TRACE-POSITIVE POLYNOMIALS

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Abstract. In this paper positivity of polynomials in free noncommuting variables in a dimension-dependent setting is considered. That is, the images of a polynomial under finite-dimensional representations of a fixed dimension are investigated. It is shown that unlike in the dimension-free case, every trace-positive polynomial is (after multiplication with a suitable denominator - a hermitian square of a central polynomial) a sum of a positive semidefinite polynomial and commutators. Together with our previous results this yields the following Positivstellensatz: every trace-positive polynomial is modulo sums of commutators and polynomial identities a sum of hermitian squares with weights and denominators. Understanding trace-positive polynomials is one of the approaches to Connes' embedding conjecture.

1. Introduction

Interest in positivity questions involving noncommutative polynomials has been recently revived by Helton's seminal paper [He], in which he proved that a polynomial is a sum of squares if and only if its values in matrices of any size are positive semidefinite. Considering polynomials with *positive trace*, Schweighofer and the author [KS, Theorem 1.6] observed that Connes' embedding conjecture [Co, §V, pp. 105–107] on type II_1 von Neumann algebras is equivalent to a problem of describing polynomials whose values at tuples of self-adjoint $d \times d$ matrices (of norm at most 1) have nonnegative trace for every $d \geq 1$. This result is the motivation for the present work. Here we investigate polynomials whose values at tuples of $d \times d$ matrices have nonnegative trace for a *fixed* $d \geq 1$. We show that such a polynomial is (after multiplication with a hermitian square of a suitable central polynomial) a sum of commutators and of a polynomial whose values at tuples of $d \times d$ matrices are positive semidefinite. The latter were characterized in [KU] leading us to the following Positivstellensatz: every polynomial with nonnegative trace on $d \times d$ matrices is modulo sums of commutators and polynomial identities for $d \times d$ matrices a sum of hermitian squares with weights and denominators; see §4 for the precise formulation.

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The organization of this paper is follows: §2 introduces main notions and interprets them in full matrix algebras, in §3 these notions are considered for free algebras, while our main results are presented in §4.

2. Basic notions and a motivating example

Let R be an associative ring with 1 and *involution* $a \mapsto a^*$ (i.e., $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $a^{**} = a$ for all $a, b \in R$). Then we denote by $\operatorname{Sym} R := \{a \in R \mid a = a^*\}$ its set of *symmetric* elements. Elements of the form a^*a and $ab - ba$ $(a, b \in A)$ are called *hermitian squares* and *commutators*, respectively. We introduce an equivalence relation (*cyclic equivalence*) on R by declaring $a \overset{\text{cyc}}{\sim} b$ if and only if $a - b$ is a sum of commutators in R. For notational convenience we write

$$\Sigma^2 R := \Big\{ \sum a_i^* a_i \mid a_i \in R \Big\} \subseteq \operatorname{Sym} R, \quad \Theta^2 R := \Big\{ a \in R \mid \exists b \in \Sigma^2 R : a \overset{\text{cyc}}{\sim} b \Big\}$$

for the sets of (finite) sums of hermitian squares, and sums of hermitian squares and commutators in R, respectively.

Throughout this paper k will denote $\mathbb{R}$ or $\mathbb{C}$.

2.1. Matrices.

For a concrete example of these notions consider the ring $R = \mathrm{M}_d(k)$ of real or complex square matrices of a fixed size $d \geq 1$ endowed with the usual (complex conjugate) transposition of matrices, denoted here by $*$. Using $\succeq$ to denote the Löwner partial order (i.e., $A \succeq B$ iff $A - B$ is positive semidefinite), it is easy to see that for $A \in \mathrm{M}_d(k)$, we have

(A) $A \succeq 0$ if and only if $A \in \Sigma^2 \mathrm{M}_d(k)$;
(B) $\operatorname{tr}(A) = 0$ if and only if $A \overset{\text{cyc}}{\sim} 0$ in $\mathrm{M}_d(k)$;
(C) $\operatorname{tr}(A) \geq 0$ if and only if $A \in \Theta^2 \mathrm{M}_d(k)$.

Let us determine multiplication by which matrices respects these properties.

Lemma 2.1. Suppose $A \in \mathrm{M}_d(k)$ is such that for all $B \in \mathrm{M}_d(k)$,

$$B \succeq 0 \quad \Rightarrow \quad AB \succeq 0. \tag{1}$$

Then $A = \lambda$ for some $\lambda \in \mathbb{R}_{\geq 0}$.

Proof. Using (1) with $B = 1$, we obtain $A \succeq 0$. In particular, $A = A^*$.

Again by (1), A commutes with all positive semidefinite matrices, hence with all symmetric matrices, which are differences of two positive semidefinite matrices by

$$B = \frac{1}{4}(B+1)^2 - \frac{1}{4}(B-1)^2.$$

So A is scalar and the desired conclusion follows. ∎

Lemma 2.2. Suppose $A \in \mathrm{M}_d(k)$ is such that for all $B \in \mathrm{M}_d(k)$,

$$\operatorname{tr}(B) = 0 \quad \Rightarrow \quad \operatorname{tr}(AB) = 0. \tag{2}$$

Then $A = \lambda$ for some $\lambda \in k$.

