $\mathbf{u}_1, \dots, \mathbf{u}_n \in \Bbbk^d$ and $\mathbf{v}_1, \dots, \mathbf{v}_n \in \Bbbk^m$. If $(\mathbf{w}^* \otimes I)(\sum_k \mathbf{u}_k \otimes \mathbf{v}_k) = 0$ for all $\mathbf{w} \in \Bbbk^d$, then $\sum_k \mathbf{u}_k \otimes \mathbf{v}_k = 0$.*

Proof. Without loss of generality we can assume that $\mathbf{v}_k$ are linearly independent. If

$$\sum_k (\mathbf{w}^* \mathbf{u}_k)\mathbf{v}_k = 0,$$

then $\mathbf{w}^*\mathbf{u}_k = 0$ for all k. Since this holds for every $\mathbf{w}$, we have $\mathbf{u}_k = 0$ and hence $\sum_k \mathbf{u}_k \otimes \mathbf{v}_k = 0$. □

The proof of the following theorem applies a specialized GNS construction that is inspired by a more general and intricate version in the proof of the matricial real Nullstellensatz in [Nel].

Theorem 2.6. *For a pencil $L = A_0 + \sum_j A_j x_j$ with $A_j \in \mathrm{M}_{d\times e}(\Bbbk)$ and $d \ge e$, the following are equivalent:*

(i) *L is free elliptic;*
(ii) *L is of full rank on $\mathcal{M}_{\mathrm{sa}}^g$;*
(iii) *$L(X)$ is of full rank for all $X \in \mathcal{M}_{\mathrm{sa}}^g$ of size at most $(g+1)e^2$.*

Remark 2.7. Square linear pencils that are nonsingular on whole $\mathcal{M}^g$ been characterized in [KV, Corollary 3.4]: if $L = A_0 + \sum_j A_j x_j$, then $\det L(X) \neq 0$ for all $X \in \mathcal{M}^g$ if and only if $A_0^{-1}A_1, \dots, A_0^{-1}A_g$ are jointly nilpotent matrices.

Remark 2.8. In the opposite direction, square linear pencils L with $\det L(X) = 0$ for all $X \in \mathcal{M}_{\mathrm{sa}}^g$ are precisely those that are not invertible as matrices over $\Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$. By [Coh95, Corollary 6.3.6], a linear pencil $A_0 + \sum_j A_j x_j$ with $A_j \in \mathrm{M}_d(\Bbbk)$ is not invertible over $\Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ if and only if there exist matrices $U, V \in \mathrm{GL}_d(\Bbbk)$ such that for $0 \le j \le g$ we have a block decomposition

$$UA_jV = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

where the zero block is of size $d_1 \times d_2$ with $d_1 + d_2 > d$. A linear bound on size of X for testing $\det L(X) = 0$ has been given in [DM].

Proof of Theorem 2.6. We prove (i)⇒(ii) and (iii)⇒(i) by induction on e, while (ii)⇒(iii) is obvious.

(i)⇒(ii) By Proposition 2.4, the claim holds if L is strongly free elliptic. If L is free elliptic but not strongly free elliptic there exists $D \in \mathrm{M}_{e\times d}(\Bbbk)$ such that $0 \neq \operatorname{Re}(DA_0) \succeq 0$, $\operatorname{Re}(DA_j) = 0$ for $j > 0$ and LV is free elliptic, where columns of V constitute a basis of $\ker \operatorname{Re}(DA_0)$. Let $X \in \mathrm{M}_n(\Bbbk)_{\mathrm{sa}}^g$ and consider the decomposition

$$\ker L(X) \subseteq \Bbbk^{en} = \Bbbk^e \otimes \Bbbk^n = \left((\ker \operatorname{Re}(DA_0))^\perp \otimes \Bbbk^n\right) \oplus \left(\ker \operatorname{Re}(DA_0) \otimes \Bbbk^n\right).$$

If $\binom{\mathbf{u}}{\mathbf{v}} \in \ker L(X)$, then

$$\begin{pmatrix}\mathbf{u}^* & \mathbf{v}^*\end{pmatrix} \operatorname{Re}(DL)(X) \begin{pmatrix}\mathbf{u} \\ \mathbf{v}\end{pmatrix} = 0$$

and so $\mathbf{u}^*(\operatorname{Re}(DA_0) \otimes I)\mathbf{u}^* = 0$. Hence $\mathbf{u} = 0$ and $(LV)(X)\mathbf{v} = 0$, therefore the induction hypothesis implies $\mathbf{v} = 0$. Thus L is of full rank on $\mathcal{M}_{\text{sa}}^g$.

(iii)⇒(i) Assume that L is not free elliptic. Therefore for every D such that $0 \neq \operatorname{Re}(DA_0) \geq 0$ and $\operatorname{Re}(DA_j) = 0$ for $j > 0$, LV is not free elliptic, where V consists of a basis of $\ker \operatorname{Re}(DA_0)$. By assumption there exists $X \in \mathcal{M}_{\text{sa}}^g$ such that $(LV)(X)$ is not of full rank, so $L(X)$ is not of full rank. Hence we need to consider the situation when L satisfies

$$\forall D\colon \quad \operatorname{Re}(DA_0) \geq 0 \text{ and } \operatorname{Re}(DA_j) = 0 \text{ for } j > 0 \;\Rightarrow\; \operatorname{Re}(DA_0) = 0. \tag{2.8}$$

For $k \in \{0, 1, 2\}$ define

$$\mathcal{V}_k = \{p \in \mathrm{M}_e(\Bbbk) \otimes \Bbbk\langle\boldsymbol{x}\rangle \colon \deg \leq k\}, \quad \mathcal{U} = \mathrm{M}_{e\times d}(\Bbbk)L + L^* \mathrm{M}_{d\times e}(\Bbbk),$$

$$\mathcal{V}_2^{\text{sa}} = \{p \in \mathcal{V}_2 \colon p^* = p\}, \quad \mathcal{S}_2 = \left\{\sum_j p_j^* p_j \colon p_j \in \mathcal{V}_1\right\}.$$

Here $\mathcal{V}_k$ and $\mathcal{U}$ are $\Bbbk$-linear spaces, while $\mathcal{V}_2^{\text{sa}}$ and $\mathcal{S}_2$ are a $\mathbb{R}$-linear space and a convex cone, respectively. It is easy to verify that $\mathcal{S}_2$ is closed in $\mathcal{V}_2$ (see e.g. the proofs of [MP05, Proposition 3.4] or Proposition 4.1 below). Observe that (2.8) implies $\mathcal{U} \cap \mathcal{S}_2 = \{0\}$. Indeed, if $DL + L^*E^* \in \mathcal{S}_2$, then $(DL + L^*E^*) + (EL + L^*D^*) \in \mathcal{S}_2$ and so $\operatorname{Re}((D+E)A_0) \geq 0$ and $\operatorname{Re}((D+E)A_j) = 0$ for $j > 0$. Hence (2.8) implies $\operatorname{Re}((D+E)A_0) = 0$ and consequently $DL + L^*E^* = 0$.