Proof. Write $A=[a_{ij}]_{i,j=1}^d$. Let $i\neq j$. Then $B=\lambda E_{ij}$ has zero trace for every $\lambda\in k$. (Here E_{ij} denotes the $d\times d$ *matrix unit* with a one in the (i,j)-position and zeros elsewhere.) By (2), this implies

$$\lambda a_{ij}=\operatorname{tr}(AB)=0.$$

Since $\lambda\in k$ was arbitrary, $a_{ij}=0$.

Now let $B=\lambda(E_{ii}-E_{jj})$. Clearly, $\operatorname{tr}(B)=0$ and hence

$$\lambda(a_{ii}-a_{jj})=\operatorname{tr}(AB)=0.$$

As before, this gives $a_{ii}=a_{jj}$. ∎

Lemma 2.3. Suppose $A\in\mathrm{M}_d(k)$ is such that for all $B\in\mathrm{M}_d(k)$,

$$\operatorname{tr}(B)\geq 0 \quad\Rightarrow\quad \operatorname{tr}(AB)\geq 0. \tag{3}$$

Then $A=\lambda$ for some $\lambda\in\mathbb{R}_{\geq 0}$.

Proof. By Lemma 2.2, A is scalar. In addition to that, $a_{ii}=\operatorname{tr}(AE_{ii})\geq 0$ by (3), showing that A must be a nonnegative multiple of the identity. ∎

Likewise we can characterize matrices which map positive semidefinite matrices into matrices with nonnegative trace:

Lemma 2.4. Suppose $A\in\mathrm{M}_d(k)$ is such that for all $B\in\mathrm{M}_d(k)$,

$$B\succeq 0 \quad\Rightarrow\quad \operatorname{tr}(AB)\geq 0. \tag{4}$$

In the case $k=\mathbb{R}$, assume moreover that $A=A^*$. Then $A\succeq 0$.

Proof. This is just a restatement of the well-known self-duality of the cone of all positive semidefinite matrices. For $v\in k^d$ let $B=vv^*\succeq 0$. Then

$$0\leq\operatorname{tr}(AB)=\operatorname{tr}(Avv^*)=\operatorname{tr}(v^*Av)=\langle Av,v\rangle$$

showing A is positive semidefinite. ∎

We remark that converses of Lemmas 2.1 - 2.4 hold as well.

3. Positivity in free algebras

3.1. **Words and polynomials.** Fix $n\in\mathbb{N}$. Let $\underline{X}:=(X_1,\dots,X_n)$ and $\underline{X}^*:=(X_1^*,\dots,X_n^*)$ denote tuples of n distinct variables (or letters). By $\langle\underline{X},\underline{X}^*\rangle$ we denote the free monoid on $\{\underline{X},\underline{X}^*\}$ (consisting of *words* in $\underline{X},\underline{X}^*$) and let $k\langle\underline{X},\underline{X}^*\rangle$ be the semigroup algebra of $\langle\underline{X},\underline{X}^*\rangle$ over k (consisting of *polynomials* in noncommuting variables $\underline{X}$ and $\underline{X}^*$ with coefficients in k). We endow $k\langle\underline{X},\underline{X}^*\rangle$ with the involution $p\mapsto p^*$ mapping $X_j\mapsto X_j^*$ and extending complex conjugation on k. Thus $k\langle\underline{X},\underline{X}^*\rangle$ is the free $*$-algebra on $\underline{X}$ over k.

3.2. **Cyclic equivalence.** It is well-known and easy to see that trace-zero matrices are sums of commutators, i.e., cyclically equivalent to 0. Cyclic equivalence can also be easily tested in $k\langle \underline{X}, \underline{X}^*\rangle$:

(a) For $v, w \in \langle \underline{X}, \underline{X}^*\rangle$, we have $v \overset{\text{cyc}}{\sim} w$ if and only if there are $v_1, v_2 \in \langle \underline{X}, \underline{X}^*\rangle$ such that $v = v_1 v_2$ and $w = v_2 v_1$. That is, $v \overset{\text{cyc}}{\sim} w$ if and only if w is a cyclic permutation of v.

(b) Polynomials $f = \sum_{w\in\langle \underline{X}, \underline{X}^*\rangle} a_w w$ and $g = \sum_{w\in\langle \underline{X}, \underline{X}^*\rangle} b_w w$ ($a_w, b_w \in k$) are cyclically equivalent if and only if for each $v \in \langle \underline{X}, \underline{X}^*\rangle$,

$$\sum_{\substack{w\in\langle \underline{X}, \underline{X}^*\rangle \\ w \overset{\text{cyc}}{\sim} v}} a_w = \sum_{\substack{w\in\langle \underline{X}, \underline{X}^*\rangle \\ w \overset{\text{cyc}}{\sim} v}} b_w. \tag{5}$$

3.3. **Evaluations and representations.** Let $d \in \mathbb{N}$. An n-tuple of matrices $\underline{A} \in (\mathrm{M}_d(k))^n$ gives rise to a $*$-*representation*

$$\mathrm{ev}_{\underline{A}} : k\langle \underline{X}, \underline{X}^*\rangle \to \mathrm{M}_d(k), \quad p \mapsto p(\underline{A}, \underline{A}^*). \tag{6}$$

We are interested in the values of a *fixed* element $f \in k\langle \underline{X}, \underline{X}^*\rangle$ under all these $*$-representations. If the size d of the matrices A_i is free, we talk about *dimension-free* properties, otherwise we call them *dimension-dependent*. We are mostly interested in the latter, but briefly review the former for the sake of completeness.

3.4. **Dimension-freeness.** Free analogs of properties (A) and (B) have been established, while a free version of (C) is closely related to an important open problem on operator algebras due to Connes; see below for further details.

Let $f \in \operatorname{Sym} k\langle \underline{X}, \underline{X}^*\rangle$.

(A)$^{\text{free}}$ $f(\underline{A}, \underline{A}^*) \succeq 0$ for all $d \in \mathbb{N}$ and all $\underline{A} \in \mathrm{M}_d(k)^n$ if and only if $f \in \Sigma^2 k\langle \underline{X}, \underline{X}^*\rangle$;

(B)$^{\text{free}}$ $\operatorname{tr}\big(f(\underline{A}, \underline{A}^*)\big) = 0$ for all $d \in \mathbb{N}$ and all $\underline{A} \in \mathrm{M}_d(k)^n$ if and only if $f \overset{\text{cyc}}{\sim} 0$ in $k\langle \underline{X}, \underline{X}^*\rangle$.

Here, (A)$^{\text{free}}$ is due to Helton [He] (see also [Mc, MP]), and (B)$^{\text{free}}$ was given by the author and Schweighofer in [KS, Theorem 2.1]; see also [CD, Lemma 2.9] for a proof inspired by free probability. For a recent study of trace-positive polynomials in a dimension-free setting see also [NT].

The obvious extension of (C) fails: there are $f \in \operatorname{Sym} k\langle \underline{X}, \underline{X}^*\rangle$ with positive trace everywhere, but still not cyclically equivalent to a sum of hermitian squares. The following is a variant of the noncommutative Motzkin polynomial [KS, Example 4.4] given in free (nonsymmetric) variables.

Example 3.1. Let X denote a single free variable and

$$\begin{aligned} M_0 := {} & 3X^4 - 3(XX^*)^2 - 4X^5X^* - 2X^3(X^*)^3 \\ & + 2X^2X^*X(X^*)^2 + 2X^2(X^*)^2XX^* + 2(XX^*)^3. \end{aligned}$$

Then the noncommutative Motzkin polynomial is

$$M := 1 + M_0 + M_0^* \in \operatorname{Sym} k\langle X, X^*\rangle.$$

It is trace-nonnegative everywhere since

$$M' := YZ^4Y + ZY^4Z - 3YZ^2Y + 1 \overset{\text{cyc}}{\sim} M\left(\frac{Y+\mathbb{i}Z}{2}, \frac{Y-\mathbb{i}Z}{2}\right) \in k\langle Y, Z\rangle$$

is trace-nonnegative on symmetric matrices [KS, Example 4.4]. Alternatively, $M(X^3, (X^*)^3) \in \Theta^2 k\langle X, X^*\rangle$. On the other hand, $M \notin \Theta^2 k\langle X, X^*\rangle$.[1]

Connes' embedding conjecture [Co, §V, pp. 105–107] states that every separable II_1-factor is embeddable in an ultrapower of the hyperfinite II_1-factor. As shown in [KS], understanding trace-positive polynomials in the dimension-free setting is the key to this problem as it is equivalent [KS, Theorem 1.6] to the following:

Conjecture 3.2 (Algebraic version of Connes' conjecture [KS, Conjecture 1.5])**.** *For $f \in \operatorname{Sym} k\langle \underline{X}, \underline{X}^*\rangle$ the following are equivalent:*

(i) $\operatorname{tr}\big(f(\underline{A}, \underline{A}^*)\big) \geq 0$ *for all $d \in \mathbb{N}$ and all tuples of contractions $\underline{A} \in \mathrm{M}_d(k)^n$;*
(ii) *for every $\varepsilon \in \mathbb{R}_{>0}$, $f + \varepsilon$ is cyclically equivalent to an element of the form*

$$\sum_j s_j^* s_j + \sum_{i,j} p_{ij}^*(1 - X_i^* X_i)p_{ij},$$

where $s_j, p_{ij} \in k\langle \underline{X}, \underline{X}^*\rangle$.

In the sequel we indicate an approach to this problem "from below". That is, we abandon the dimension-free setting and solve a Hilbert 17-type problem characterizing polynomials with nonnegative trace in a dimension-dependent setting. It is our belief that this might constitute an important step towards (a positive or negative resolution of) Connes' embedding conjecture.