By [Kle55, Theorem 2.5] there exists a $\mathbb{R}$-linear functional $\lambda : \mathcal{V}_2^{\text{sa}} + \mathcal{U} \to \mathbb{R}$ such that $\lambda(\mathcal{S}_2 \setminus \{0\}) \subseteq \mathbb{R}_{>0}$ and $\lambda(\mathcal{U}) = \{0\}$, which we extend to a $\Bbbk$-linear functional $\Lambda : \mathcal{V}_2 \to \Bbbk$ by setting $\Lambda(p) := \lambda(\operatorname{Re} p) + i\lambda(\operatorname{Im} p)$ if $\Bbbk = \mathbb{C}$ and $\Lambda(p) := \lambda(\operatorname{Re} p)$ if $\Bbbk = \mathbb{R}$. Consequently we obtain a scalar product $\langle p_1, p_2\rangle := \Lambda(p_2^* p_1)$ on $\mathcal{V}_1$. Note that

$$\langle ap, q\rangle = \Lambda(q^* ap) = \langle p, a^* q\rangle$$

for all $a \in \mathcal{V}_0$ and $p, q \in \mathcal{V}_1$. Let $\pi : \mathcal{V}_1 \to \mathcal{V}_0$ be the orthogonal projection. For every $a, b \in \mathcal{V}_0$ and $1 \leq j \leq g$ we have

$$\langle \pi(ax_j), b\rangle = \langle ax_j, b\rangle = \langle x_j, a^* b\rangle = \langle \pi(x_j), a^* b\rangle = \langle a\pi(x_j), b\rangle,$$

so

$$\pi(ap) = a\pi(p) \qquad \forall a \in \mathcal{V}_0,\ p \in \mathcal{V}_1. \tag{2.9}$$

Now define

$$\begin{aligned} X_j &\colon \mathcal{V}_0 \to \mathcal{V}_0, \qquad b \mapsto \pi(x_j b), \\ \ell_a &\colon \mathcal{V}_0 \to \mathcal{V}_0, \qquad b \mapsto ab. \end{aligned}$$

It is easy to check that X_j is a self-adjoint operator that commutes with ℓ_a by (2.9). Let $D \in \mathrm{M}_{e\times d}(\Bbbk)$ be arbitrary and consider $I \in \mathrm{M}_e(\Bbbk)$ as a vector in $\mathcal{V}_0$. Then DL determines a linear operator $(DL)(X) : \mathcal{V}_0 \to \mathcal{V}_0$ and

$$\langle (DL)(X)I, b\rangle = \langle \pi(DL), b\rangle = \langle DL, b\rangle = \Lambda((b^* D)L) = 0$$

for every $b \in \mathcal{V}_0$ by (2.9) and the definition of Λ. Therefore $(DL)(X)I = 0$ and consequently $L(X)I = 0$ by Lemma 2.5.

Finally, the bound from the statement follows from Proposition 2.4 and the fact that $\dim \mathcal{V}_0 = e^2 < (g+1)e^2$. ☐

Remark 2.9. Let $L = A_0 + \sum_{j>0} A_j x_j$ be given and assume D satisfies

$$\operatorname{Re}(DA_0) \succeq 0, \qquad \operatorname{Re}(DA_j) = 0 \quad \text{for } j > 0. \tag{2.10}$$

If $\operatorname{Re}(DA_0) \neq 0$, then Theorem 2.6 implies that L is free elliptic if and only if LV is free elliptic, where V comprises a basis of $\ker \operatorname{Re}(DA_0)$. This fact simplifies the ellipticity testing: we can do the recursion with an arbitrary D which non-trivially solves (2.10) in the sense that $\operatorname{Re}(DA_0) \neq 0$.

3. Regular rational functions

In this section we turn our attention to regular nc rational functions, i.e., those without singularities. The main result, Theorem 3.7, shows that $\mathbb{r} \in \Bbbk\mathopen{(\!<}\boldsymbol{x}\mathclose{>\!)}$ is regular (strongly bounded) if and only if it admits a realization with a (strongly) free elliptic pencil.

3.1. Preliminaries.

We introduce noncommutative rational functions using matrix evaluations of formal rational expressions following [HMV06, K-VV12]. Originally they were defined ring-theoretically, cf. [Ami66, Coh95]. **Noncommutative (nc) rational expressions** are syntactically valid combinations of elements in $\Bbbk$, freely noncommuting variables $\{x_1, \dots, x_g\}$, arithmetic operations $+, \cdot, ^{-1}$ and parentheses $($, $)$. For example, $(1 + x_2^{-1}x_1)^{-1} + 1$, $x_1 + (-1)x_1$ and 0^{-1} are nc rational expressions. Their set is $\mathcal{R}_\Bbbk(\boldsymbol{x})$.

Given $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ and $X \in \mathrm{M}_n(\Bbbk)^g$, the evaluation $r(X)$ is defined in the obvious way if all inverses appearing in r exist at X. The set of all $X \in \mathcal{M}^g$ such that r is defined at X is is called the **domain of** r and denoted $\operatorname{dom} r$. Note that $\operatorname{dom} r \cap \mathrm{M}_n(\Bbbk)^g \subseteq \mathrm{M}_n(\Bbbk)^g$ is a Zariski open set for every $n \in \mathbb{N}$ and therefore either empty or dense in $\mathrm{M}_n(\Bbbk)^g$ with respect to Euclidean topology. A nc rational expression r is **non-degenerate** if $\operatorname{dom} r \neq \varnothing$. On the set of all non-degenerate nc rational expressions we define an equivalence relation $r_1 \sim r_2$ if and only if $r_1(X) = r_2(X)$ for all $X \in \operatorname{dom} r_1 \cap \operatorname{dom} r_2$. The equivalence classes with respect to this relation are called **noncommutative (nc) rational functions**. By [K-VV12, Proposition 2.1] they form a skew field denoted $\Bbbk\mathopen{(\!<}\boldsymbol{x}\mathclose{>\!)}$, which is the universal skew field of fractions of $\Bbbk\langle\boldsymbol{x}\rangle$ by [Coh95, Section 4.5]. The equivalence class of a nc rational expression $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ is written as $\mathbb{r} \in \Bbbk\mathopen{(\!<}\boldsymbol{x}\mathclose{>\!)}$. The previously defined involution on $\Bbbk\langle\boldsymbol{x}\rangle$ naturally extends to $\Bbbk\mathopen{(\!<}\boldsymbol{x}\mathclose{>\!)}$.

We define the **domain** of a nc rational function $\mathbb{r} \in \Bbbk\mathopen{(\!<}\boldsymbol{x}\mathclose{>\!)}$ as the union of $\operatorname{dom} r$ over all representatives $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ of $\mathbb{r}$. Lastly, let

$$\operatorname{dom}_{\mathrm{sa}} r = \operatorname{dom} r \cap \mathcal{M}^g_{\mathrm{sa}}, \qquad \operatorname{dom}_{\mathrm{sa}} \mathbb{r} = \operatorname{dom} \mathbb{r} \cap \mathcal{M}^g_{\mathrm{sa}}.$$

For a non-degenerate $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ we have $\operatorname{dom}_{\mathrm{sa}} r \neq \varnothing$ if and only if $\operatorname{dom} r \neq \varnothing$, see e.g. [Vol, Remark 6.8].