4. Dimension-dependent positivity

The properties (A) and (B) for free algebras in a dimension-dependent setting are well understood due to our previous work [BK, KU]. Roughly speaking, a trace-zero polynomial is cyclically equivalent to a polynomial identity [BK, §4], and a positive semidefinite polynomial is a sum of hermitian squares with denominators and weights [KU, §5]. In this section property (C) is explored and we present our main result, a Positivstellensatz characterizing polynomials with nonnegative trace on all tuples of $d \times d$ matrices for *fixed* d. This is done in §4.3. Before that we recall generic matrices and universal division algebras with involution in §4.1 and take a look at polynomial preservers of the various notions of positivity in §4.2.

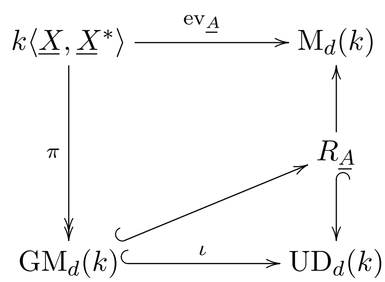
4.1. **Generic matrices and universal division algebras.** We assume the reader is familiar with the theory of polynomial identities as presented e.g. in [Pr1, Ro]. We review the notion of generic matrices and universal division algebras with involution and refer the reader to [Pr2, PS] for details.

Let $\zeta := (\zeta_{ij}^{(\ell)} \mid 1 \leq i,j \leq d, 1 \leq \ell \leq n)$ and $\bar{\zeta} := (\bar{\zeta}_{ij}^{(\ell)} \mid 1 \leq i,j \leq d, 1 \leq \ell \leq n)$ denote commuting variables. To keep the notation uniform, let

[1] Some of these computations were done with the aid of two computer algebra systems: `NCSOStools` [CKP] and `NCAlgebra` [HOMS].

$\underline{\zeta} := \zeta$ if $k = \mathbb{R}$ and $\underline{\zeta} := (\zeta, \bar{\zeta})$ otherwise. Form the polynomial $*$-algebra $k[\underline{\zeta}]$ endowed with the involution extending complex conjugation on k and fixing $\zeta_{ij}^{(\ell)}$ pointwise (if $k = \mathbb{R}$), respectively sending $\zeta_{ij}^{(\ell)} \mapsto \bar{\zeta}_{ij}^{(\ell)}$ (if $k = \mathbb{C}$). Consider the $d \times d$ matrices $Y_\ell := \big[\zeta_{ij}^{(\ell)}\big]_{1 \leq i,j \leq d} \in \mathrm{M}_d(k[\underline{\zeta}])$, $\ell \in \mathbb{N}$. Each Y_ℓ is called a *generic matrix*. The (unital) k-subalgebra of $\mathrm{M}_d(k[\underline{\zeta}])$ generated by the Y_ℓ and their (complex conjugate) transposes is the *ring of generic matrices with involution* $\mathrm{GM}_d(k)$. Equivalently, $\mathrm{GM}_d(k) \cong k\langle \underline{X}, \underline{X}^* \rangle / \mathfrak{t}_d$, where $\mathfrak{t}_d \subseteq k\langle \underline{X}, \underline{X}^* \rangle$ is the T-ideal of polynomial identities for $d \times d$ matrices.

For $d \geq 2$, $\mathrm{GM}_d(k)$ is a prime PI algebra and a domain (cf. [PS, §II]). Hence its central localization is a central simple algebra $\mathrm{UD}_d(k)$ with involution, which we call (by an abuse of notation) the *universal division algebra*. Relating these notions to $*$-representations of the free $*$-algebra is the following commutative diagram: for $d \in \mathbb{N}$ and $\underline{A} \in \mathrm{M}_d(k)^n$ let $R_{\underline{A}}$ denote all the elements of $\mathrm{UD}_d(k)$ which are regular at $\underline{A}$. Then:



For a more geometric viewpoint of the ring of generic matrices and the universal division algebra we refer the reader to [Pr2, Sa]. The standard textbook on central simple algebras with involution is [KMRT].

4.2. **Polynomial preservers.** In this subsection we present versions of Lemmas 2.1 - 2.4 in the context of free $*$-algebras. To avoid trivialities, we assume throughout that $d \geq 2$.

Lemma 4.1. Suppose $f \in k\langle \underline{X}, \underline{X}^* \rangle$ is such that for all $g \in k\langle \underline{X}, \underline{X}^* \rangle$,

$$g \succeq 0 \text{ on } d \times d \text{ matrices} \quad \Rightarrow \quad fg \succeq 0 \text{ on } d \times d \text{ matrices.} \tag{7}$$

Then f is a central polynomial positive semidefinite on $d \times d$ matrices.

Proof. Using (7) with $g = 1$, we see f is positive semidefinite on $d \times d$ matrices. Thus there is no harm in assuming $f = f^*$.

Again by (7), $fg - gf$ vanishes on all $d \times d$ matrices for all polynomials g of the form $g = h^*h$. That is, $[f, g]$ is a polynomial identity of $d \times d$ matrices. Now the same holds true for all symmetric g as

$$2[f, g] + [f, g^2] = [f, (1 + g)^2]$$

is a polynomial identity by the above. Hence f commutes (modulo the T-ideal of identities) with all symmetric polynomials.

Every element of $\mathrm{UD}_d(k)$ can be represented as rs^{-1} for some $r, s \in \mathrm{GM}_d(k)$ with $s = s^* \in Z(\mathrm{GM}_d(k))$. Such an element is symmetric iff $r = r^*$. So $\pi(f)$ commutes with all symmetric elements of $\mathrm{UD}_d(k)$. By Dieudonné's theorem

[Di, Lemma 1], the latter generate $\mathrm{UD}_d(k)$. Hence $\pi(f) \in Z(\mathrm{UD}_d(k))$ and f is indeed a central polynomial.

(Note: once we have established that f commutes with all symmetric polynomials, an easier argument is available if $k = \mathbb{C}$. In this case one immediately obtains that f also commutes with all skew symmetric polynomials as these are all of the form $\mathbb{i}g$ for symmetric g.) ∎

Lemma 4.2. Suppose $f \in k\langle \underline{X}, \underline{X}^* \rangle$ is such that for all $g \in k\langle \underline{X}, \underline{X}^* \rangle$,

$$\mathrm{tr}(g) = 0 \text{ on } d \times d \text{ matrices} \quad \Rightarrow \quad \mathrm{tr}(fg) = 0 \text{ on } d \times d \text{ matrices.} \tag{8}$$

Then f is a central polynomial.

Proof. Let $g = [h_1, h_2]$ for some $h_i \in k\langle \underline{X}, \underline{X}^* \rangle$. Then

$$fg = f[h_1, h_2] = [f, h_1 h_2] + [h_1, f h_2] + h_1[h_2, f]. \tag{9}$$

Since $\mathrm{tr}(g) = 0$ on all $d \times d$ matrices, this implies $\mathrm{tr}(h_1[h_2, f]) = 0$ on $d \times d$ matrices. Fix h_2 and denote $r := [h_2, f]$. Then r satisfies the following:

$$\mathrm{tr}(pr) = 0 \text{ on } d \times d \text{ matrices}$$

for all $p \in k\langle \underline{X}, \underline{X}^* \rangle$. Taking $p = -r^*$ leads to $-\mathrm{tr}(r^* r) = 0$ and hence, $r = 0$ on all $d \times d$ matrices. That is, r is an identity of $d \times d$ matrices. As $r = [h_2, f]$ and h_2 was arbitrary, this implies f is a central polynomial. ∎

Lemma 4.3. Suppose $f \in k\langle \underline{X}, \underline{X}^* \rangle$ is such that for all $g \in k\langle \underline{X}, \underline{X}^* \rangle$,

$$\mathrm{tr}(g) \geq 0 \text{ on } d \times d \text{ matrices} \quad \Rightarrow \quad \mathrm{tr}(fg) \geq 0 \text{ on } d \times d \text{ matrices.} \tag{10}$$

Then f is a central polynomial positive semidefinite on $d \times d$ matrices.

Proof. If $\mathrm{tr}(g) = 0$, then by (10), $\mathrm{tr}(fg) \geq 0$ and $\mathrm{tr}(-fg) \geq 0$ on $d \times d$ matrices. That is, $\mathrm{tr}(fg) = 0$. Now by Lemma 4.2, f is a central polynomial.

Applying (10) with $g = 1$ yields $f(\underline{A}, \underline{A}^*) = \mathrm{tr}\big(f(\underline{A}, \underline{A}^*)\big) \geq 0$ for all $\underline{A} \in \mathrm{M}_d(k)^n$ showing f is positive semidefinite on $d \times d$ matrices. ∎

Likewise we can characterize polynomials which map positive semidefinite polynomials into trace-nonnegative ones. At the same time this indicates how to build examples of trace-nonnegative polynomials. As we shall see in the next subsection, the procedure is essentially exhaustive.

Lemma 4.4. Suppose $f \in \mathrm{Sym}\, k\langle \underline{X}, \underline{X}^* \rangle$ is such that for all $g \in k\langle \underline{X}, \underline{X}^* \rangle$,

$$g \succeq 0 \text{ on } d \times d \text{ matrices} \quad \Rightarrow \quad \mathrm{tr}(fg) \geq 0 \text{ on } d \times d \text{ matrices.} \tag{11}$$

Then f is positive semidefinite on $d \times d$ matrices.

Proof. Assume f is not positive semidefinite on $d \times d$ matrices. Then there exists an n-tuple $\underline{A} = (A_1, \ldots, A_n) \in \mathrm{M}_d(k)^n$ with

$$f(\underline{A}, \underline{A}^*) \not\succeq 0. \tag{12}$$

Let $\mathcal{A} \subseteq \mathrm{M}_d(k)$ denote the $*$-subalgebra generated by the $A_1, \ldots, A_n$. Since the hermitian square of a nonzero matrix is not nilpotent, $\mathcal{A}$ is semisimple. By the Artin-Wedderburn theorem, $\mathcal{A}$ is $*$-isomorphic to a direct sum of full matrix algebras. We distinguish two cases.