3.1.1. *Realizations of nc rational functions.*

For every $\mathbb{r} \in \Bbbk\mathopen{(\!<}\boldsymbol{x}\mathclose{>\!)}$ there exist $d \in \mathbb{N}$, $\mathbf{b}, \mathbf{c} \in \Bbbk^d$ and a linear pencil L of size d such that

$$\mathbb{r} = \mathbf{c}^* L^{-1} \mathbf{b}, \tag{3.1}$$

cf. [Coh95, Section 4.2]. We say that (3.1) is a **realization** of $\mathbb{r}$ of size d; we refer to [Coh95, BR11] for good expositions on classical realization theory.

Fix $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ and suppose $0 \in \operatorname{dom} \mathbb{r}$. In general, $\mathbb{r}$ admits various realizations. A realization of $\mathbb{r}$ whose size is smallest among all realizations of $\mathbb{r}$ is called **minimal**. These are unique up to basis change [BR11, Theorem 2.4], and if $\mathbf{c}^* L^{-1} \mathbf{b}$ is a minimal realization of $\mathbb{r}$, then $\operatorname{dom} \mathbb{r} \subseteq \{X \in \mathcal{M}^g : \det L(X) \neq 0\}$, see [K-VV09, Theorem 3.1].

3.2. Regularity.

Definition 3.1. We say that $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ is:

(1) **regular** if $\operatorname{dom}_{\rm sa} \mathbb{r} = \mathcal{M}^g_{\rm sa}$;
(2) **bounded** if there exists $M > 0$ such that $\|\mathbb{r}(X)\| \leq M$ for all $X \in \operatorname{dom}_{\rm sa} \mathbb{r}$.
(3) **strongly bounded** if there exist $\varepsilon > 0$ and $M > 0$ such that $\|\mathbb{r}(X)\| \leq M$ for all $X \in \operatorname{dom} \mathbb{r}$ with $\|\operatorname{Im} X\| < \varepsilon$.

Analogously, we say that $r \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$ is **regular** if $\operatorname{dom}_{\rm sa} r = \mathcal{M}^g_{\rm sa}$.

This definition is naturally extended to matrices over $\Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$. Obviously a regular expression yields a regular function and (3) implies (2). Using Riemann's removable singularities theorem [Kra01, Theorem 7.3.3] it is not hard to deduce that (3) implies (1); however, this is also a consequence of Theorem 3.7.

Example 3.2. Examples of regular but not bounded nc rational functions are nonconstant nc polynomials. An example of a bounded but not regular (and hence also not strongly bounded) function is $\mathbb{r} = (1+x_1x_2^{-2}x_1)^{-1}$: indeed, we have $\|\mathbb{r}(X_1, X_2)\| \leq 1$ for all $(X_1, X_2) \in \operatorname{dom}_{\rm sa} \mathbb{r}$ and $(0, 0) \notin \operatorname{dom} \mathbb{r}$. On the other hand, $(\frac{1}{2} + x_1^2 + x_2^2 + x_1^2x_2^2)^{-1}$ is an example of a strongly bounded rational function.

Proposition 3.3. *Let M be a square matrix over $\Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ and assume each of its entries admits a regular rational expression. If $M(X)$ is nonsingular for every $X \in \mathcal{M}^g_{\rm sa}$, then every entry of M^{-1} admits a regular expression.*

Proof. We prove the statement by induction on the size d of M. If $d = 1$, then $M = \mathbb{r}$ is an everywhere invertible regular rational function with a corresponding regular expression r; hence M^{-1} is given by r^{-1}.

Assume the statement holds for matrices of size $d - 1$ and let $\mathbf{m}$ be the first column of M. Then $\mathbf{m}(X)$ is of full rank for every $X \in \mathcal{M}^g_{\rm sa}$, so the regular rational function $\mathbf{m}^*\mathbf{m}$ is everywhere invertible and its inverse is given by a regular expression. Hence the entries of the Schur complement of $\mathbf{m}^*\mathbf{m}$ in M^*M admit regular rational expressions. Since M^*M is nonsingular on $\mathcal{M}^g_{\rm sa}$, the same holds for the Schur complement, which is a matrix of size $d - 1$. By the induction hypothesis, the entries of the inverse of this Schur complement admit regular rational expressions, hence the same holds for the inverse of M^*M. Finally, the entries of $M^{-1} = (M^*M)^{-1}M^*$ admit regular rational expressions. □

Corollary 3.4. *If $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ is regular, then it arises from a regular rational expression.*

Proof. Let $\mathbb{r} = \mathbf{c}^* L^{-1} \mathbf{b}$ be a minimal realization of $\mathbb{r}$. Since $\mathbb{r}$ is regular, L is nonsingular on $\mathcal{M}^g_{\rm sa}$ by [K-VV09, Theorem 3.1]. Since L is a matrix of polynomials, entries of L^{-1} admit regular rational expressions by Proposition 3.3. Hence $\mathbb{r}$ admits a regular rational expression. □

Lemma 3.5. *Let L be a square linear pencil. The following are equivalent:*

(i) L^{-1} *is strongly bounded;*
(ii) L^{-1} *is bounded;*
(iii) *There exists $\eta > 0$ such that $L(X)^* L(X) \succ \eta^2 I$ for all $X \in \mathcal{M}_{\mathrm{sa}}^g$.*

Proof. (i)⇒(ii) Trivial.

(ii)⇒(iii). Assume that $\|L^{-1}(X)\| \le M$ for $X \in \operatorname{dom}_{\mathrm{sa}} L^{-1}$; this means that the largest eigenvalue of $(L^*L)^{-1}(X)$ is at most M^2, so the smallest eigenvalue of $(L^*L)(X)$ is at least $\frac{1}{M^2}$. Since $\operatorname{dom}_{\mathrm{sa}} L^{-1} \cap \mathrm{M}_n(\Bbbk)_{\mathrm{sa}}^g$ is dense in $\mathrm{M}_n(\Bbbk)_{\mathrm{sa}}^g$ for infinitely many $n \in \mathbb{N}$, we conclude that the smallest eigenvalue of $(L^*L)(X)$ is at least $\frac{1}{M^2}$ for every $X \in \mathcal{M}_{\mathrm{sa}}^g$ and hence $L(X)^* L(X) \succ \frac{1}{M^2} I$ for all $X \in \mathcal{M}_{\mathrm{sa}}^g$.

(iii)⇒(i) Let $L = A_0 + \sum_j A_j x_j$ and assume $L(X)^* L(X) \succ \eta^2 I$ for all $X \in \mathcal{M}_{\mathrm{sa}}^g$. Choose $\varepsilon = \frac{\eta}{2}(\sum_{j>0} \|A_j\|)^{-1}$. If $X \in \operatorname{dom} L^{-1}$ and $\|\operatorname{Im} X\| < \varepsilon$, then

$$\left\| L(\operatorname{Re} X)^{-1} \sum_{j>0} A_j \otimes \operatorname{Im} X_j \right\| \le \frac{1}{\eta} \left(\sum_{j>0} \|A_j\| \right) \varepsilon = \frac{1}{2},$$

so

$$L(X) = L(\operatorname{Re} X)^{-1} \left(I + L(\operatorname{Re} X)^{-1} \sum_{j>0} A_j \otimes \operatorname{Im} X_j \right)$$

is invertible and $\|L(X)^{-1}\| \le \frac{2}{\eta}$. □

Lemma 3.6. *A nc rational function is strongly bounded if and only if the inverse of the pencil from its minimal realization is strongly bounded.*

Proof. The implication (⇐) is clear, so consider (⇒). For $1 \le j \le g$ let

$$\Delta_j : \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle \to \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle \otimes \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$$

be the difference-differential operator as in [K-VV12, Section 4]; this is the noncommutative counterpart of both the partial finite difference and partial differential operator. If $X, X' \in \operatorname{dom} \mathbb{r} \cap \mathrm{M}_n(\Bbbk)^g$ and $\Delta(\mathbb{r})(X, X')$ is interpreted as an element of $\operatorname{End}_{\Bbbk}(\mathrm{M}_n(\Bbbk)) \cong \mathrm{M}_n(\Bbbk) \otimes \mathrm{M}_n(\Bbbk)$, then

$$\begin{pmatrix} X & W \\ 0 & X' \end{pmatrix} \in \operatorname{dom} \mathbb{r} \cap \mathrm{M}_{2n}(\Bbbk)^g$$

for every $W \in M_n(\Bbbk)^g$ and

$$\mathbb{r} \begin{pmatrix} X & W \\ 0 & X' \end{pmatrix}$$

is up to conjugation by a permutation matrix equal to

$$\begin{pmatrix} \mathbb{r}(X) & \sum_j \Delta(\mathbb{r})(X, X')(W_j) \\ 0 & \mathbb{r}(X') \end{pmatrix}$$

by [K-VV12, Theorem 4.8]. In particular, if $\mathbb{r}$ is strongly bounded, then $(\Delta_j \mathbb{r})(x, 0)$ and $(\Delta_j \mathbb{r})(0, x)$ are also strongly bounded nc rational functions. Indeed, suppose $\|\mathbb{r}(X)\| \le M$ for all $X \in \operatorname{dom} \mathbb{r}$ with $\|\operatorname{Im} X\| < \varepsilon$. Then for every $X \in \operatorname{dom} \mathbb{r}$ with $\|\operatorname{Im} X\| < \frac{\varepsilon}{2}$ we have

$$\left\| \operatorname{Im} \begin{pmatrix} X & \varepsilon I \\ 0 & 0 \end{pmatrix} \right\| < \varepsilon$$

and therefore

$$\left\|\begin{pmatrix} \mathbb{r}(X) & \varepsilon(\Delta_j\mathbb{r})(X,0) \\ 0 & \mathbb{r}(0)\end{pmatrix}\right\| \le M$$

by assumption. Consequently

$$\|(\Delta_j\mathbb{r})(X,0)\| \le \frac{M}{\varepsilon}.$$

Since $\operatorname{dom}\mathbb{r}$ is dense in $\operatorname{dom}(\Delta_j\mathbb{r})(x,0)$, we conclude that $(\Delta_j\mathbb{r})(x,0)$ is strongly bounded.

Now let $\mathbb{r}\in\Bbbk\langle\!\langle \boldsymbol{x}\rangle\!\rangle$ be a strongly bounded nc rational function with minimal realization $\mathbb{r}=\mathbf{c}^*L^{-1}\mathbf{b}$ which can be chosen such that $L(0)=I$. By [K-VV12, Example 4.7] we have

$$\Delta_j(\mathbb{r})(x,0)=\mathbf{c}^*L^{-1}A_j\mathbf{b},\qquad \Delta_j(\mathbb{r})(0,x)=\mathbf{c}^*A_jL^{-1}\mathbf{b}.$$

Minimality of the realization implies

$$\operatorname{span}_{\Bbbk}\{A^w\mathbf{b}\colon w\in\langle\boldsymbol{x}\rangle\}=\Bbbk^d,\qquad \operatorname{span}_{\Bbbk}\{(A^*)^w\mathbf{c}\colon w\in\langle\boldsymbol{x}\rangle\}=\Bbbk^d$$

by [BR11, Proposition 2.1]. Therefore we conclude that every entry of L^{-1} is strongly bounded. □

3.3. Free elliptic realizations and regular rational functions.

Theorem 3.7. *Let $\mathbb{r}\in\Bbbk\langle\!\langle \boldsymbol{x}\rangle\!\rangle$.*

1. *$\mathbb{r}$ is strongly bounded if and only if it admits a (minimal) realization with a strongly free elliptic pencil.*
2. *$\mathbb{r}$ is regular if and only if it admits a (minimal) realization with a free elliptic pencil.*

Proof. (1) If $\mathbb{r}$ is strongly bounded and $\mathbf{c}^*L^{-1}\mathbf{b}'$ is its minimal realization, then L^{-1} is strongly bounded by Lemma 3.6. Hence L is strongly free elliptic by Lemma 3.5 and Proposition 2.4. Conversely, if $\mathbb{r}$ admits a realization with a strongly free elliptic pencil L, then L^{-1} is bounded by Proposition 2.4 and hence strongly bounded by Lemma 3.5. Therefore $\mathbb{r}$ is strongly bounded.

(2) Let $\mathbf{c}^*L^{-1}\mathbf{b}$ be a minimal realization of $\mathbb{r}$. Then L is invertible on $\mathcal{M}^g_{\mathrm{sa}}$ by [K-VV09, Theorem 3.1] and hence free elliptic by Theorem 2.6. The converse implication also follows by Theorem 2.6. □

Remark 3.8. One may also ask an analogous question about functions $\mathbb{r}\in\Bbbk\langle\!\langle \boldsymbol{x}\rangle\!\rangle$ satisfying $\operatorname{dom}\mathbb{r}=\mathcal{M}^g$. The answer to this question is much simpler [KV, Theorem 4.2]: a nc rational function is defined at every point in $\mathcal{M}^g$ if and only if it is a nc polynomial.

3.4. **Functions in $\boldsymbol{x}$ and $\boldsymbol{x}^*$.** We briefly discuss nc rational functions in $\boldsymbol{x}$ and $\boldsymbol{x}^*$, i.e., elements of the free skew field with involution $\mathbb{C}\langle\!\langle \boldsymbol{x},\boldsymbol{x}^*\rangle\!\rangle$. They are naturally evaluated at g-tuples of matrices X by replacing x_j with X_j and x_j^* with X_j^*. We refer to [KŠ] for analytic properties of these $*$-evaluations. On the other hand, if $\boldsymbol{y}$ is a copy of $\boldsymbol{x}$, then we have skew field isomorphisms

$$\mathbb{C}\langle\!\langle \boldsymbol{x},\boldsymbol{x}^*\rangle\!\rangle\to\mathbb{C}\langle\!\langle \boldsymbol{x},\boldsymbol{y}\rangle\!\rangle,\qquad x_j\mapsto x_j+iy_j,\ x_j^*\mapsto x_j-iy_j$$

$$\mathbb{C}\langle\!\langle \boldsymbol{x},\boldsymbol{y}\rangle\!\rangle\to\mathbb{C}\langle\!\langle \boldsymbol{x},\boldsymbol{x}^*\rangle\!\rangle,\qquad x_j\mapsto \frac{1}{2}(x_j+x_j^*),\ y_j\mapsto\frac{1}{2i}(x_j-x_j^*).$$

Thus we get a natural correspondence between $*$-evaluations of elements in $\mathbb{C}\langle\!\langle \boldsymbol{x},\boldsymbol{x}^*\rangle\!\rangle$ and Hermitian evaluations of elements in $\mathbb{C}\langle\!\langle \boldsymbol{x},\boldsymbol{y}\rangle\!\rangle$. Our main results on rational functions and pencils in self-adjoint variables can be easily adapted to this setup. We leave this as an exercise for the reader.