CASE 1: if $k=\mathbb{C}$, then there is a $*$-isomorphism

$$\mathcal{A} \cong \bigoplus_{j=1}^{s} \mathrm{M}_{d_j}(\mathbb{C}) \tag{13}$$

for some $d_j \in \mathbb{C}$, and $\sum_j d_j \le d$. This induces a block diagonalization of the A_j as follows:

$$A_j = \begin{bmatrix} A_{j,1} & & \\ & \ddots & \\ & & A_{j,s} \end{bmatrix}, \quad A_{j,k} \in \mathrm{M}_{d_k}(\mathbb{C}).$$

By (12), there is a j such that $\underline{A}_{(j)} = (A_{1,j},\dots,A_{n,j}) \in \mathrm{M}_{d_j}(\mathbb{C})^n$ satisfies $f(\underline{A}_{(j)},\underline{A}^*_{(j)}) \not\succeq 0$. Choose $u \in \mathbb{C}^{d_j}$ with

$$\langle f(\underline{A}_{(j)},\underline{A}^*_{(j)})u,u\rangle < 0. \tag{14}$$

There is a $B \in \mathrm{M}_{d_j}(\mathbb{C})$ with $Be_{i,d_j} = u$ for all $i=1,\dots d_j$. (Here e_{i,d_j} are the standard basis vectors for $\mathbb{C}^{d_j}$.) By the construction of $\mathcal{A}$ and (13), there is a $h \in \mathbb{C}\langle \underline{X},\underline{X}^*\rangle$ with $h(\underline{A}_{(j)},\underline{A}^*_{(j)}) = B$. Let $g = hh^*$. Then

$$\begin{aligned}
\operatorname{tr}\big((fg)(\underline{A}_{(j)},\underline{A}^*_{(j)})\big) &= \operatorname{tr}\big((h^*fh)(\underline{A}_{(j)},\underline{A}^*_{(j)})\big) \\
&= \sum_{i=1}^{d_j} \langle h^*(\underline{A}_{(j)},\underline{A}^*_{(j)}) f(\underline{A}_{(j)},\underline{A}^*_{(j)}) h(\underline{A}_{(j)},\underline{A}^*_{(j)}) e_{i,d_j}, e_{i,d_j}\rangle \\
&= \sum_{i=1}^{d_j} \langle f(\underline{A}_{(j)},\underline{A}^*_{(j)}) Be_{i,d_j}, Be_{i,d_j}\rangle \qquad (15) \\
&= \sum_{i=1}^{d_j} \langle f(\underline{A}_{(j)},\underline{A}^*_{(j)}) u,u\rangle < 0.
\end{aligned}$$

As this contradicts our assumption (11), we conclude $f \succeq 0$ on $d \times d$ matrices.

CASE 2: if $k=\mathbb{R}$, the reasoning is the same with a minor technical modification. Let

$$\mathcal{A} \cong \bigoplus_{j=1}^{s} \mathrm{M}_{d_j}(\mathbb{R}) \oplus \bigoplus_{k=1}^{r} \mathrm{M}_{e_k}(\mathbb{C}) \oplus \bigoplus_{\ell=1}^{p} \mathrm{M}_{f_\ell}(\mathbb{H}) \tag{16}$$

for some $d_j, e_k, f_\ell \in \mathbb{N}$. If there is a tuple $\underline{A} \in \mathrm{M}_{d_j}(\mathbb{R})^n$ with $f(\underline{A},\underline{A}^*) \not\succeq 0$, we proceed as in Case 1. If there is a $\underline{A} \in \mathrm{M}_{e_k}(\mathbb{C})^n$ with $0 \not\preceq f(\underline{A},\underline{A}^*) \in \mathrm{M}_{e_k}(\mathbb{C})$, we proceed as follows. Let V be the invariant subspace of $\mathbb{R}^d$ corresponding to the action of $\mathrm{M}_{e_k}(\mathbb{C})$. There is a $u \in V$ with $\langle f(\underline{A},\underline{A}^*)u,u\rangle < 0$. Pick a basis (over $\mathbb{C}$) $\{v_1,\dots,v_{e_k}\}$ of V and let $B \in \mathrm{M}_{e_k}(\mathbb{C})$ satisfy $Bv_j = u$ for all j. Choose $h \in \mathbb{R}\langle \underline{X},\underline{X}^*\rangle$ with $h(\underline{A},\underline{A}^*) = B$ and $g = hh^*$. Then the *complex* trace z of $(fg)(\underline{A},\underline{A}^*)$ is negative by the same computation as in (15). Hence the *real* trace satisfies

$$\operatorname{tr}\big((fg)(\underline{A},\underline{A}^*)\big) = \frac{z+\bar{z}}{2} < 0.$$

The remaining case of quaternion matrices is dealt with similarly. We leave this as an exercise for the reader. ■

It is clear that converses of Lemmas 4.1 - 4.4 hold true. Also, with the exception of (11) which is satisfied when f is a sum of hermitian squares, there are no nonconstant dimension-free polynomial preservers.

4.3. **The dimension-dependent tracial Positivstellensatz.** Our main tool for describing trace-nonnegative polynomials is the following proposition deduced from the properties of the reduced trace [KMRT, §1] on $\mathrm{UD}_d(k)$.