4. Positive rational functions

In this section we solve a noncommutative analog of Hilbert's 17th problem: every positive regular nc rational function is a sum of hermitian squares, see Theorem 4.5. For this we shall require a description of complexity of nc rational expressions. A **sub-expression** of $r \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$ is any nc rational expression which appears during the construction of r. For example, if $r = ((2+x_1)^{-1}x_2)x_1^{-1}$, then all its sub-expressions are

$$2, x_1, 2+x_1, (2+x_1)^{-1}, x_2, (2+x_1)^{-1}x_2, x_1^{-1}, ((2+x_1)^{-1}x_2)x_1^{-1}$$

We recursively define a complexity-measuring function $\tau : \mathcal{R}_{\Bbbk}(\boldsymbol{x}) \to \mathbb{N}_0$ as follows:

(a) $\tau(\alpha) = 0$ for $\alpha \in \Bbbk$;
(b) $\tau(x_j) = 1$ for $1 \le j \le g$;
(c) $\tau(r_1 + r_2) = \max\{\tau(r_1), \tau(r_2)\}$ for $r_1, r_2 \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$;
(d) $\tau(r_1 r_2) = \tau(r_1) + \tau(r_2)$ for $r_1, r_2 \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$;
(e) $\tau(r^{-1}) = 2\tau(r)$ for $r \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$.

Note that there is also a well-defined map $r \mapsto r^*$ on $\mathcal{R}_{\Bbbk}(\boldsymbol{x})$ that mimics the involution on $\Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ and $\tau(r^*) = \tau(r)$ for all $r \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$.

4.1. **A sum of squares cone associated with a rational expression.** Throughout the rest of this section fix a non-degenerate expression $r \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$ and the following notation. Let $Q \subset \mathcal{R}_{\Bbbk}(\boldsymbol{x})$ be the finite set of all sub-expressions of r and their images under the map $q \mapsto q^*$. Then define $\tilde{Q} = \{\mathbb{q} : q \in Q\} \subset \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ and

$$\mathcal{V}_k = \sum_{j=0}^{k} \Bbbk\, \overbrace{\tilde{Q}\cdots\tilde{Q}}^{j}, \qquad \mathcal{V}_k^{\mathrm{sa}} = \{\mathbb{s} \in \mathcal{V}_k : \mathbb{s} = \mathbb{s}^*\}, \qquad \mathcal{S}_{2k} = \left\{\sum_j \mathbb{s}_j^* \mathbb{s}_j : \mathbb{s}_j \in \mathcal{V}_k\right\}$$

for $k \in \mathbb{N}$. Then $\mathcal{S}_{2k} \subseteq \mathcal{V}_{2k}^{\mathrm{sa}} \subseteq \mathcal{V}_{2k}$ are a convex cone, $\mathbb{R}$-linear space and $\Bbbk$-linear space, respectively. Since $\mathcal{V}_{2k}^{\mathrm{sa}}$ is finite-dimensional, every norm on $\mathcal{V}_{2k}^{\mathrm{sa}}$ yields the usual Euclidean topology. We note that Q and $\tilde{Q}$ are rational analogs of Newton chips [BKP16] for free polynomials.

Proposition 4.1. *$\mathcal{S}_{2k}$ is closed in $\mathcal{V}_{2k}^{\mathrm{sa}}$.*

Proof. Since $\mathcal{V}_{2k}^{\mathrm{sa}}$ is finite-dimensional, there exists $X \in \operatorname{dom}_{\mathrm{sa}} r$ such that

$$\forall \mathbb{s} \in \mathcal{V}_{2k}^{\mathrm{sa}} : \ \mathbb{s}(X) = 0 \ \Rightarrow \ \mathbb{s} = 0$$

by the CHSY Lemma ([CHSY03, Corollary 3.2] or [BPT13, Corollary 8.87]). Hence we can define a norm on $\mathcal{V}_{2k}^{\mathrm{sa}}$ by $\|\mathbb{s}\|_\bullet := \|\mathbb{s}(X)\|$. Also, finite-dimensionality of $\mathcal{V}_{2k}^{\mathrm{sa}}$ implies that every element of $\mathcal{S}_{2k}$ can be written as a sum of $N = 1 + \dim \mathcal{V}_{2k}^{\mathrm{sa}}$ hermitian squares by Carathéodory's theorem [Bar02, Theorem I.2.3]. Assume that a sequence $\{\mathbb{r}_n\}_n \subset \mathcal{S}_{2k}$ converges to $\mathbb{s} \in \mathcal{V}_{2k}^{\mathrm{sa}}$ with respect to $\|\cdot\|_\bullet$. If

$$\mathbb{r}_n = \sum_{j=1}^{N} \mathbb{s}_{n,j}^* \mathbb{s}_{n,j}, \qquad \mathbb{s}_{n,j} \in \mathcal{V}_k,$$

then the definition of our norm implies $\|\mathbb{s}_{n_j}\|^2 \le \|\mathbb{r}_n\|$. In particular, the sequences $\{\mathbb{s}_{n,j}\}_n \subset \mathcal{V}_k$ for $1 \le j \le N$ are bounded. Hence, after restricting to subsequences, we may assume that they are convergent: $\mathbb{s}_j = \lim_n \mathbb{s}_{n,j}$ for $1 \le j \le N$. Consequently we have

$$\mathbb{s} = \lim_n \mathbb{r}_n = \sum_{j=1}^{N} \lim_n \left(\mathbb{s}_{n,j}^* \mathbb{s}_{n,j}\right) = \sum_{j=1}^{N} \mathbb{s}_j^* \mathbb{s}_j \in \mathcal{S}_{2k}. \qquad \square$$

4.2. **Moore-Penrose evaluations.** In this subsection we generalize our notion of an evaluation of a nc rational expression. For $A \in \mathrm{M}_n(\Bbbk)$ let $A^\dagger \in \mathrm{M}_n(\Bbbk)$ be its Moore-Penrose pseudoinverse [HJ85, Section 7.3]. Its properties that will be used in this section are

$$(A^\dagger)^* = (A^*)^\dagger, \qquad A^* = A^\dagger A A^*.$$

Given $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ and $X \in \mathcal{M}^g$ we recursively define the **Moore-Penrose evaluation** of r at X, denoted $r^{\mathrm{mp}}(X)$:

(a) $\alpha^{\mathrm{mp}}(X) = \alpha I$ for $\alpha \in \Bbbk$;
(b) $x_j^{\mathrm{mp}}(X) = X_j$ for $1 \le j \le g$;
(c) $(r_1 + r_2)^{\mathrm{mp}}(X) = r_1^{\mathrm{mp}}(X) + r_2^{\mathrm{mp}}(X)$ and $(r_1 r_2)^{\mathrm{mp}}(X) = r_1^{\mathrm{mp}}(X) r_2^{\mathrm{mp}}(X)$ for $r_1, r_2 \in \mathcal{R}_\Bbbk(\boldsymbol{x})$;
(d) $(r^{-1})^{\mathrm{mp}}(X) = (r^{\mathrm{mp}}(X))^\dagger$ for $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$.

Loosely speaking, with this kind of evaluation we replace all the inverses in an expression with Moore-Penrose pseudoinverses and the evaluation is then defined at any matrix point. Moore-Penrose evaluations of nc rational expressions frequently appear in control theory; see e.g. [BEFB94]. We warn the reader that in general these evaluations do not respect the equivalence relation defining nc rational functions; also, Moore-Penrose evaluation is defined even for degenerate expressions. For example, $(0^{-1})^{\mathrm{mp}}(X) = 0$ for all $X \in \mathcal{M}^g$. However, if $r \in \mathcal{R}_\Bbbk(\boldsymbol{x})$ is non-degenerate and $X \in \operatorname{dom} r$, then

$$r^{\mathrm{mp}}(X) = r(X) = \mathbb{r}(X).$$

Proposition 4.2. *Let r be a non-degenerate expression and assume the notation from the beginning of the section. If $\lambda : \mathcal{V}_{2k+2}^{\mathrm{sa}} \to \mathbb{R}$ is a $\mathbb{R}$-linear functional satisfying $\lambda(\mathcal{S}_{2k+2} \setminus \{0\}) \subseteq \mathbb{R}_{>0}$, then there exists a scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{V}_k$ and self-adjoint operators X_j on $\mathcal{V}_k$ such that*

$$\lambda(\mathbb{q}) = \langle q^{\mathrm{mp}}(X) 1, 1 \rangle$$

for every $q = q^* \in Q$ with $4\tau(q) \le k$.

Proof. Let $\Lambda : \mathcal{V}_{2k+2} \to \Bbbk$ be the $\Bbbk$-linear functional given by $\Lambda(\mathbb{s}) := \lambda(\operatorname{Re} \mathbb{s}) + i\lambda(\operatorname{Im} \mathbb{s})$ if $\Bbbk = \mathbb{C}$ and $\Lambda(\mathbb{s}) := \lambda(\operatorname{Re} \mathbb{s})$ if $\Bbbk = \mathbb{R}$. Now $\langle \mathbb{s}_1, \mathbb{s}_2 \rangle := \Lambda(\mathbb{s}_2^* \mathbb{s}_1)$ defines a scalar product on $\mathcal{V}_{k+1}$. Let $\pi : \mathcal{V}_{k+1} \to \mathcal{V}_k$ be the orthogonal projection. If $x_j \in Q$, then define

$$X_j : \mathcal{V}_k \to \mathcal{V}_k, \qquad \mathbb{s} \mapsto \pi(x_j \mathbb{s}).$$

It is clear that X_j is a self-adjoint operator. We claim the following:

($\star$) *if $q \in Q$, then $q^{\mathrm{mp}}(X)\mathbb{s} = \mathbb{q}\mathbb{s}$ holds for $s \in \bigcup_{j=0}^{k} \overbrace{Q \cdots Q}^{j}$ satisfying $4\tau(q) + \tau(s) \le k$.*

We prove ($\star$) by the induction on the construction of q. Firstly, ($\star$) obviously holds for $q \in \Bbbk$ or $q \in Q \cap \{x_1 \dots, x_g\}$. Next, if ($\star$) holds for $q_1, q_2 \in Q$ such that $q_1 + q_2 \in Q$ or $q_1 q_2 \in Q$, then it also holds for the latter. Finally, suppose that $(\star)$ holds for $q \in Q$ and assume $q^{-1} \in Q$. If $s \in \bigcup_{j=0}^{k} Q \cdots Q$ and $4\tau(q^{-1}) + \tau(s) \leq k$, then $4\tau(q) + (\tau(q^{-*}) + \tau(q^{-1}) + \tau(s)) \leq k$ and $q^{-*}q^{-1}s \in \bigcup_{j=0}^{k} Q \cdots Q$, so

$$q^{\mathrm{mp}}(X)^{*}(\mathbb{q}^{-*}\mathbb{q}^{-1}\mathbb{s}) = (q^{*})^{\mathrm{mp}}(X)(\mathbb{q}^{-*}\mathbb{q}^{-1}\mathbb{s}) = \mathbb{q}^{*}\mathbb{q}^{-*}\mathbb{q}^{-1}\mathbb{s} = \mathbb{q}^{-1}\mathbb{s}, \tag{4.1}$$

$$q^{\mathrm{mp}}(X)(\mathbb{q}^{-1}\mathbb{s}) = \mathbb{q}\mathbb{q}^{-1}\mathbb{s} = \mathbb{s}. \tag{4.2}$$

Since $\mathbb{q}^{-1}\mathbb{s}$ lies in the image of $q^{\mathrm{mp}}(X)^{*}$ by (4.1), we have

$$\mathbb{q}^{-1}\mathbb{s} = (q^{\mathrm{mp}}(X)^{\dagger}q^{\mathrm{mp}}(X))(\mathbb{q}^{-1}\mathbb{s}) = (q^{\mathrm{mp}}(X))^{\dagger}\mathbb{s} = (q^{-1})^{\mathrm{mp}}(X)\mathbb{s}$$

by (4.2).

In particular, if $q = q^{*}$ satisfies $4\tau(q) \leq k$, then

$$\langle q^{\mathrm{mp}}(X)1, 1\rangle = \Lambda\left(q^{\mathrm{mp}}(X)1\right) = \lambda(\mathbb{q})$$

by $(\star)$. □

Proposition 4.3. *Let $r \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$ be a non-degenerate expression and $t = \tau(r)$. If $\mathbb{r} \notin \mathcal{S}_{8t+2}$, then there exists $X \in \mathcal{M}_{\mathrm{sa}}^{g}$ of size $\dim \mathcal{V}_{4t}$ such that $r^{\mathrm{mp}}(X)$ is not positive semidefinite.*

Proof. If $\mathbb{r} \neq \mathbb{r}^{*}$, then there clearly exists $X \in \mathcal{M}_{\mathrm{sa}}^{g}$ such that $\mathbb{r}(X)$ is not self-adjoint; hence we assume that $\mathbb{r} = \mathbb{r}^{*}$. By Proposition 4.1 and the Hahn-Banach separation theorem [Bar02, Theorem III.1.3] there exists a $\mathbb{R}$-linear functional $\lambda : \mathcal{V}_{8t+2}^{\mathrm{sa}} \to \mathbb{R}$ such that $\lambda(\mathcal{S}_{8t+2} \setminus \{0\}) \subseteq \mathbb{R}_{>0}$ and $\lambda(\mathbb{r}) < 0$. By Proposition 4.2 there exists $X \in \mathcal{M}_{\mathrm{sa}}^{g}$ and a vector $\mathbf{v}$ of size $\dim \mathcal{V}_{4t}$ such that

$$\langle r^{\mathrm{mp}}(X)\mathbf{v}, \mathbf{v}\rangle = \lambda(\mathbb{r}) < 0.$$ □

Remark 4.4. The converse of Proposition 4.3 does not hold. For example, if $r = x^{-1}x - 1$, then $\mathbb{r} = 0$ is a sum of hermitian squares, but $r^{\mathrm{mp}}(0) = -1$ is not positive semidefinite.

4.3. Regular positive rational functions.

As a consequence of Proposition 4.3 we obtain the following version of Artin's theorem. It is a rational function analog of Helton's sum of hermitian squares theorem [Hel02].