Proposition 4.5. *For every $f \in k\langle \underline{X}, \underline{X}^*\rangle$ and $d \in \mathbb{N}$ there exists a nonvanishing central polynomial for $d\times d$ matrices $c \in k\langle \underline{X}, \underline{X}^*\rangle$ such that cf is cyclically equivalent to a central polynomial. That is,*

$$cf \overset{\mathrm{cyc}}{\sim} c' \tag{17}$$

for some central polynomial c'.

Proof. Consider $F := \iota(\pi(f)) \in \mathrm{UD}_d(k)$. So $\mathrm{Trd}(F) \in Z(\mathrm{UD}_d(k))$ and there is a nonvanishing central polynomial $c_0 \in k\langle \underline{X}, \underline{X}^*\rangle$ and a central polynomial c_0' with

$$\mathrm{Trd}(F) = \pi(c_0')\pi(c_0)^{-1}. \tag{18}$$

Since Trd is $Z(\mathrm{UD}_d(k))$-linear, this yields $\mathrm{Trd}(\pi(c_0 f - c_0')) = 0$. By the Amitsur-Rowen result [AR, Theorem 2.4], $\pi(c_0 f - c_0') \overset{\mathrm{cyc}}{\sim} 0$ in $\mathrm{UD}_d(k)$. Clearing denominators shows

$$\pi(cf - c'') \overset{\mathrm{cyc}}{\sim} 0 \tag{19}$$

in $\mathrm{GM}_d(k)$ for a nonvanishing central polynomial c and a central polynomial c''. Lifting (19) to $k\langle \underline{X}, \underline{X}^*\rangle$ gives the desired conclusion: $cf \overset{\mathrm{cyc}}{\sim} c'$. ■

Remark 4.6. Instead of the Amitsur-Rowen result used in this proof, we can apply the tracial Nullstellensatz [BK, Theorem 5.2]: once $\mathrm{Trd}(\pi(c_o f - c_0')) = 0$ has been established, by clearing denominators we obtain $\mathrm{tr}(\pi(c_0 c'' f - c_0' c'')) = 0$ for some nonvanishing central polynomial c''. Hence $\pi(c_0 c'' f - c' c'') \overset{\mathrm{cyc}}{\sim} 0$ in $\mathrm{GM}_d(k)$ by [BK, Theorem 5.2]. As before, lifting this relation to $k\langle \underline{X}, \underline{X}^*\rangle$ yields the desired conclusion.

We are now ready to give our main results characterizing trace-nonnegative polynomials.

Theorem 4.7. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ and suppose $f \in \mathrm{Sym}\, k\langle \underline{X}, \underline{X}^*\rangle$ satisfies*

$$\mathrm{tr}\left(f(\underline{A}, \underline{A}^*)\right) \geq 0 \tag{20}$$

for all $\underline{A} \in \mathrm{M}_d(k)^n$. Then there is a nonvanishing central polynomial for $d \times d$ matrices $c \in k\langle \underline{X}, \underline{X}^*\rangle$ such that cfc^* is cyclically equivalent to a polynomial $g \in k\langle \underline{X}, \underline{X}^*\rangle$ that is positive semidefinite on $d \times d$ matrices, i.e.,

$$cfc^* \overset{\mathrm{cyc}}{\sim} g \quad \text{and} \quad g \succeq 0 \text{ on } d \times d \text{ matrices.} \tag{21}$$

Proof. This is a consequence of Proposition 4.5. Indeed, there is a nonvanishing central polynomial c with

$$cf \overset{\mathrm{cyc}}{\sim} c' \tag{22}$$

for a central polynomial c'. Multiplying (22) with c^* (from the right) shows

$$cfc^* \overset{\mathrm{cyc}}{\sim} c'c^*. \tag{23}$$

For any $\underline{A} \in \mathrm{M}_d(k)^n$,

$$\begin{aligned} 0 &\le \mathrm{tr}(c(\underline{A},\underline{A}^*)f(\underline{A},\underline{A}^*)c(\underline{A},\underline{A}^*)^*) = \mathrm{tr}(c'(\underline{A},\underline{A}^*)c(\underline{A},\underline{A}^*)^*) \\ &= \mathrm{tr}((c'c^*)(\underline{A},\underline{A}^*)) = (c'c^*)(\underline{A},\underline{A}^*). \end{aligned} \tag{24}$$

So $g := c'c^*$ is a (central) polynomial positive semidefinite on $d \times d$ matrices satisfying

$$cfc^* \overset{\mathrm{cyc}}{\sim} g. \qquad \blacksquare$$

Remark 4.8. The proof shows that g in Theorem 4.7 can actually be taken to be a central polynomial.