Theorem 4.5. *Let $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ be regular. Then $\mathbb{r}(X) \succeq 0$ for all $X \in \mathcal{M}_{\mathrm{sa}}^{g}$ if and only if*

$$\mathbb{r} = \sum_{j} \mathbb{s}_j^{*}\mathbb{s}_j$$

for some regular $\mathbb{s}_j \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$.

Proof. The implication $(\Leftarrow)$ is clear, so consider $(\Rightarrow)$. Since $\mathbb{r}$ is regular, it admits a regular rational expression r by Corollary 3.4. If $\mathbb{r}$ is not a sum of hermitian squares, then there exists $X \in \mathcal{M}_{\mathrm{sa}}^{g}$ such that $r^{\mathrm{mp}}(X)$ is not positive semidefinite by Proposition 4.3. Since $X \in \mathrm{dom}_{\mathrm{sa}}\, r$, we have $r^{\mathrm{mp}}(X) = \mathbb{r}(X)$. □

Remark 4.6. Let $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ be a regular rational function that is not a sum of hermitian squares. If $r \in \mathcal{R}_{\Bbbk}(\boldsymbol{x})$ is its regular representative and $t = \tau(r)$, then there exists $X \in \mathcal{M}_{\mathrm{sa}}^{g}$ of size $\dim \mathcal{V}_{2t}$ such that $r^{\mathrm{mp}}(X)$ is not positive semidefinite. Indeed, this improved bound follows by replacing 4 with 2 in Proposition 4.2 since (4.1) becomes unnecessary when dealing with proper inverses.

For strictly positive regular nc rational functions also see Remark 5.1. Moreover, by the same reasoning as in Subsection 3.4, a suitably modified version of Theorem 4.5 holds for nc rational functions in $\boldsymbol{x}$ and $\boldsymbol{x}^{*}$ over $\mathbb{C}$.

5. Examples and algorithms

In this section we present efficient algorithms to check whether $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ is regular and whether it is a sum of hermitian based on semidefinite programming. We finish the section with worked out examples.

5.1. **Testing regularity.** Our main results are effective and enable us to devise an algorithm to check for regularity of a nc rational function.

5.1.1. *Free elliptic pencils.* Let $L = A_0 + \sum_{j>0} A_j x_j \in \mathrm{M}_{d\times e}(\Bbbk) \otimes \Bbbk\langle \boldsymbol{x} \rangle$ with $d \ge e$. (If $d < e$, simply replace L by L^*.) Now solve the following feasibility semidefinite program (SDP) for $D \in \mathrm{M}_{e\times d}(\Bbbk)$:

$$\begin{aligned} \mathrm{Re}(DA_0) &\succeq 0 \\ \mathrm{tr}(DA_0) &= 1 \\ \mathrm{Re}(DA_j) &= 0 \quad \text{for } j > 0. \end{aligned} \tag{5.1}$$

(See e.g. [WSV12, BPT13] for more on SDPs.) If (5.1) is infeasible, then L is not free elliptic. If the output is a D with $\mathrm{Re}(DA_0) \succ 0$, then L is (strongly) free elliptic. Otherwise replace L by the linear pencil LV, where the columns of V form a basis for $\ker \mathrm{Re}(DA_0)$, and repeat the algorithm. Since LV is of smaller size than L, the procedure will eventually terminate.

5.1.2. *Regular nc rational functions.* Given $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$, we use an efficient (linear-algebra-based) algorithm, cf. [BR11, Section II.3], to construct a realization of $\mathbb{r}$ and then reducing it to a minimal one, say $\mathbb{r} = \mathbf{c}^* L^{-1}\mathbf{b}$, where L is a $d\times d$ pencil. Now use the algorithm in Subsection 5.1.1 below to check whether L is free elliptic. By Theorem 3.7, $\mathbb{r}$ is regular if and only if L is free elliptic.

Remark 5.1. This algorithm also yields a procedure to check whether a regular rational function is strictly positive everywhere: for a regular $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ we have $\mathbb{r}(X) \succ 0$ for all $X \in \mathcal{M}^g_{\mathrm{sa}}$ if and only if $\mathbb{r}(0) \succ 0$ and $\mathbb{r}^{-1}$ is regular. In particular, this can be applied for testing whether a nc polynomial is positive everywhere.

5.2. **Testing positivity.** In this subsection we present an efficient algorithm based on SDP to check whether a regular rational function is a sum of hermitian squares, i.e., whether it is positive everywhere. We point out this is in sharp contrast to the classical commutative case [BCR98] where no efficient algorithms exist (in > 2 variables) to check whether a rational function $r \in \mathbb{R}(X)$ is globally positive.

Let $\mathbb{r} \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$. A **positively free elliptic realization** is one of the form

$$\mathbb{r} = \mathbf{b}^* \left(A_0 + \sum_j A_j x_j \right)^{-1} \mathbf{b}, \qquad \mathrm{Re}\, A_0 \succeq 0, \qquad \mathrm{Re}\, A_j = 0 \quad \text{for } j > 0.$$

Proposition 5.2. *A nc rational function $\mathbb{r} = \mathbb{r}^* \in \Bbbk\langle\!\langle \boldsymbol{x} \rangle\!\rangle$ is a sum of hermitian squares if and only if it admits a positively free elliptic realization.*

Proof. ($\Rightarrow$) Let $\mathbb{r} = \sum_j \mathbb{s}_j^* \mathbb{s}_j$ and $\mathbb{s}_j = \mathbf{c}_j^* L_j^{-1} \mathbf{b}_j$. Then

$$\mathbb{s}_j^* \mathbb{s}_j = \begin{pmatrix} 0 & b_j^* \end{pmatrix} \begin{pmatrix} \mathbf{c}_j \mathbf{c}_j^* & L_j^* \\ -L_j & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ b_j \end{pmatrix}$$

is a positively free elliptic realization and the direct sum of these pencils and vectors yields a positively free elliptic realization of $\mathbb{r}$.

($\Leftarrow$) Let $\mathbb{r} = \mathbf{b}^* L^{-1}\mathbf{b}$ be a positively free elliptic realization. Then

$$\operatorname{Re}\left(L^{-1}\right) = L^{-*}\operatorname{Re}(L)L^{-1} = L^{-*}R^*RL^{-1}$$

for some constant matrix R by assumption. Since $\mathbb{r}$ is self-adjoint we have

$$\mathbb{r} = \operatorname{Re}\left(\mathbf{b}^* L^{-1}\mathbf{b}\right) = \mathbf{b}^*\operatorname{Re}\left(L^{-1}\right)\mathbf{b} = \left(RL^{-1}\mathbf{b}\right)^*\left(RL^{-1}\mathbf{b}\right),$$

so $\mathbb{r}$ is a sum of hermitian squares. □

Remark 5.3. If $\Bbbk = \mathbb{R}$, then a positively free elliptic realization automatically yields a symmetric nc rational function.

5.2.1. *Sum of squares testing.* Retain the notation of Section 4. Let $t = \tau(r)$ and suppose that $\mathbb{r} = \mathbb{r}^*$. To test whether $\mathbb{r}$ is a sum of hermitian squares, we proceed as follows. Pick a basis for $\mathcal{V}_{2t+1}$ and stack the elements of the basis into a vector, say W. Then $\mathbb{r}$ is a sum of squares if and only if the SDP (5.2) is feasible.

$$\begin{aligned} &G \succeq 0 \\ \text{s.t.}\quad &\mathbb{r} = W^* G W. \end{aligned} \tag{5.2}$$

To implement the equality in (5.2) we evaluate $\mathbb{r}$ and W^*GW at sufficiently many tuples of random self-adjoint matrices $X \in \mathcal{M}_{\mathrm{sa}}^g$ of size

$$\kappa(r)(1 + (2t+1)(\dim \mathcal{V}_{2t+1})^2), \tag{5.3}$$

where

$$\kappa(r) = \#(\text{constant terms in } r) + 2\cdot\#(\text{symbols in } r) + \#(\text{inverses in } r).$$

We refer to [Vol, Subsection 6.1] for the bound (5.3).

Each solution G to (5.2) yields a sum of squares decomposition of $\mathbb{r}$. Namely, letting $G = H^*H$ and $\mathbb{s} = HW$, we have $\mathbb{r} = \mathbb{s}^*\mathbb{s} = \sum_j \mathbb{s}_j^*\mathbb{s}_j$, where $\mathbb{s}_j$ are the entries of the vector $\mathbb{s}$. Finally, as in the proof of Proposition 5.2, such a sum of squares decomposition can be employed to construct a positively free elliptic realization for $\mathbb{r}$.

Example 5.4. A sum of hermitian squares does not necessarily admit a positively free elliptic realization that is also a minimal one. For example, x_1 has a realization of size 2, so x_1^2 has a positively free elliptic realization of size 4 by the proof of Proposition 5.2. However, it can be checked that x_1^2 admits a realization of size 3 but does not admit a positively free elliptic realization of size 3.

5.3. **Examples.**

Example 5.5. Let $\Bbbk = \mathbb{R}$ and

$$\mathbb{r} = \left((1 - x_1x_2)(1 - x_2x_1) + x_1^2\right)^{-1}.$$

While one can show that $\mathbb{r}$ is regular using elementary arguments, we demonstrate this fact by applying our algorithm. Firstly we take a minimal realization of $\mathbb{r}$ with the corresponding pencil

$$L = \begin{pmatrix} 1 & 0 & 0 & -x_2 \\ 0 & 1 & -x_2 & 0 \\ 0 & -x_1 & 1 & -x_1 \\ -x_1 & 0 & x_1 & 1 \end{pmatrix} = A_0 + A_1x_1 + A_2x_2.$$

The system $\operatorname{Re}(DA_1) = \operatorname{Re}(DA_2) = 0$ has a solution space of dimension 2. Adding the constraint $\operatorname{Re}(DA_0) \succeq 0$ we obtain a one-dimensional salient convex cone $\mathcal{C}$. For every $D \in \mathcal{C}$ we have $\operatorname{rk}\operatorname{Re}(DA_0) = 2$, so $\mathbb{r}$ is not strongly bounded. By choosing

$$D = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

we get

$$\operatorname{Re}(DA_1) = \operatorname{Re}(DA_2) = 0, \qquad \operatorname{Re}(DA_0) = \operatorname{diag}(0,0,1,1).$$

Hence let

$$L' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -x_1 \\ -x_1 & 0 \end{pmatrix}.$$

By choosing

$$D' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

we verify that L' is free elliptic. Therefore L is free elliptic and so $\mathbb{r}$ is regular.

Example 5.6. The rational function

$$\mathbb{r} = \left(2 + (x_1x_2 - x_1 - 2x_2)\left(1 + x_2^2\right)^{-1} + (x_1 + x_2 - 1)\left(1 + x_2^2\right)^{-1} x_1\right)^{-1}$$

is strongly bounded since it admits a realization with the strongly free elliptic pencil

$$\begin{pmatrix} 1 & -x_2 & 0 \\ x_2 & 1 & -x_1 - x_2 + 1 \\ 0 & x_1 + x_2 - 1 & 1 \end{pmatrix}.$$

Example 5.7. Let $\mathbb{k} = \mathbb{C}$ and consider

$$\mathbb{r} = \left(1 + x_2^2 - ((1-i)x_1 + x_2)\left(1 + 2x_1^2\right)^{-1}((1+i)x_1 + x_2)\right)^{-1}.$$

Note that $(X_1, X_2) \in \operatorname{dom}_{\mathrm{sa}} \mathbb{r}$ for every pair of commuting hermitian matrices X_1 and X_2. It can be checked that $\mathbb{r}$ admits a minimal realization with the pencil

$$L = \begin{pmatrix} 1 & (-1-i)x_1 & -x_1 & 0 \\ 0 & 1 & 0 & -x_1 \\ (-1+i)x_2 & 0 & 1 & -x_2 \\ 0 & (-1-i)x_2 & 0 & 1 \end{pmatrix} = A_0 + A_1x_1 + A_2x_2.$$

If

$$D=\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 1+i & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & \frac{1}{2}+\frac{i}{2} \end{pmatrix},$$

then

$$\operatorname{Re}(DA_1)=\operatorname{Re}(DA_2)=0, \qquad \operatorname{Re}(DA_0)=\operatorname{diag}\left(0,1,0,\frac{1}{2}\right).$$

Therefore we are left with

$$L'=\begin{pmatrix} 1 & -x_1 \\ 0 & 0 \\ (-1+i)x_2 & 1 \\ 0 & 0 \end{pmatrix}=A_0'+A_1'x_1+A_2'x_2.$$

But for every D' satisfying $\operatorname{Re}(D'A_1')=\operatorname{Re}(D'A_2')=0$ we have $\operatorname{Re}(D'A_0')=\left(\begin{smallmatrix} 0 & \alpha \\ \bar\alpha & 0 \end{smallmatrix}\right)$ for $\alpha\in\mathbb{C}$, so L' and L are not free elliptic. Hence $\mathbb{r}$ is not regular. To find $X\in\mathcal{M}_{\mathrm{sa}}^2\setminus\operatorname{dom}\mathbb{r}$ consider the structure of L'. We see that if $\frac{1}{2}+\frac{i}{2}$ is an eigenvalue of X_2X_1, then $(X_1,X_2)\notin\operatorname{dom}_{\mathrm{sa}}\mathbb{r}$. For a concrete example, take

$$X_1=\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}, \qquad X_2=\begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}.$$

Example 5.8. Again let $\Bbbk=\mathbb{R}$. Another nontrivial example is the following inverse of a sum of hermitian squares:

$$\mathbb{r}=\left(\left(1+(x_2x_1)^2x_2^2\right)\left(1+x_2^2(x_1x_2)^2\right)-(x_1x_2-x_2x_1)^2\right)^{-1}.$$

The size of its minimal realization is 15. With routine computation one observes that

$$\operatorname{Re}(DA_1)=\operatorname{Re}(DA_2)=0 \text{ and } \operatorname{Re}(DA_0)\succeq 0 \;\Rightarrow\; \operatorname{Re}(DA_0)=0,$$

so $\mathbb{r}$ is not regular. However, it is not apparent which concrete tuple of symmetric matrices is not contained in $\operatorname{dom}_{\mathrm{sa}}\mathbb{r}$; using brute force one can check that $\operatorname{dom}_{\mathrm{sa}}\mathbb{r}=\mathrm{M}_2^{\mathrm{sa}}(\mathbb{R})$ and $\operatorname{dom}_{\mathrm{sa}}\mathbb{r}\neq\mathrm{M}_3^{\mathrm{sa}}(\mathbb{R})$.

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