Combining Theorem 4.7 with the dimension-dependent Positivstellensatz for positive semidefinite polynomials ([PS, Theorem 5.4] or [KU, Theorem 5.4]) yields:

Corollary 4.9. *Choose $\alpha_1, \dots, \alpha_m \in k\langle \underline{X}, \underline{X}^*\rangle$ whose images in $\mathrm{GM}_d(k)$ form a diagonalization of the quadratic form $\mathrm{Trd}(x^*x)$ on $\mathrm{UD}_d(k)$. Then for $f \in \mathrm{Sym}\, k\langle \underline{X}, \underline{X}^*\rangle$ the following are equivalent:*

(i) $\mathrm{tr}(f(\underline{A},\underline{A}^*)) \ge 0$ *for every* $\underline{A} \in \mathrm{M}_d(k)^n$*;*
(ii) *there exists a nonvanishing central polynomial $c \in k\langle \underline{X}, \underline{X}^*\rangle$, a polynomial identity $h \in k\langle \underline{X}, \underline{X}^*\rangle$ for $d \times d$ matrices, and $p_{i,\varepsilon} \in k\langle \underline{X}, \underline{X}^*\rangle$ with*

$$cfc^* \overset{\mathrm{cyc}}{\sim} h + \sum_{\varepsilon \in \{0,1\}^m} \underline{\alpha}^{\varepsilon} \sum_i p_{i,\varepsilon}^* p_{i,\varepsilon}. \tag{25}$$

Remark 4.10. For experts we mention that (by applying the reduced trace to (25)) we can reformulate (25) as follows:

$$cfc^* \overset{\mathrm{cyc}}{\sim} h + t, \tag{26}$$

where c, h are as above, and t belongs to the preordering in $Z(\mathrm{UD}_d(k))$ generated by the α_j.

If $d = 2$ the weights α_j are superfluous since the reduced trace of a hermitian square is a sum of hermitian squares in this case (cf. [PS, p. 405] or [KU, §4]) and Corollary 4.9 simplifies as follows:

Corollary 4.11. *For $f \in \mathrm{Sym}\, k\langle \underline{X}, \underline{X}^*\rangle$ the following are equivalent:*

(i) $\mathrm{tr}(f(\underline{A},\underline{A}^*)) \ge 0$ *for every* $\underline{A} \in \mathrm{M}_2(k)^n$*;*
(ii) *there exists a nonvanishing central polynomial $c \in k\langle \underline{X}, \underline{X}^*\rangle$, and a polynomial identity $h \in k\langle \underline{X}, \underline{X}^*\rangle$ for 2×2 matrices, such that*

$$cfc^* \in h + \Theta^2\, k\langle \underline{X}, \underline{X}^*\rangle. \tag{27}$$

Example 4.12. We finish this presentation with an example showing denominators are necessary for these results to hold. First of all, the Motzkin polynomial M from Example 3.1 is not cyclically equivalent to a sum of hermitian squares modulo a T-ideal of identities. Indeed, if

$$M \overset{\mathrm{cyc}}{\sim} h + \sum g_j^* g_j \tag{28}$$

for some $g_j \in k\langle \underline{X}, \underline{X}^*\rangle$ and a polynomial identity $h \in k\langle \underline{X}, \underline{X}^*\rangle$ for $d \times d$ matrices ($d \geq 2$), then

$$M_{\mathrm{cc}} = \operatorname{tr}\left(M\left(\begin{bmatrix} Y/2 & Z/2 \\ -Z/2 & Y/2 \end{bmatrix}\right)\right) = \sum \operatorname{tr}\left((g_j^* g_j)\left(\begin{bmatrix} Y/2 & Z/2 \\ -Z/2 & Y/2 \end{bmatrix}\right)\right),$$

where $M_{\mathrm{cc}} \in \mathbb{R}[Y,Z]$ denotes the commutative collapse $Y^4Z^2+Y^2Z^4-3Y^2Z^2+1$ of the noncommutative variant M' of the Motzkin polynomial (in symmetric variables). Since M_{cc} is not a sum of squares in $\mathbb{R}[Y,Z]$, and the trace of a hermitian square is a sum of squares, M does not satisfy a relation of the form (28). Hence a denominator is needed in Corollaries 4.9 and 4.11.

A little more work is required to show the necessity of the denominator in Theorem 4.7. Let $d \in \mathbb{N}$ be sufficiently large (any $d \geq 127 =$ the dimension of the vector space of all polynomials in X, X^* of degree ≤ 6 will do). Suppose M is cyclically equivalent to a polynomial g that is positive semidefinite on $d \times d$ matrices. Without loss of generality, $g \in \operatorname{Sym} k\langle \underline{X}, \underline{X}^*\rangle$. Choose g of smallest possible degree. If this degree is > 6, then the highest homogeneous component $g^{(\infty)}$ of g is positive semidefinite on $d \times d$ matrices and at the same time $g^{(\infty)} \overset{\mathrm{cyc}}{\sim} 0$. Hence $\operatorname{tr}\left(g^{(\infty)}\right) = 0$ on $d \times d$ matrices implying $g^{(\infty)}$ is a polynomial identity. Then $M \overset{\mathrm{cyc}}{\sim} (g - g^{(\infty)})$, $g - g^{(\infty)}$ is positive semidefinite and of degree smaller than g. This contradicts the minimality of g, so $\deg(g) \leq 6$.

Now g is positive semidefinite on $d \times d$ matrices for some $d \geq 127$ and is thus a sum of hermitian squares by Helton's sum of squares theorem [He]. But M is not cyclically equivalent to a sum of hermitian squares by the first part of this example.

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